PMATH 764: Assignment 4

Due: Friday, 28 June, 2013.

1. Let X be a variety and $f \in k(X)$.
   (a) Prove that zero set of f is the intersection of the pole set of $1/f$ with the domain of f. This implies, in particular, that the zero set of f is a closed subset of its domain $\mathrm{dom}(f)$.
   (b) Show that f is continuous with respect to the induced Zariski topology on $\mathrm{dom}(f)$ and the Zariski topology on $\mathbb{A}^1 \cong k$.
2. Let $X = V(y^2 - x^2(x+1)) \subset \mathbb{A}^2$. Let $z = \bar{y}/\bar{x} \in k(X)$. What are the pole sets of z and z^2? Are z and z^2 in $\Gamma(X)$? Justify your answer.
3. *Classification of irreducible conics in* $\mathbb{A}^2$. The zero set of an irreducible polynomial $f \in k[x,y]$ of degree two is called an *irreducible conic* in $\mathbb{A}^2$. Suppose that the field k is algebraically closed.
   (a) Show that any irreducible conic in $\mathbb{A}^2$ is isomorphic to $V(y - x^2)$ or $V(x^2 + y^2 - 1)$ under an appropriate affine coordinate change.
   (b) Show that $V(y - x^2)$ is isomorphic to $\mathbb{A}^1$, but that $V(x^2 + y^2 - 1)$ is not.
   (c) Consider the unit circle $X = V(x^2 + y^2 - 1)$. The *stereographic projection* of X from the north pole $N = (0,1) \in X$ onto the x-axis maps a point $P \in X$ to the point of intersection P' of the x-axis with the line passing through N and P (see). Verify that the stereographic projection is given by the rational map $\phi : X \to \mathbb{A}^1, (x,y) \mapsto x/(1-y)$, where the x-axis is identified with $\mathbb{A}^1$ by sending $(x,0)$ to x. Show that ϕ is a birational equivalence, thus proving that X is a rational curve. Consequently, all irreducible conics in $\mathbb{A}^2$ are rational.
4. Decompose into irreducible components the closed set $X \subset \mathbb{A}^3$ defined by $y^2 = xz, z^2 = y^3$. Prove that all its components are rational.
5. *Germs of functions.* In this problem, we examine a new but equivalent interpretation of the notion of rational function on a variety.

   Let $X \subseteq \mathbb{A}^n$ be a variety and U be an open subset of X. A function $f : U \to k$ is said to be *regular at* $p \in U$ if there exits an open neighbourhood $W \subseteq U$ of p, and polynomials $a, b \in k[x_1, \ldots, x_n]$ such that $b(q) \neq 0$ for all $q \in W$ and $f = a/b$ on W. Moreover, f is *regular on* U if it is regular at every point of U.

   (a) Consider pairs (U, f) with U a nonempty open subset of X and $f : U \to k$ a regular function on U. Two such pairs (U, f) and (V, g) are considered equivalent if and only if $f|_{U\cap V} = g|_{U\cap V}$, which we denote $(U, f) \sim (V, g)$. Show that the set of equivalence classes

   $$\{(U, f) : U \neq \emptyset \text{ is open in } X \text{ and } f : U \to k \text{ is regular on } U\}/\sim$$

   is nothing but the quotient field $k(X)$ of $\Gamma(X)$.
   (b) Let $p \in X$. A *germ of function at* p is a pair (U, f) where U is an open subset of X containing p and $f : U \to k$ is a regular function on U. Two germs of functions at p, say (U, f) and (V, g), are said to be equivalent if and only if $f|_{U\cap V} = g|_{U\cap V}$, which we again denote $(U, f) \sim (V, g)$. Verify that $\sim$ is an equivalence relation. Moreover, show that the set of equivalence classes of germs of functions at p coincides with the local ring $\mathcal{O}_p(X)$ of X at p.

   Note: $\mathcal{O}_p(X)$ is defined as the subring of $k(X)$ consisting of rational functions on X that are regular at p, and is called the *local ring of* X *at* p.
6. Let X and Y be affine varieties. Show that if there is a dominant rational map from X to Y, then $\dim Y \leq \dim X$.

7. Let $I \subset k[x_1, \cdots, x_n]$ be an ideal that can be generated by r elements. Show that every irreducible component of $V(I)$ has dimension $\geq n - r$.

8. Let X be an affine variety and let $p \in X$.

   (a) Show that there is a one-to-one correspondence between prime ideals in $\Gamma(X)$ contained in the maximal ideal M_p of p and prime ideals in $\mathcal{O}_p(X)$.

   (b) Use (a) to show that there is a one-to-one correspondence between the prime ideals of the local ring $\mathcal{O}_p(X)$ and the subvarieties of X containing p.

   (c) Use (b) to show that $\dim \mathcal{O}_p(X) = \dim X$, where $\dim \mathcal{O}_p(X)$ denotes the Krull dimension of $\mathcal{O}_p(X)$.

   Note: The *Krull dimension* of a Noetherian ring R is the number n of strict inclusions in the longest chain of prime ideals $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ in R.

9. (Optional) *Local isomorphisms.* Let X and Y be two affine varieties. Suppose that there are points $p \in X$ and $q \in Y$ such that the local rings $\mathcal{O}_p(X)$ and $\mathcal{O}_q(Y)$ are isomorphic as k-algebras. Show that there are open sets $p \in U \subseteq X$ and $q \in V \subseteq Y$ and an isomorphism of U to V that sends p to q. This is the local version of the statement "$X \simeq Y$ if and only if $\Gamma(X) \simeq \Gamma(Y)$".

   Note: A map $\phi : U \to V$ between open sets $U \subseteq X \subseteq \mathbb{A}^n$ and $V \subseteq Y \subseteq \mathbb{A}^m$ is said to be *regular* if there exist rational functions $f_1, \ldots, f_m \in k(x_1, \ldots, x_n)$ that are regular at every point in U and are such that $\phi(x) = (f_1(x), \ldots, f_m(x))$ for all $x \in U$. Moreover, U and V are called *isomorphic* if there exists a bijective regular map $\phi : U \to V$ whose inverse is regular.