Chapter 1

Affine Varieties

In this chapter, we will assume that $\Bbbk$ is infinite, since when $\Bbbk$ is finite the only irreducible algebraic sets in $\mathbb{A}^n(\Bbbk)$ are singletons.

1.0.1 Definition. An irreducible affine algebraic set is called an *affine (algebraic) variety*, or simply a *variety* if the context is clear. Any variety X is endowed with the induced (Zariski) topology, whose open sets are of the form $X \bigcap U$ for some open subset $U \subseteq \mathbb{A}^n$.

1.1 Coordinate Rings

Since affine varieties are defined by polynomials over a field, the most natural functions to consider on an affine variety are those that come from evaluating polynomials over the base field.

1.1.1 Definition. Let $X \subseteq \mathbb{A}^n$ be an affine variety. A function $F : X \to \Bbbk$ is called a *polynomial function* if there is an $f \in \Bbbk[x_1, \ldots, x_n]$ such that $F(x) = f(x)$ for all $x \in X$.

If we wish to consider all polynomial functions on X, we can not simply take the entire polynomial ring $\Bbbk[x_1, \ldots, x_n]$, because two polynomials may give the same function when restricted to X. If $\Bbbk$ is finite, this is no surprise, because many polynomials in $\Bbbk[x_1, \ldots, x_n]$ determine the same function on $\mathbb{A}^n$. However, if $\Bbbk$ is infinite then polynomials in $\Bbbk[x_1, \ldots, x_n]$ determine unique functions on $\mathbb{A}^n$, so if $f, g \in \Bbbk[x_1, \ldots, x_n]$, then f and g determine the same polynomial function on X if and only if $f - g \in \mathrm{I}(X)$. Therefore, at least when $\Bbbk$ is infinite, we can identify the polynomial functions on X with the quotient ring $\Bbbk[x_1, \ldots, x_n]/\,\mathrm{I}(X)$.

1.1.2 Definition. Let $X \subseteq \mathbb{A}^n$ be an affine variety. The quotient ring $\Gamma(X) = \Bbbk[x_1, \ldots, x_n]/\,\mathrm{I}(X)$ is called the *coordinate ring* of X. Other common notations are $\Bbbk[X]$ and $A(X)$.

One can look at $\Bbbk[x_1, \ldots, x_n]/\operatorname{I}(X)$ to determine the irreducibility of X, since $\Bbbk[x_1, \ldots, x_n]/\operatorname{I}(X)$ is a domain if and only if $\operatorname{I}(X)$ is prime, which happens if and only if X is irreducible.

1.1.3 Examples.

(i) If $X = \mathbb{A}^n$ then $\operatorname{I}(X) = 0$, so $\Gamma(X) = \Bbbk[x_1, \ldots, x_n]$.

(ii) If X is a single point then $\Gamma(X) = \Bbbk$ since defining a function on a point is the same as fixing a value for that point.

(iii) If $X = \operatorname{V}(y - x^2) \subseteq \mathbb{A}^2$ then $\Gamma(X) = \Bbbk[x, y]/\langle y - x^2 \rangle \cong \Bbbk[\overline{x}]$, where $\overline{x}$ is the residue class of x in $\Gamma(X)$.

1.1.4 Theorem. *If X is an affine variety then $\Gamma(X)$ is Noetherian.*

Proof: Suppose $X \subseteq \mathbb{A}^n$. Let $q : \Bbbk[x_1, \ldots, x_n] \to \Gamma(X)$ be the quotient map, and let I be an ideal of $\Gamma(X)$. Then $q^{-1}(I)$ is an ideal of $\Bbbk[x_1, \ldots, x_n]$, which is Noetherian, so $q^{-1}(I) = \langle f_1, \ldots, f_k \rangle$ for some $f_1, \ldots, f_k \in \Bbbk[x_1, \ldots, x_n]$. Therefore, $I = \langle q(f_1), \ldots, q(f_k) \rangle$, showing that $\Gamma(X)$ is Noetherian. □

The coordinate ring $\Gamma(X)$ has additional structure besides its ring structure. It is also a vector space over $\Bbbk$, where the vector space addition is the same as addition in the ring, and scalar multiplication coincides with multiplication in the ring. Such a ring is called a $\Bbbk$-*algebra*.

1.1.5 Examples.

(i) $\Bbbk[x_1, \ldots, x_n]$ is a $\Bbbk$-algebra.

(ii) Let A be a $\Bbbk$-algebra and I an ideal of A. Then I is also a vector subspace of A, and the ring quotient of A by I agrees with the vector space quotient of A by I. Hence A/I is also a $\Bbbk$-algebra.

1.2 Polynomial Maps

Continuing the theme of the previous section, we will now examine maps between two affine varieties. Since affine varieties are defined by the vanishing of polynomials, the natural functions between affine varieties are those whose components are polynomial functions.

1.2.1 Definition. Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be two affine varieties. A function $\varphi : X \to Y$ is called a *polynomial map* if there are polynomials $f_1, \ldots, f_m \in \Bbbk[x_1, \ldots, x_n]$ such that $\varphi(x) = (f_1(x), \ldots, f_m(x))$ for every $x \in X$.

1.2.2 Examples.

(i) Polynomial functions $F : X \to \Bbbk \cong \mathbb{A}^1$ are polynomial maps.

(ii) Any linear map $\mathbb{A}^n \to \mathbb{A}^m$ is a polynomial map.

(iii) Affine maps $\mathbb{A}^n \to \mathbb{A}^m : x \mapsto Ax + b$, where A is an $m \times n$ matrix over $\mathbb{k}$ and $b \in \mathbb{A}^m$, are polynomial maps. If A is invertible then the map $x \mapsto Ax + b$ is called a *affine coordinate change*.
(iv) Projections and inclusions are polynomial maps.
(v) The map $\varphi : \mathbb{A}^1 \to \mathrm{V}(y^2 - x^3) \subseteq \mathbb{A}^2$ given by $\varphi(t) = (t^2, t^3)$ is a polynomial map.

Polynomial maps give the natural notion of equivalence for affine varieties.

1.2.3 Definition. Let X and Y be affine varieties. A polynomial map $\varphi : X \to Y$ is said to be an *isomorphism* if it is a bijection and its inverse is a polynomial map. We say that X and Y are *isomorphic* if there exists an isomorphism $\varphi : X \to Y$, in which case we write $X \cong Y$.

1.2.4 Examples.
(i) $\varphi : \mathrm{V}(y-x^2) \subseteq \mathbb{A}^2 \to \mathbb{A}^1 : (x, x^2) \mapsto x$ has a polynomial inverse $\varphi^{-1}(t) = (t, t^2)$, so $\mathrm{V}(y - x^2) \cong \mathbb{A}^1$.
(ii) $\varphi : \mathbb{A}^1 \to X = V(y^2 - x^3) \subseteq \mathbb{A}^2 : t \mapsto (t^2, t^3)$ is a bijective polynomial map. Indeed, in the metric topology over $\mathbb{C}$, φ is a homeomorphism. However, φ does not have a polynomial inverse. Suppose that $\varphi^{-1} : X \to \mathbb{A}^1$ is polynomial. Then φ^{-1} is a polynomial function on X, so it is an element of $\Gamma(X)$. Moreover, $\Gamma(X) = \mathbb{k}[x, y]/\langle y^2 - x^3\rangle$. Since $\overline{y}^2 = \overline{x}^3$ in $\Gamma(X)$, any polynomial $f(x, y)$ can be written as $p(\overline{x}) + \overline{y}q(\overline{x})$ in $\Gamma(X)$. Therefore $\varphi^{-1}(x, y) = p(x)+yq(x)$ for some $p, q \in \mathbb{k}[x]$, and the composition $t \mapsto (t^2, t^3) \mapsto p(t^2)+t^3q(t^2)$, an expression of degree at least 2 in t. In particular, $\varphi^{-1}(t^2, t^3) \neq t$, so φ does not have a polynomial inverse.
(iii) Any two varieties which are isomorphic via an affine coordinate change are said to be *affinely equivalent*. For example, any conic (a curve given by a polynomial of degree 2) is affinely equivalent to a parabola, a hyperbola, an ellipse, the union of two lines, or a double line.

