PMATH 764: Assignment 2

Due: Monday, 31 January, 2011

1. Let k be an infinite field. Prove that any non-empty Zariski open subset of $\mathbb{A}^n(k)$ is dense.

2. Let V and W be algebraic sets in $\mathbb{A}^n(k)$ such that $V \subset W$. Show that each irreducible component of V is contained in some irreducible component of W.

3. Let k be an algebraically closed field, which may have characteristic 2. Find the irreducible components of the following algebraic sets.

   (a) $V(x^3 + x^2y - 2xy - 2y^2)$ in $\mathbb{A}^2(k)$.

   (b) $V(x^2 - xy, xy - x^4)$ in $\mathbb{A}^2(k)$.

   (c) $V(x^2 - yz, xz - x)$ in $\mathbb{A}^3(k)$.

4. Let E be the curve in $\mathbb{R}^2$ given in polar coordinates by $r = \theta$. Show that E is dense in $\mathbb{R}^2$ in the Zariski topology.

5. Sketch the algebraic curve $X = V(y^2 - x^2(x-1))$ in $\mathbb{R}^2$. Is X irreducible in the Zariski topology on $\mathbb{R}^2$? Is your answer consistent with the sketch you drew? Explain. What happens if you consider the curve in $\mathbb{C}^2$ instead?

6. Let R be a ring with identity and I be an ideal in R. Then there is a one-to-one correspondence between radical (resp. prime, resp. maximal) ideals of R containing I and radical (resp. prime, resp. maximal) ideals of R/I.

7. *Subvarieties.* Let $X \subset \mathbb{A}^n$ be a variety. A subvariety of X is then a variety $Y \subset \mathbb{A}^n$ that is contained in X.

   (a) Verify that the induced Zariski topology on X coincides with the topology on X whose closed sets are the algebraic sets of $\mathbb{A}^n$ contained in X. The same is of course true for any subvariety Y of X.

   (b) Suppose that k is algebraically closed. Prove that there is a one-to-one correspondence between algebraic subsets (resp. subvarieties, resp. points) of X and radical ideals (resp. prime ideals, resp. maximal ideals) of $\Gamma(X)$.

   (c) Let Y be a subvariety of X, and let $I_X(Y)$ be the ideal of $\Gamma(X)$ corresponding to Y. Show that $\Gamma(Y)$ is isomorphic to $\Gamma(X)/I_X(Y)$.

8. Determine whether or not the following are varieties.

   (a) The orthogonal group $O(n, k)$, where k is an algebraically closed field with $\mathrm{char}(k) \neq 2$.

(b) The special unitary group $SU(2, \mathbb{C})$.

(c) $V(xz - y^2, yz - x^3, z^2 - x^2y) \subset \mathbb{C}^3$.

9. Let X and Y be two varieties and $\phi : X \to Y$ be a polynomial map.

   (a) Show that ϕ^* is injective if and only if $\overline{\phi(X)} = Y$.

   (b) Show that ϕ^* is surjective if and only if ϕ has a polynomial left inverse.