PMATH 764: Assignment 3

Due: Monday, 14 February, 2011

1. Let k be an algebraically closed field with characteristic $p > 0$. Consider the map $\phi : \mathbb{A}^1 \to \mathbb{A}^1$ defined by $t \mapsto t^p$; this is called the *Frobenius morphism*. Show that ϕ is bijective but not an isomorphism.

2. Let X be a variety and $f \in k(X)$.

   (a) Show that zero set of f is the intersection of the pole set of $1/f$ with the domain of f.

   (b) Show that f is continuous in the Zariski topology.

3. Let $X = V(y^2 - x^2(x+1)) \subset \mathbb{A}^2$. Let $z = \bar{y}/\bar{x} \in k(X)$. What are the pole sets of z and z^2? Are z and z^2 in $\Gamma(X)$? Justify your answer.

4. *Alpha curves.* Consider the curve $X = V(y^2 - x^2(x-m)) \subset \mathbb{C}^2$, $m \in \mathbb{C}$, which is an example of alpha curve (a visualisation of such curves can be found, for example, at www.mi.sanu.ac.rs/vismath/lip/index.html).

   (a) Recall that the projection in $\mathbb{C}^2$ from the origin to the line $x = 1$ sends the point $p = (x_0, y_0) \neq (0,0)$ to the point q of intersection of $x = 1$ with the line joining p to the origin; verify that $q = (1, y_0/x_0)$. The rational map $\phi : X \to \mathbb{C}, (x, y) \mapsto y/x$ is therefore the projection of X to $x = 1$. Show that ϕ is a birational equivalence.

   (b) A point in $\mathbb{C}^2$ is called rational if its coordinates are rational numbers. Show that all the rational points on X (except the origin) are of the form $((a^2 + mb^2)/b^2, a(a^2 + mb^2)/b^3)$ with $a, b \in \mathbb{Z}$.

   (c) Show that although X is rational, it is not isomorphic to $\mathbb{C}$.

5. Consider the unit circle $X = V(x^2 + y^2 - 1) \subset \mathbb{C}^2$.

   (a) Show that X is not isomorphic to $\mathbb{C}$.

   (b) Verify that the stereographic projection of the unit circle X onto the x-axis is given by the rational map $\phi : X \to \mathbb{C}, (x, y) \mapsto x/(1-y)$. Show that ϕ is a birational equivalence, thus proving that X is a rational curve.

6. Let X be an affine variety and $p \in X$. The *local ring of X at p*, denoted by $\mathcal{O}_p(X)$, is the subring of $k(X)$ consisting of all rational functions defined at p. Moreover, the *maximal ideal of p* is $M_p(X) := \{f \in \mathcal{O}_p(X) \mid f(p) = 0\}$.

   Let X and Y be two affine varieties and $\varphi : X \to Y$ be a polynomial map. Let $p \in X$ and $q = \varphi(p)$. Prove that the pullback map $\varphi^* : \Gamma(Y) \to \Gamma(X)$ extends uniquely to a ring homomorphism (also written φ^*) from $\mathcal{O}_q(Y)$ to $\mathcal{O}_p(X)$ such that $\varphi^*(M_q(Y)) \subset M_p(X)$. However, note that φ^* may not extend to all of $k(Y)$: explain why that is true.

7. Let X and Y be affine varieties. Show that if there is a dominant rational map from X to Y, then $\dim Y \leq \dim X$.

8. Let $I \subset k[x_1, \cdots, x_n]$ be an ideal that can be generated by r elements. Show that every irreducible component of $V(I)$ has dimension $\geq n - r$.