1.2.5 Proposition. *Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties and $\varphi : X \to Y$ a polynomial map. Then:*

(i) *$\varphi^{-1}(Z) \subseteq X$ is an algebraic set for every algebraic set $Z \subseteq Y$.*
(ii) *If X is irreducible, then $\overline{\varphi(X)}$ is irreducible.*

PROOF:
(i) This is simply that statement that φ is continuous in the Zariski topology. Indeed, if $Z = \mathrm{V}(g_1, \dots, g_r)$ then $\varphi^{-1}(Z) = \mathrm{V}(g_1 \circ \varphi, \dots, g_r \circ \varphi)$.
(ii) Let $Z = \overline{\varphi(X)}$. Suppose that $Z = Z_1 \cup Z_2$, where Z_1 and Z_2 are algebraic. Then $X = \varphi^{-1}(Z) = \varphi^{-1}(Z_1) \cup \varphi^{-1}(Z_2)$. By part (i), $\varphi^{-1}(Z_1)$ and $\varphi^{-1}(Z_2)$ are algebraic subsets of X. Therefore, by the irreducibility of X, either $X = \varphi^{-1}(Z_1)$ or $X = \varphi^{-1}(Z_2)$. Without loss of generality, assume that $X = \varphi^{-1}(Z_1)$. Then $\varphi(X) \subseteq Z_1$, so $\overline{\varphi(X)} \subseteq \overline{Z_1} = Z_1$, and $\overline{\varphi(X)} = Z_1$. Therefore, $\overline{\varphi(X)}$ is irreducible. □

So far we have three ways to test whether an algebraic set $X \subseteq \mathbb{A}^n$ is irreducible. We may ask:

(i) Is $\mathrm{I}(X)$ prime?
(ii) Is $\Bbbk[x_1, \ldots, x_n]/\,\mathrm{I}(X)$ an integral domain?
(iii) Is X the closure of the image of an irreducible algebraic set under a polynomial map?

1.2.6 Example. Consider $X = \mathrm{V}(y-x^2, z-x^3) \subseteq \mathbb{A}^3$, the twisted cubic. Note that $\mathrm{I}(X) = \langle y - x^2, z - x^3 \rangle$. One inclusion is obvious, and for any $f \in \mathrm{I}(X)$, by applying the division algorithm twice (once with respect to y and once with respect to z), we can write $f(x, y, z) = (y-x^2)g(x, y, z)+(z-x^3)h(x, z)+r(x)$. For all $x \in \Bbbk$, $(x, x^2, x^3) \in X$, so $r(x) = 0$ for all $x \in \Bbbk$, hence $r = 0$ and $f \in \langle y - x^2, z - x^3 \rangle$. In the quotient ring $\overline{y} = \overline{x}^2$ and $\overline{z} = \overline{x}^3$, so $\Bbbk[x, y, z]/I(X)$ is isomorphic to $\Bbbk[x]$, an integral domain. Therefore X is irreducible. On the other hand, $\varphi : \mathbb{A}^1 \to X, t \mapsto (t, t^2, t^3)$, is a surjective polynomial map. Therefore, since $\mathbb{A}^1$ is irreducible, so is $X = \varphi(\mathbb{A}^1)$.

A polynomial map between affine varieties acts naturally on their coordinate rings. Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties and $\varphi : X \to Y$ a polynomial map. Pick $f_1, \ldots, f_m \in \Bbbk[x_1, \ldots, x_n]$ such that $\varphi(x) = (f_1(x), \ldots, f_m(x))$. If $g \in \Bbbk[x_1, \ldots, x_n]$ then $g \circ \varphi = g(f_1, \ldots, f_m)$ is a polynomial in $\Bbbk[x_1, \ldots, x_n]$. If $g \in \mathrm{I}(Y)$, then

$$(g \circ \varphi)(x) = g(f_1(x), \ldots, f_m(x)) = 0$$

for every $x \in X$ because $\varphi(x) \in Y$. Thus $g\circ\varphi \in \mathrm{I}(X)$. It follows that φ induces a well-defined map $\varphi^* : \Gamma(Y) \to \Gamma(X)$ given by

$$\varphi^*(g + \mathrm{I}(Y)) = (g \circ \varphi) + \mathrm{I}(X).$$

We call φ^* the *pullback* of φ. As is shown by the next proposition, the pullback completely determines the original polynomial map.

1.2.7 Proposition. *Let X and Y be affine varieties. If $\varphi : X \to Y$ and $\psi : X \to Y$ are polynomial maps such that $\varphi^* = \psi^*$, then $\varphi = \psi$.*

PROOF: Consider $\Gamma(Y)$ as a quotient of $\Bbbk[y_1, \ldots, y_m]$. We have $\varphi^*(\overline{y_i}) = \psi^*(\overline{y_i})$, so $y_i \circ \varphi = y_i \circ \psi$. Let $\varphi = (f_1, \ldots, f_m)$ and $\psi = (g_1, \ldots, g_m)$ for some $f_1, \ldots, f_m, g_1, \ldots, g_m \in \Bbbk[x_1, \ldots, x_n]$. Then $y_i \circ \varphi = f_i$ and $y_i \circ \psi = g_i$, showing that $f_i = g_i$. Therefore, $\varphi = \psi$. □

Since coordinate rings naturally carry the additional structure of a $\Bbbk$-algebra, we would hope that the pullback of a polynomial map between affine varieties preserves this structure. Given $\Bbbk$-algebras A and B, we define a $\Bbbk$-*algebra homomorphism* from A to B to be a $\Bbbk$-linear ring homomorphism $\Phi : A \to B$, i.e. a ring homomorphism $\Phi : A \to B$ such that $\Phi(\alpha) = \alpha$ for every $\alpha \in \Bbbk$. Similarly, a $\Bbbk$-*algebra isomorphism* is a bijective $\Bbbk$-algebra homomorphism whose inverse is also a $\mathbb{k}$-algebra homomorphism. If there exists a $\mathbb{k}$-algebra isomorphism $\Phi : A \to B$, we say that A and B are *isomorphic*, in which case we write $A \cong B$.

1.2.8 Examples.

(i) Let A be a $\mathbb{k}$-algebra and I an ideal of A. Then the quotient map $q : A \to A/I$ is a $\mathbb{k}$-algebra homomorphism.

(ii) The map $\Phi : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]$ defined by

$$\Phi(a_n x^n + \cdots + a_1 x + a_0) = \overline{a_n} x^n + \cdots + \overline{a_1} x + \overline{a_0}$$

is a ring homomorphism that is not a $\mathbb{C}$-algebra homomorphism.

One important property of $\mathbb{k}$-algebra homomorphisms is that they preserve the evaluation of polynomials. If A is a $\mathbb{k}$-algebra and $f \in \mathbb{k}[x_1, \ldots, x_n]$, then we can view f as a function from A^n to A, simply by substituting elements of A into the expression for f. If $\Phi : A \to B$ is a $\mathbb{k}$-algebra homomorphism and $f \in \mathbb{k}[x_1, \ldots, x_n]$, then $\Phi(f(a_1, \ldots, a_n)) = f(\Phi(a_1), \ldots, \Phi(a_n))$ for all $a_1, \ldots, a_n \in A$. Indeed, this property is equivalent to Φ being a $\mathbb{k}$-algebra homomorphism.

