PMATH 764: Assignment 2

Due: Friday, 7 June, 2013.

1. Let k be an infinite field. Prove that any non-empty Zariski open subset of $\mathbb{A}^n(k)$ is dense.

   Note. A subset Y of a topological space X is called *dense* if $\overline{Y} = X$.

2. Let V and W be algebraic sets in $\mathbb{A}^n(k)$ such that $V \subset W$. Show that each irreducible component of V is contained in some irreducible component of W.

3. Let k be a infinite field (which may have characteristic 2). Are the following ideals prime, radical, or closed in $k[x, y]$? Justify your answers.

   (a) $< x, y^2 - 1 >$

   (b) $< x + y, xy >$

   (c) $< x^3 - y^2 >$

4. Let k be an algebraically closed field (which may have characteristic 2). Find the irreducible components of the following algebraic sets.

   (a) $V(x^3 + x^2y - 2xy - 2y^2)$ in $\mathbb{A}^2(k)$.

   (b) $V(x^2 - xy, xy - x^4)$ in $\mathbb{A}^2(k)$.

   (c) $V(x^2 - yz, xz - x)$ in $\mathbb{A}^3(k)$.

5. Let E be the curve in $\mathbb{R}^2$ given in polar coordinates by $r = \theta$. Show that E is dense in $\mathbb{R}^2$ in the Zariski topology.

6. Let R be a ring with identity and I be an ideal in R. Prove that there is a one-to-one correspondence between radical (resp. prime, resp. maximal) ideals of R containing I and radical (resp. prime, resp. maximal) ideals of R/I.

7. *Subvarieties.* Let $X \subset \mathbb{A}^n$ be a variety. A *subvariety* of X is then defined as a variety $Y \subset \mathbb{A}^n$ that is contained in X.

   (a) Verify that the induced Zariski topology on X coincides with the topology on X whose closed sets are the algebraic sets of $\mathbb{A}^n$ contained in X. The same is of course true for any subvariety Y of X.

   (b) Suppose that k is algebraically closed. Show that there is a one-to-one correspondence between algebraic subsets (resp. subvarieties, resp. points) of X and radical ideals (resp. prime ideals, resp. maximal ideals) of $\Gamma(X) := k[x_1, \ldots, x_n]/I(X)$.

   (c) Let Y be a subvariety of X, and let $I_X(Y)$ be the ideal of $\Gamma(X)$ corresponding to Y. Show that $\Gamma(Y)$ is isomorphic to $\Gamma(X)/I_X(Y)$.