PMATH 764: Assignment 1

Due: Monday, 17 January, 2011

1. Suppose that X is an algebraic set and L is a line in $\mathbb{A}^n(k)$ such that $L \not\subset X$. Show that $L \cap X$ is either empty or a finite set of points.

2. Determine whether or not the following sets are algebraic.

   (a) The set of points in $\mathbb{R}^2$ whose polar coordinates (r, θ) satisfy the equation $r = \theta$.

   (b) The set of points in $\mathbb{R}^2$ whose polar coordinates (r, θ) satisfy $r = \cos\theta$.

   (c) $\{(\cos t, 1, \sin t) \mid t \in \mathbb{R}\} \subset \mathbb{R}^3$.

   (d) $\{(\cos t, t, \sin t) \mid t \in \mathbb{R}\} \subset \mathbb{R}^3$.

   (e) $\{v \in \mathbb{R}^4 \mid |v|^2 = 1\} \subset \mathbb{R}^4$,where $|\cdot|$ represents length.

   (f) $\{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} \subset \mathbb{C}^2$, where $|x + iy|^2 = x^2 + y^2$ for $x, y \in \mathbb{R}$.

3. Consider the real line $\mathbb{R}$ endowed with the Zariski topology. Verify that, given any two points $p, q \in \mathbb{R}$, any two open neighbourhoods of p and q have a non-empty intersection, thus proving that the Zariski topology on $\mathbb{R}$ is not Hausdorff.

4. If k is a finite field, show that every subset of $\mathbb{A}^n(k)$ is both open and closed in the Zariski topology. Is the Zariski topology Hausdorff in this case?

5. Let k be any field and $M_{n\times n}(k)$ be the set of $n \times n$ matrices with entries in k. This set can naturally be identified with $\mathbb{A}^{n^2}(k)$, by considering the (ordered) n^2 entries of a matrix as a point in $\mathbb{A}^{n^2}(k)$, and is thus endowed with the Zariski topology. Show that the group $GL(n, k)$ of all invertible matrices in $M_{n\times n}(k)$ is open in the Zariski topology on $M_{n\times n}(k)$.

6. Let $V \subset \mathbb{A}^n(k)$ and $W \subset \mathbb{A}^m(k)$ be algebraic sets. Show that $V \times W := \{(a_1, ..., a_n, b_1, ..., b_m) \mid (a_1, ..., a_n) \in V, (b_1, ..., b_m) \in W\}$ is an algebraic set in $\mathbb{A}^{n+m}(k)$. It is called the *product* of V and W.

7. The *product topology* on the Cartesian product $X \times Y$ of two topological spaces is defined as the topology whose open sets are the unions of subsets $A \times B$, where A and B are open subsets of X and Y, respectively. Let k be any field and consider the affine line $\mathbb{A}^1 = \mathbb{A}^1(k)$ and the affine plane $\mathbb{A}^2 = \mathbb{A}^2(k)$.

   (a) Describe the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$ in $\mathbb{A}^1 \times \mathbb{A}^1$.

   (b) Show that although $\mathbb{A}^2$ can be naturally identified with $\mathbb{A}^1 \times \mathbb{A}^1$, the Zariski topology on $\mathbb{A}^2$ is not equal to the product Zariski topology on $\mathbb{A}^1 \times \mathbb{A}^1$ you described in (a), if k is an infinite field.