This next proposition shows that the association of a coordinate ring to an affine variety and the pullback of polynomial maps define a contravariant functor from the category of affine varieties with polynomial maps as morphisms to the category of $\mathbb{k}$-algebras with $\mathbb{k}$-algebra homorphisms as morphisms.

1.2.9 Proposition (Functorality). *Let X, Y, and Z be affine varieties. Then:*

(i) *if $\varphi = \mathrm{id}_X$ then $\varphi^* = \mathrm{id}_{\Gamma(X)}$;*

(ii) *if $\varphi : X \to Y$ and $\psi : Y \to Z$ are polynomial maps, $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$;*

(iii) *if $\varphi : X \to Y$ is a polynomial map, $\varphi^* : \Gamma(Y) \to \Gamma(X)$ is a $\mathbb{k}$-algebra homomorphism.*

Proof:

(i) For every $g \in \Gamma(Y)$, $\mathrm{id}_X^*(g) = g \circ \mathrm{id}_X = g$.

(ii) For every $g \in \Gamma(Z)$, $(\psi \circ \varphi)^*(g) = g \circ \psi \circ \varphi = \varphi^*(g \circ \psi) = \varphi^* \circ \psi^*(g)$.

(iii) Let $f, g \in \Gamma(Y)$ and $\alpha \in \mathbb{k}$. Then

$$\varphi^*(\alpha f + g) = (\alpha f + g) \circ \varphi = \alpha f \circ \varphi + g \circ \varphi = \alpha \varphi^*(f) + \varphi^*(g),$$

$$\varphi^*(fg) = (fg) \circ \varphi = (f \circ \varphi)(g \circ \varphi) = \varphi^*(f)\varphi^*(g),$$

and φ^* clearly sends the identity in $\Gamma(Y)$ to the identity in $\Gamma(X)$. □

What is the range of the functor that takes an affine variety to its coordinate ring and takes polynomial maps to $\mathbb{k}$-algebra homomorphisms of the coordinate rings? More precisely:

(i) If $\Gamma(X) \cong \Gamma(Y)$, is $X \cong Y$? More generally, which $\mathbb{k}$-algebra homomorphisms from $\Gamma(Y)$ to $\Gamma(X)$ are pullbacks of polynomial maps?

(ii) Which $\Bbbk$-algebras are coordinate rings of affine varieties defined over $\Bbbk$?

The answer to the first question very simple: *every* $\Bbbk$-algebra homomorphism between coordinate rings is the pullback of a unique polynomial map.

1.2.10 Proposition. *Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties, and let $\Phi : \Gamma(Y) \to \Gamma(X)$ be a $\Bbbk$-algebra homomorphism. Then there exists a unique polynomial map $\varphi : X \to Y$ such that $\varphi^* = \Phi$.*

PROOF: Let $I = \mathrm{I}(X)$ and $J = \mathrm{I}(Y)$. Thus $\Gamma(X) = \Bbbk[x_1, \dots, x_n]/I$ and $\Gamma(Y) = \Bbbk[y_1, \dots, y_m]/J$. Let $\tilde{\Phi} : \Bbbk[y_1, \dots, y_m] \to \Gamma(X)$ be the map defined by $\tilde{\Phi}(g) = \Phi(g + J)$, i.e. the lift of Φ to $\Bbbk[y_1, \dots, y_m]$. Then $\tilde{\Phi}$ is a $\Bbbk$-algebra homomorphism, as it is the composition of two $\Bbbk$-algebra homomorphisms, Φ and the quotient map from $\Bbbk[y_1, \dots, y_m]$ to $\Gamma(Y)$. For $i = 1, \dots, m$, let f_i be a representative in $\Bbbk[x_1, \dots, x_n]$ for $\Phi(y_i + J)$, so that $\tilde{\Phi}(y_i) = \Phi(y_i + J) = f_i + I$. Then for any $g \in \Bbbk[y_1, \dots, y_m]$ we have

$$\begin{aligned} g(f_1, \dots, f_m) + I &= g(\tilde{\Phi}(y_1), \dots, \tilde{\Phi}(y_m)) \\ &= \tilde{\Phi}(g(y_1, \dots, y_m)) \\ &= \tilde{\Phi}(g). \end{aligned}$$

Let $\varphi : X \to \mathbb{A}^m$ be the map defined by $\varphi(x) = (f_1(x), \dots, f_m(x))$. In order to show that φ restricts to a polynomial map from X to Y, we only need to show that $\varphi(X) \subseteq Y$. Since $Y = \mathrm{V}(J)$, we want to show that every polynomial in J vanishes at $\varphi(x)$ for every $x \in X$. Fix $g \in J$. Then from the above,

$$\begin{aligned} g(f_1 + I, \dots, f_m + I) &= g(f_1, \dots, f_m) + I \\ &= \tilde{\Phi}(g) \\ &= 0 + I, \end{aligned}$$

since $g \in J$, so $g(f_1, \dots, f_m) \in I$. It follows that $g(f_1(x), \dots, f_m(x)) = 0$ for every $x \in X$. Therefore, $\varphi(X) \subseteq Y$, and φ defines a polynomial map from X to Y. It is clear that $\varphi^* = \Phi$, because for $g \in \Bbbk[y_1, \dots, y_m]$,

$$\begin{aligned} \varphi^*(g + J) &= (g \circ \varphi) + I \\ &= g(f_1, \dots, f_m) + I \\ &= \tilde{\Phi}(g) \\ &= \Phi(g + J). \end{aligned}$$

Finally, uniqueness follows from Proposition 1.2.7. □

1.2.11 Proposition. *Let $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ be affine varieties, and let $\varphi : X \to Y$ be a polynomial map. Then φ is an isomorphism if and only if φ^* is an isomorphism, in which case $(\varphi^*)^{-1} = (\varphi^{-1})^*$.*

PROOF: Suppose that φ is an isomorphism. Then there exists a polynomial map $\varphi^{-1} : Y \to X$ such that $\varphi \circ \varphi^{-1} = \mathrm{id}_Y$ and $\varphi^{-1} \circ \varphi = \mathrm{id}_X$. Taking pullbacks, we get $(\varphi^{-1})^* \circ \varphi^* = \mathrm{id}_{\Gamma(Y)}$ and $\varphi^* \circ (\varphi^{-1})^* = \mathrm{id}_{\Gamma(X)}$, so φ^* is an isomorphism, and $(\varphi^{-1})^*$ is its inverse.

Conversely, suppose φ^* is an isomorphism. Then there exists a $\Bbbk$-algebra homomorphism $(\varphi^*)^{-1} : \Gamma(X) \to \Gamma(Y)$ such that $\varphi^* \circ (\varphi^*)^{-1} = \mathrm{id}_{\Gamma(X)}$ and $(\varphi^*)^{-1} \circ \varphi = \mathrm{id}_{\Gamma(Y)}$. By Proposition 1.2.10, there exists a polynomial map $\psi : Y \to X$ such that $(\varphi^*)^{-1} = \psi^*$. Then

$$(\varphi \circ \psi)^* = \varphi^* \circ \psi^* = \varphi \circ (\varphi^*)^{-1} = \mathrm{id}_{\Gamma(X)},$$

and

$$(\psi \circ \varphi)^* = \psi^* \circ \varphi^* = (\varphi^*)^{-1} \circ \varphi = \mathrm{id}_{\Gamma(Y)}.$$

Hence by the uniqueness in Proposition 1.2.10 we have $\varphi \circ \psi = \mathrm{id}_X$ and $\psi \circ \varphi = \mathrm{id}_Y$. Therefore, φ is an isomorphism. □

1.2.12 Corollary. *Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties. Then $X \cong Y$ if and only if $\Gamma(X) \cong \Gamma(Y)$.*

PROOF: Suppose $X \cong Y$. Then there exists an isomorphism $\varphi : X \to Y$. By Proposition 1.2.11, $\varphi^* : \Gamma(Y) \to \Gamma(X)$ is an isomorphism, so $\Gamma(X) \cong \Gamma(Y)$. Conversely, suppose $\Gamma(X) \cong \Gamma(Y)$. Then there exists a $\Bbbk$-algebra isomorphism $\Phi : \Gamma(Y) \to \Gamma(X)$. By Proposition 1.2.10, there exists a polynomial map $\varphi : X \to Y$ such that $\varphi^* = \Phi$. By Proposition 1.2.11, φ is an isomorphism, so $X \cong Y$. □

1.2.13 Example. Is $X = \mathrm{V}(yx - 1) \subseteq \mathbb{A}^2$ isomorphic to $\mathbb{A}^1$? No, since $\Gamma(\mathbb{A}^1) = \Bbbk[t]$ while $\Gamma(X)$ is the ring of Laurent polynomials, $\Bbbk[\overline{x}, \overline{x}^{-1}]$, and these $\Bbbk$-algebras are not isomorphic. To see this, suppose that $\Phi : \Bbbk[\overline{x}, \overline{x}^{-1}] \to \Bbbk[t]$ is an isomorphism. In particular, Φ is surjective, so $\Phi(1) = 1$ and $\Phi(\overline{x})\Phi(\overline{x}^{-1}) = \Phi(1) = 1$. Hence $\Phi(\overline{x})$ and $\Phi(\overline{x}^{-1})$ are units in $\Bbbk[t]$, so they must be scalars. But this implies that the entire range of Φ is contained in the scalars, a contradiction.

We now return to the question of which $\Bbbk$-algebras are isomorphic to coordinate rings of affine varieties. We will restrict ourselves to the case when $\Bbbk$ is algebraically closed, which is certainly reasonable, as when $\Bbbk$ is not algebraically closed we can not even describe the closed ideals $\Bbbk[x_1, \dots, x_n]$ in any elementary manner.

We say that a $\Bbbk$-algebra A is *finitely generated* if there exist $a_1, \dots, a_n \in A$ such that $A = \Bbbk[a_1, \dots, a_n]$, or equivalently, if there exists a surjective homomorphism $\varphi : \Bbbk[x_1, \dots, x_n] \to A$ for some $n \in \mathbb{N}$.

1.2.14 Examples.

(i) $\Bbbk[x_1, \dots, x_n]$ is finitely generated.

(ii) Any quotient of a finitely generated $\Bbbk$-algebra is finitely generated. In particular, if X is an affine variety then $\Gamma(X)$ is finitely generated.

1.2.15 Proposition. *Suppose $\Bbbk$ is algebraically closed, and let A be a finitely generated $\Bbbk$-algebra that is an integral domain. Then there exists an affine variety X such that $A \cong \Gamma(X)$.*

PROOF: Since A is finitely generated, there exists an $n \in \mathbb{N}$ and a surjective $\Bbbk$-algebra homomorphism $\varphi : \Bbbk[x_1, \ldots, x_n] \to A$. Let $I = \ker(\varphi)$. By the First Isomorphism Theorem, $\Bbbk[x_1, \ldots, x_n]/I \cong A$. Hence $\Bbbk[x_1, \ldots, x_n]/I$ is an integral domain, and I is prime. Let $X = \mathrm{V}(I)$. Then X is an affine variety and

$$\Gamma(X) = \Bbbk[x_1, \ldots, x_n]/\,\mathrm{I}(X) = \Bbbk[x_1, \ldots, x_n]/I \cong A. \qquad \square$$

Therefore, when $\Bbbk$ is algebraically closed, the contravariant functor from the category of affine varieties with polynomial maps as morphisms to the category of finitely generated $\Bbbk$-algebras that are integral domains with $\Bbbk$-algebra homomorphisms as morphisms is an equivalence of categories. Indeed, we have the following correspondence:

Geometry	**Algebra**
affine variety X	finitely generated $\Bbbk$-algebra and integral domain $\Gamma(X)$
algebraic subset of X	radical ideal of $\Gamma(X)$
irreducible algebraic subset of X	prime ideal of $\Gamma(X)$
point of X	maximal ideal of $\Gamma(X)$
polynomial map $\varphi : X \to Y$	$\Bbbk$-algebra homomorphism $\varphi^* : \Gamma(Y) \to \Gamma(X)$

1.3 Rational Functions

1.3.1 Definition. Let $X \subseteq \mathbb{A}^n$ be a variety and $\Gamma(X)$ its coordinate ring. Since $\Gamma(X)$ is a domain, we may consider its field of fractions, which we will denote $\Bbbk(X)$. In this context, $\Bbbk(X)$ is called the *field of rational functions* on X, or the *function field* of X.

In contrast to polynomial functions, rational functions are not necessarily defined at every point in X, e.g. $f = 1/x$ is not defined at $x = 0$ on $\mathbb{A}^1$. However, at the same time, even though the expression $f = x^2/x$ is not defined at $x = 0$ on $\mathbb{A}^1$, by expressing f as $x/1$, we can extend it to all of $\mathbb{A}^1$.

1.3.2 Definition. A rational function F is said to be *regular* (or *defined*) at $p \in X$ if f can be written as a/b for some $a, b \in \Gamma(X)$ such that $b(p) \neq 0$. The *value of a rational function* f at p is defined to by $f(p) = a(p)/b(p)$. A point where f is not defined is called a *pole* and the set of all such points is called the *pole set* of f.

Remark. There may be more than one way of writing f as a ratio of polynomial functions; f is defined at p if we can find a "denominator" for f that does not vanish at p. Nevertheleess, the value of f at p is independent of a and b. Indeed, if $f = a/b = a'/b'$ with $b(p), b'(p) \neq 0$, then $ab' = a'b$ if and only if $ab' - a'b \in \mathrm{I}(X)$, so

$$\frac{a(p)}{b(p)} = \frac{a'(p)}{b'(p)},$$

since $p \in X$.

1.3.3 Examples.

(i) Consider $f = x/y$ on $\mathbb{A}^2$. Then the pole set of f is $\{(x,y) \in \mathbb{A}^2 : y = 0\}$. But if one restricts f to $X = \mathrm{V}(x - y^2) \subseteq \mathbb{A}^2$ then $\overline{x}/\overline{y} = \overline{y^2}/\overline{y} = \overline{y}$ on X, so $\overline{f}$ is defined everywhere on X.

(ii) Consider $f = (1-y)/x$ on $X = \mathrm{V}(x^2 + y^2 - 1) \subseteq \mathbb{A}^2$. If $\mathrm{char}(\mathbb{k}) = 2$, then $x^2 + y^2 + 1 = (x + y - a)^2$, so $X = \mathrm{V}(x + y - 1)$. Therefore, on X, $1 - \overline{y} = \overline{x}$, so $f = 1$ is defined everywhere. If $\mathrm{char}(\mathbb{k}) \neq 2$, then f has pole set $\{(0,-1)\}$. Indeed, there are two points on X with x coordinate equal to 0, namely $(0,1)$ and $(0,-1)$, but since we have

$$\frac{1-\overline{y}}{\overline{x}} = \frac{(1-\overline{y})(1+\overline{y})}{\overline{x}(1+\overline{y})} = \frac{\overline{x}^2}{\overline{x}(1+\overline{y})} = \frac{\overline{x}}{1+\overline{y}}$$

on X, the point $(0,1)$ is not a pole of f. If $(0,-1)$ were not a pole then there would be $\overline{a}, \overline{b} \in \Gamma(X)$ such that $(1-\overline{y})/\overline{x} = \overline{a}/\overline{b}$ with $b(0,-1) \neq 0$. Hence $(1-\overline{y})\overline{b} - \overline{ax} = 0$, so lifting to $\mathbb{k}[x,y]$ we get $(1-y)b - ax = h$, where $h \in \mathrm{I}(X)$ and $b(0,-1) \neq 0$. But then at $(0,-1) \in X$ we have that $h(0,-1) = 0$ and $2b(0,-1) = 0$, a contradiction since $\mathrm{char}(\mathbb{k}) \neq 2$.

1.3.4 Proposition. *Let X be an affine variety. Then the pole set of a rational function on X is an algebraic subset of X.*

PROOF: Suppose $f \in \mathbb{k}(X)$. The pole set of f is

$$X \cap \bigcap_{\substack{b \in \mathbb{k}[x_1,\dots,x_n], \\ f = a/b}} \mathrm{V}(b),$$

which is clearly algebraic, as it is the intersection of algebraic sets. □

Therefore, the set of all points where $f \in \mathbb{k}(X)$ is defined is an open subset of X, called the *domain* of f. We denote the domain of f by $\mathrm{dom}(f)$.

Remark. If $f \in \Bbbk(X)$ is such that $\operatorname{dom}(f)$ is closed and non-empty, then $\operatorname{dom}(f) = X$. Indeed, if $\operatorname{dom}(f)$ is closed, then it is both open in closed in X. Thus $X = \operatorname{dom}(f) \cup (X \setminus \operatorname{dom}(f))$, where both $\operatorname{dom}(f)$ and $X \setminus \operatorname{dom}(f)$ are closed, implying that $\operatorname{dom}(f) = \varnothing$ or $\operatorname{dom}(f) = X$. But $\operatorname{dom}(f) \neq \varnothing$, so $\operatorname{dom}(f) = X$.

We have seen that polynomial functions are continuous with respect to the Zariski topology. Similarly, rational functions are continuous with respect to the induced Zariski topology on their domain.

1.3.5 Proposition. *Let $X \subseteq \mathbb{A}^n$ be an affine variety. If $f \in \Bbbk(X)$ then f is continuous with respect to the induced Zariski topology on* $\operatorname{dom}(f)$ *and the Zariski topology on $\mathbb{A}^1 \cong \Bbbk$.*

PROOF: Exercise. □

1.3.6 Proposition. *Let $X \subseteq \mathbb{A}^n$ be an affine variety. If $f \in \Bbbk(X)$ is zero on a non-empty open set $U \subseteq X$, then f is zero on all of X.*

PROOF: Choose $p \in U$ and $a, b \in \Gamma(X)$ such that $f = a/b$ and $b(p) \neq 0$. Note that although $b(p) \neq 0$, it may be zero at other points in U; let us then shrink the open set U. Consider $V = X \setminus \mathrm{V}(b)$. Then V is open in the induced Zariski topology on X, and $p \in U \cap V$, so $U \cap V$ is a non-empty open subset of X. Since $b(x) \neq 0$ for all $x \in U \cap V$ and $f(x) = 0$ for all $x \in U$, $a(x) = f(x)b(x) = 0$ for all $x \in U \cap V$. Moreover, since $U \cap V$ is a non-empty open subset of X with X is irreducible, $\overline{U \cap V} = X$. Hence $a(x) = 0$ for all $x \in X$ because polynomials are continuous with respect to the Zariski topology, so that $a^{-1}(0)$ is closed in X, implying that

$$X = \overline{U \cap V} \subseteq \overline{a^{-1}(0)} = a^{-1}(0) \subseteq X.$$

Therefore, $a = 0$, and $f = a/b = 0/b = 0$. □

1.3.7 Corollary (Identity Theorem). *Let X be an affine variety. If $f, g \in \Bbbk(X)$ agree on a non-empty open subset of X then $f = g$.*

PROOF: Consider $f - g$ and apply the previous proposition. □

In particular, this corollary tells us that rational functions are completely determined by their restriction to some open set $U \subseteq X$.

1.4 Rational Maps

1.4.1 Definition. Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be varieties. A map $\varphi : X \to Y$ such that

$$\varphi(x) = (f_1(x), \ldots, f_m(x))$$

for some $f_1, \ldots, f_m \in \Bbbk(X)$ is called a *rational map*. If $p \in X$, we say that φ is *regular* (or *defined*) at p if each f_i is regular at p. The set of all points where φ is defined is called the *domain* of φ, and is denoted by $\operatorname{dom}(f)$.

Note that the domain of φ is an open subset of X, since it is the intersection of the domains of the rational functions f_i, which are each open sets.

1.4.2 Examples.

(i) Rational functions $f : X \to \Bbbk \cong \mathbb{A}^1$ are rational maps.

(ii) Any polynomial map is a rational map.

(iii) Suppose $\operatorname{char}(\Bbbk) \neq 2$, and let $X = \operatorname{V}(x^2 + y^2 - 1) \subseteq \mathbb{A}^2$. The parameterization $\varphi : \mathbb{A}^1 \to X$ of X given by

$$\varphi(t) = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1} \right)$$

is a rational map. Its domain is $\mathbb{A}^1 \setminus \{i, -i\}$, and its range is $X \setminus \{(0, 1)\}$.

1.4.3 Definition. A rational map $\varphi : X \to Y$ is called *dominant* if $\overline{\varphi(X)} = Y$.

1.4.4 Examples.

(i) Suppose $\operatorname{char}(\Bbbk) \neq 2$, and let $X = \operatorname{V}(x^2 + y^2 - 1) \subseteq \mathbb{A}^2$. The rational map $\varphi : \mathbb{A}^1 \to X$ given by

$$\varphi(t) = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1} \right)$$

is dominant, because $\overline{\varphi(\mathbb{A}^1)} = \overline{X \setminus \{(0, 1)\}} = X$.

(ii) A proper inclusion between affine varieties is not a dominant map.

Given two rational maps $\varphi : X \to Y$ and $\psi : Y \to Z$, the composition of ψ and φ is not defined if $\varphi(X) \cap (\text{domain } \psi)$ is empty. This problem can be bypassed if we assume that φ is dominant. Indeed, if $\overline{\varphi(X)} = Y$, then if $\varphi(X) \cap \operatorname{dom}(\psi) = \varnothing$, this would imply that $\varphi(X) \subseteq (X \setminus \operatorname{dom}(\psi))$ is a proper closed subset of Y (since $\operatorname{dom}(\psi)$ is open), and thus $\overline{\varphi(X)} \subseteq X \setminus \operatorname{dom}(\psi)$, contradicting the assumption that φ is dominant. Therefore, if φ is dominant there is a well-defined composition $\psi \circ \varphi : X \to Z$.

1.4.5 Definition. A dominant rational map $\varphi : X \to Y$ is *birational* or a *birational equivalence* if φ has an inverse rational map that is also dominant. In this case, X and Y are said to be *birational* or *birationally equivalent*, denoted $X \sim Y$.

1.4.6 Examples.

(i) Every isomorphism is a birational equivalence.

(ii) Let $X = \mathrm{V}(xy-1) \subseteq \mathbb{A}^2$. The map $\varphi : X \to \mathbb{A}^1$ defined by $\varphi(x,y) = x$ is a polynomial map that is injective but not surjective, because $0 \notin \varphi(X)$. However, it is a dominant rational map, and it has a rational inverse on the open subset $U = \mathbb{A}^1 \setminus \{(0,0)\}$ of $\mathbb{A}^1$, given by $\varphi^{-1}(t) = (t, 1/t)$. This inverse is dominant because φ is defined on all of X, so X is birationally equivalent to $\mathbb{A}^1$, even though X is not isomorphic to $\mathbb{A}^1$.

(iii) Let $X = \mathrm{V}(y^2 - x^3) \subseteq \mathbb{A}^2$. The rational map $\varphi : \mathbb{A}^1 \to X$ defined by $\varphi(t) = (t^2, t^3)$ is bijective, and thus dominant, but not an isomorphism. Nevertheless, it has a rational inverse $\varphi^{-1} : X \to \mathbb{A}^1$, defined on the open subset $U = X \setminus \{(0,0)\}$ and given by $\varphi^{-1}(x,y) = y/x$. Since the domain of φ is all of X, φ^{-1} is dominant. Thus X is birational to $\mathbb{A}^1$.

1.4.7 Definition. A variety $X \subseteq \mathbb{A}^n$ that is birationally equivalent to $\mathbb{A}^m$, for some m, is said to be *rational*.

The last two examples above are both rational. We will later see that there are curves that are not rational.

How do rational maps act on rational functions? That is, given a rational map $\varphi : X \subseteq \mathbb{A}^n \to Y \subseteq \mathbb{A}^m$, can we define a pullback $\varphi^* : \Bbbk(Y) \to \Bbbk(X)$ that is well-defined as a $\Bbbk$-algebra homomorphism? This imposes the following two conditions on φ:

(i) φ must be dominant to ensure that one can compose φ with any rational function on Y;

(ii) φ^* must be a non-zero field homomorphism and would therefore be injective (as units are mapped to units).

For any rational function $g \in \Bbbk(Y)$, let $\varphi^*(g) = g \circ \varphi \in \Bbbk(X)$. Clearly, $\varphi^* : \Gamma(Y) \to \Bbbk(X)$ is a well-defined ring homomorphism. Moreover, φ^* is injective. Indeed, suppose on the contrary that there exists a non-zero $f \in \Gamma(Y)$ such that $\varphi * (f) = 0$. Then $\varphi(X) \subseteq \mathrm{V}(f) \subsetneq Y$, so $\overline{\varphi(X)} \subseteq \overline{\mathrm{V}(f)}$, contradicting the fact that $\varphi(X)$ is dense in Y. Hence φ^* is injective on $\Gamma(Y)$ so that it extends in the obvious way to a $\Bbbk$-algebra homomorphism $\varphi^* : \Bbbk(Y) \to \Bbbk(X)$, i.e. $\varphi^*(a/b) = \varphi^*(a)\varphi^*(b)^{-1}$.

As in the case of the pullback of a polynomial map, the next proposition shows that the association of a function field to an affine variety and the pullback of dominant rational maps defines a contravariant functor from the category of affine varieties with dominant rational maps as morphisms to the category of field extensions of $\Bbbk$ with $\Bbbk$-algebra homomorphisms as morphisms.

1.4.8 Proposition (Functoriality). *Let X, Y, and Z be affine varieties. Then:*

(i) *if $\varphi : X \to Y$ and $\psi : X \to Y$ are dominant rational maps such that $\varphi^* = \psi^*$ then $\varphi = \psi$;*

(ii) $\mathrm{id}_X^* = \mathrm{id}_{\Bbbk(X)}$;
(iii) *if* $\varphi : X \to Y$ *and* $\psi : Y \to Z$ *are both dominant rational maps then* $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$;
(iv) φ^* *is an injective* $\Bbbk$*-algebra homomorphism.*

PROOF:
(i) Choose $f_1, \ldots, f_m, g_1, \ldots, g_m \in \Bbbk(X)$ such that $\varphi = (f_1, \ldots, f_m)$ and $\psi = (g_1, \ldots, g_m)$. Then $y_i \circ \varphi = f_i$ and $y_i \circ \psi = g_i$, showing that $f_i = g_i$. Therefore, $\varphi = \psi$.
(ii) For every $g \in \Bbbk(Y)$, $\mathrm{id}_X^*(g) = g \circ \mathrm{id}_X = g$.
(iii) For every $g \in \Bbbk(Z)$, $(\psi \circ \varphi)^*(g) = g \circ \psi \circ \varphi = \varphi^*(g \circ \psi) = \varphi^* \circ \psi^*(g)$.
(iv) We showed that it is injective above, and it is clearly a $\Bbbk$-algebra homomorphism. □

As in the case of the pullback of a polynomial map, every $\Bbbk$-algebra homomorphism between function fields is the pullback of a dominant rational map.

1.4.9 Proposition. *Let* $X \subseteq \mathbb{A}^n$ *and* $Y \subseteq \mathbb{A}^m$ *be affine varieties, and let* $\Phi : \Bbbk(Y) \to \Bbbk(X)$ *be a* $\Bbbk$*-algebra homomorphism. Then there exists a unique dominant rational map* $\varphi : X \to Y$ *such that* $\varphi^* = \Phi$.

PROOF: Let $I = \mathrm{I}(X)$ and $J = \mathrm{I}(Y)$. Thus $\Gamma(X) = \Bbbk[x_1, \ldots, x_n]/I$ and $\Gamma(Y) = \Bbbk[y_1, \ldots, y_m]/J$. Since $\Phi : \Bbbk(Y) \to \Bbbk(X)$ is an injective $\Bbbk$-algebra homomorphism, it restricts to an injective $\Bbbk$-algebra homomorphism from $\Gamma(Y)$ to $\Bbbk(X)$. Let $\tilde{\Phi} : \Bbbk[y_1, \ldots, y_m] \to \Bbbk(X)$ be the map defined by $\tilde{\Phi}(g) = \theta(g + J)$, i.e. the lift of $\Phi_{|\Gamma(Y)}$ to $\Bbbk[y_1, \ldots, y_m]$. Then $\tilde{\Phi}$ is a $\Bbbk$-algebra homomorphism, as it is the composition of two $\Bbbk$-algebra homomorphisms, Φ and the quotient map from $\Bbbk[y_1, \ldots, y_m]$ to $\Gamma(Y)$. For $i = 1, \ldots, m$, let $f_i = \tilde{\Phi}(y_i)$. Then for any $g \in \Bbbk[y_1, \ldots, y_m]$ we have

$$\begin{aligned} g(f_1, \ldots, f_m) &= g(\tilde{\Phi}(y_1), \ldots, \tilde{\Phi}(y_m)) \\ &= \tilde{\Phi}(g(y_1, \ldots, y_m)) \\ &= \tilde{\Phi}(g). \end{aligned}$$

Let $\varphi : X \to \mathbb{A}^m$ be the rational map defined by $\varphi(x) = (f_1(x), \ldots, f_m(x))$. In order to show that φ is a dominant rational map from X to Y, we need to show that $\varphi(X) \subseteq Y$ and $\overline{\varphi(X)} = \varphi(Y)$. Fix $g \in J$. Then from the above,

$$g(f_1, \ldots, f_m) = \tilde{\Phi}(g) = \Phi(g + J) = \Phi(0) = 0,$$

so $g(f_1(x), \ldots, f_m(x)) = g(f_1, \ldots, f_m)(x) = 0$ for every $x \in X$. Therefore, $\varphi(X) \subseteq Y$, and φ defines a rational map from X to Y. Since Y is closed, $\overline{\varphi(X)} \subseteq Y$. We need to show that

$$Y \subseteq \overline{\varphi(X)} = \mathrm{V}(\mathrm{I}(\varphi(X))).$$

Fix $p \in Y$ and $g \in \mathrm{I}(\varphi(X))$. Since Φ is a $\Bbbk$-algebra homomorphism, we have that for all $x \in X$, $\Phi(g)(x) = g(\varphi(x)) = 0$. Thus $\Phi(g) = 0$, and $g = 0$ since Φ is injective. In particular, $g(p) = 0$. Since the choice of g was arbitrary, $p \in \mathrm{V}(\mathrm{I}(\varphi(X))) = \overline{\varphi(X)}$, showing that $\overline{\varphi(X)} = Y$. It is clear that $\varphi^* = \Phi$, because for $g \in \Bbbk[y_1, \ldots, y_m]$,

$$\varphi^*(g + J) = g(f_1, \ldots, f_m) = \tilde{\Phi}(g).$$

Finally, uniqueness follows from Proposition 1.4.9 (i). □

1.4.10 Proposition. *Let $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ be affine varieties, and let $\varphi : X \to Y$ be a dominant rational map. Then φ is birational if and only if φ^* is an isomorphism, in which case $(\varphi^*)^{-1} = (\varphi^{-1})^*$.*

PROOF: Analogous to the proof of Proposition 1.2.11. □

1.4.11 Corollary. *Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be varieties. Then $X \sim Y$ if and only if $\Bbbk(X) \cong \Bbbk(Y)$.*

PROOF: Analogous to the proof of Corollary 1.2.12. □

1.5 Dimension

There are two natural ways to define dimension for affine varieties varieties, which are both very similar to the case of linear algebra, where the dimension of a finite-dimensional vector space V is equal to the maximum size of a linearly independent family of linear maps to the scalar field. In the case of of an affine variety X, it would make sense to consider instead algebraically independent rational functions on X. Thus we arrive at a definition of the dimension of X as the transcendence degree of $\Bbbk(X)$ over $\Bbbk$ (for the definition of transcendence degree, see Appendix A).

The dimension of V can also be characterized as the maximum length of a descending chain of proper subspaces. We will see that the dimension of a variety can be described in a similar fashion as the maximum length of a descending chain of proper subvarieties, and that this notion of dimension agrees with the transcendence degree of the function field of the variety.

1.5.1 Definition. Let X be an affine variety. The *dimension* of X is the transcendence degree of the function field $\Bbbk(X)$ over $\Bbbk$, and is denoted by $\dim X$. If $Y \subseteq X$ is a subvariety of X then the *codimension* of Y in X is $\operatorname{codim}_X Y = \dim X - \dim Y$. A variety of dimension 1 is a *curve*, a variety of dimension 2 is a *surface*, and a variety of dimension n is called an *n-fold*.

By Corollary 1.4.12, dimension is invariant under birational equivalence.

1.5.2 Examples.

(i) $\mathbb{A}^n$ has dimension n since in $\Bbbk(x_1,\dots,x_n)\cong\Bbbk(\mathbb{A}^n)$ the coordinate functions $x_1,\dots,x_n$ are algebraically independent. It follows that $\mathbb{A}^n$ and $\mathbb{A}^m$ cannot be birational if $m\neq n$.

(ii) If X consists of a single point then $\Bbbk(X)=\Bbbk$, so $\dim X=0$.

The dimension of a variety has the particularly strong property that any proper subvariety must have strictly smaller dimension. This is certainly not the case if one considers manifolds instead of varieties.

1.5.3 Theorem. *If Y is a proper subvariety of $X\subseteq\mathbb{A}^m$ then $\dim Y<\dim X$.*

PROOF: Let $n=\dim X$. Then any $n+1$ of the coordinate functions $x_1,\dots,x_m$ are algebraically dependent as elements of $\Bbbk(X)$, and also as elements of $\Bbbk(Y)$. Therefore $\dim Y\leq\dim X$.

Assume that $\dim Y=\dim X$. We will derive the contradiction $Y=X$ by showing that $\mathrm{I}(Y)\subseteq\mathrm{I}(X)$. Since $\dim Y=n$ there are coordinate functions $x_{i_1},\dots,x_{i_n}$ whose images are algebraically independent in $\Bbbk(Y)$. Then $x_{i_1},\dots,x_{i_n}$ must be algebraically independent in $\Bbbk(X)$. Let $u\in\Gamma(X)$ be non-zero. Then u depends algebraically on $x_{i_1},\dots,x_{i_n}$, i.e. there is a polynomial $a\in\Bbbk[t_1,\dots,t_{n+1}]$ such that

$$\begin{aligned}&a(u,x_{i_1},\dots,x_{i_n})\\&\qquad=a_k(x_{i_1},\dots,x_{i_n})u^k+\cdots+a_1(x_{i_1},\dots,x_{i_n})u+a_0(x_{i_1},\dots,x_{i_n})=0\end{aligned}$$

on X. Since $\Gamma(X)$ is a domain we may assume a is irreducible and $a_0(x_{i_1},\dots,x_{i_n})$ is non-zero on X. Then $a(u,x_{i_1},\dots,x_{i_n})=0$ on Y since $Y\subseteq X$ so if $u=0$ on Y then $a_0(x_{i_1},\dots,x_{i_n})=0$ on Y, a contradiction since $x_{i_1},\dots,x_{i_n}$ are algebraically independent in $\Bbbk(Y)$. Since $u\neq 0$ on X implies $u\neq 0$ on Y we have that $\mathrm{I}(Y)\subseteq\mathrm{I}(X)$. □

1.5.4 Corollary. *Let X be an affine variety. Then $\dim X=0$ if and only if X is a single point.*

PROOF: Clearly, if X is a single point, then $\Bbbk(X)\cong\Bbbk$, so $\dim X=0$. Conversely, suppose that $\dim X=0$, but X has more than one point. Then there exists a $p\in X$ such that $\{p\}\subsetneq X$. Then $0=\dim\{p\}<\dim X=0$, contradicting to the previous theorem. □

1.5.5 Example. Plane curves given by irreducible polynomials have dimension 1. Suppose $X=\mathrm{V}(f)$ for some irreducible polynomial $f\in\Bbbk[x,y]$. The algebraically independent elements x and y of $\Bbbk[x,y]$ descend to algebraically dependent elements $\overline{x}$ and $\overline{y}$ in $\Bbbk(X)$. Indeed, $f(\overline{x},\overline{y})=0$.

Since $\dim\mathbb{A}^2=2$, the previous example shows that plane curves have codimension 1 in $\mathbb{A}^2$. Similarly, any hypersurface defined by a non-constant irreducible polynomial has codimension 1.

1.5.6 Theorem. *Let $f \in \Bbbk[x_1, \ldots, x_n]$ be a non-constant irreducible polynomial. Then $\mathrm{V}(f) \subseteq \mathbb{A}^n$ has codimension 1.*

PROOF: Let $X = \mathrm{V}(f)$. Suppose that x_n appears in the expression of f. Then $\overline{x}_1, \ldots, \overline{x}_{n-1}$ are algebraically independent in $\Bbbk(X)$. Indeed, if they are not then there is a polynomial g involving only the variables $x_1, \ldots, x_{n-1}$ that is zero on X. Then $g \in \mathrm{I}(X) = \langle f \rangle$, so $f \mid g$ and x_n appears in the expression for g, a contradiction. Therefore $\dim X \geq n - 1$, and Theorem 1.5.3 implies that $\dim X = n - 1$, so $\operatorname{codim} X = 1$. $\square$

1.5.7 Example. Consider $X = \mathrm{V}(y^2 - x^3$. By Example 1.4 (ii), $f = \overline{x} - 1 \in \Gamma(X)$ is irreducible, but $\mathrm{V}(f) \cap X = \{(1, 1), (1, -1)\}$ is reducible. However, both of the irreducible components of $\mathrm{V}(f) \cap X$, { (1, 1) } and $\{(1, -1)\}$ have codimension 1 in X.

In fact, the following general result holds, but its proof uses more complicated algebraic techniques.

1.5.8 Theorem. *Let $X \subseteq \mathbb{A}^n$ be a affine variety, and let $f \in \Bbbk[x_1, \ldots, x_n]$ be a polynomial such that $\mathrm{V}(f) \cap X \neq X$. Then each of the irreducible components of $\mathrm{V}(f) \cap X$ has codimension 1 in X.*

1.5.9 Corollary. *If $Y \subsetneq X \subseteq \mathbb{A}^n$ has codimension r in X then there exist irreducible closed subsets $Y_0, \ldots, Y_r$ of X of codimension $0, \ldots, r$, respectively, such that*

$$Y = Y_r \subsetneq Y_{r-1} \subsetneq \cdots \subsetneq Y_0 = X.$$

PROOF: We prove the statement by induction on r. If $r = 1$, then there is nothing to prove. Suppose $r > 1$. Since $Y \subsetneq X$, there exists a non-zero $f \in \mathrm{I}(Y)$ such that f does not vanish on X, i.e. such that $f \neq \mathrm{I}(X)$. Moreover, since $\mathrm{I}(Y)$ is prime, we can assume that f is irreducible. Then $Y_i = \mathrm{V}(f)$ is a subvariety of X of codimension 1 that contains Y. Then $Y \subsetneq Y_I \subsetneq X$. Repeat the construction with $Y \subsetneq Y_1$ to get a subvariety Y_2 of Y_1 of codimension 1 and such that $Y \subsetneq Y_2 \subsetneq Y_1 \subsetneq X$, and continue inductively to prove the statement. $\square$

The last corollary suggests the following characterization of the dimension of a variety, which could be called topological, as it involves only the Zariski topology of the variety and no additional structure.

1.5.10 Corollary. *The dimension of an affine variety X is the largest integer d for which there exists a chain of irreducible closed subsets*

$$X_1 \subsetneq \cdots \subsetneq X_d = X.$$

PROOF: If $X = \{p\}$, then this is clear from Theorem 1.5.3. Otherwise, this follows from the previous corollary with $Y = \{p\}$ for some $p \in X$. $\square$

The *Krull dimension* of a ring R is defined as the length of the longest chain of prime ideals in R. Let $X \subseteq \mathbb{A}^n$ be an affine variety. Since prime ideals of $\Gamma(X)$ correspond to prime ideals of $\Bbbk[x_1, \ldots, x_n]$ that contain $\mathrm{I}(X)$, the chains of prime ideals

$$I_1 \subsetneq \cdots \subsetneq I_d$$

in $\Gamma(X)$ correspond to chains of prime ideals

$$\mathrm{I}(X) \subseteq J_1 \subsetneq \cdots \subsetneq J_d$$

of $\Bbbk[x_1, \ldots, x_n]$. In turn, such chains of prime ideals of $\Bbbk[x_1, \ldots, x_n]$ correspond to chains of irreducible closed sets

$$\mathrm{V}(J_d) \subsetneq \mathrm{V}(J_{d-1}) \subsetneq \cdots \subsetneq \mathrm{V}(J_1) \subseteq X.$$

Therefore, by Corollary 1.5.10, we see that the Krull dimension of $\Gamma(X)$ is equal to the dimension of X.

Appendix A

Transcendence Bases

A.0.11 Definition. Let K be a field, and let F be a subfield of K. A subset $U \subseteq K$ is said to be *algebraically independent* over F if for every $n \geq 1$, every non-zero $f \in F[x_1, \ldots, x_n]$, and all $u_1, \ldots, u_n \in U$, we have that $f(u_1, \ldots, u_n) \neq 0$. A *transcendence basis* of K over F is an algebraically independent subset of K that is maximal with respect to inclusion.

A.0.12 Examples.

(i) The empty set is algebraically independent. If $K = F$, it is also a transcendence basis of K over F.

(ii) Let F a fixed field and let $K = F(x_1, \ldots, x_n)$ be the fraction field of the ring $F[x_1, \ldots, x_n]$. We claim that $\{x_1, \ldots, x_n\}$ is a transcendence basis of K over F. It is clearly algebraically independent, as if $f \in F[t_1, \ldots, t_n]$ is such that $f(x_1, \ldots, x_n) = 0$, we have that $f = f(t_1, \ldots, t_n) = 0$. To show that $\{x_1, \ldots, x_n\}$ is a maximal algebraically independent set, we will show that $\{x_1, \ldots, x_n, p/q\}$ is algebraically dependent over F for any $p, q \in F[x_1, \ldots, x_n]$, $q \neq 0$. Define $f \in F[t_1, \ldots, t_{n+1}]$ by

$$f(t_1, \ldots, t_{n+1}) = p(t_1, \ldots, t_n) - q(t_1, \ldots, t_n)t_{n+1}.$$

Then $f \neq 0$, but $f(x_1, \ldots, x_n, p/q) = 0$, showing that $\{x_1, \ldots, x_n, p/q\}$ is algebraically dependent over F. Therefore, $\{x_1, \ldots, x_n\}$ is a transcendence basis of K over F.

A.0.13 Theorem. *Let K be a field, and let F be a subfield of K. Then:*

(i) *Every algebraically independent subset U of K is contained in some transcendence basis. In particular, since the empty set is algebraically independent, K has a transcendence basis.*

(ii) *If B_1 and B_2 are both transcendence bases of K over F, then* $\operatorname{card}(B_1) = \operatorname{card}(B_2)$.

Proof:

(i) Let P be the partial order of algebraically independent subsets of K that contain U, ordered by inclusion. If C is a chain in P, then $\bigcup C$ is clearly algebraically independent, as any possible algebraic dependence involves finitely many elements of $\bigcup C$, which could all be chosen to be in the same member of C. Therefore, by Zorn's Lemma, P has a maximal element. However, by definition, such a maximal element is a transcendence basis of K containing U.

(ii) For the sake of sanity, we will assume that B_1 is finite. In the infinite case, it is argued using either multiple applications of Zorn's Lemma or transfinite induction. Suppose $B_1 = \{x_1, \dots, x_m\}$, where $m \geq 1$ is the minimal cardinality of any transcendence basis. It suffices to show that if $w_1, \dots, w_n$ are algebraically independent elements of K then $n \leq m$, as we could then swap B_1 and B_2 to get the opposite inequality. If every w_i is an x_j, there is nothing to prove, so by possibly reordering the w_i's, we can assume that $w_1 \neq x_i$ for $i = 1, \dots, m$. Since $\{x_1, \dots, x_m\}$ is a transcendence basis, $\{w_1, x_1, \dots, x_m\}$ is algebraically dependent, so there is a non-zero polynomial $f_1 \in F[t_1, \dots, t_{m+1}]$, which can clearly be chosen to be irreducible, such that $f_1(w_1, x_1, \dots, x_m) = 0$. After possibly renumbering the x_j's we may write

$$f_1 = \sum_{j=1}^{k} g_j(w_1, x_2, \dots, x_m) x_1^j$$

for some $k \geq 1$ and $g_1, \dots, g_k \in F[t_1, \dots, t_{m+1}$. No irreducible factor of g_k vanishes on $(w_1, x_2, \dots, x_n)$, otherwise w_1 would be a root of two distinct irreducible polynomials over $F(x_1, \dots, x_m)$. Hence x_1 is algebraic over $F(w_1, x_2, \dots, x_m)$ and $w_1, x_2, \dots, x_m$ are algebraically independent over F, as otherwise the minimality of m would be contradicted. Continuing inductively, suppose that after a suitable renumbering of $x_1, \dots, x_m$ we have found $w_1, \dots, w_r$, $r < n$, sch that K is algebraiic over $F(w_1, \dots, w_r, x_{r+1}, \dots, x_m)$. Then there exists a non-zero $f \in F[t_1, \dots, t_{m+1}]$ such that

$$f(w_{r+1}, w_1, \dots, w_r, x_{r+1}, \dots, x_m) = 0.$$

Since the w_i's are algebraically independent over F, it follows by the same argument as in the case above that some x_j, which we can assume to be x_{r+1}, is algebraic over $F(w_1, \dots, w_{r+1}, x_{r+2}, \dots, x_m)$. Since a tower of algebraic extensions is algebraic, it follows that K is algebraic over $F(w_1, \dots, w_{r+1}, x_{r+2}, \dots, x_m)$. If $n \geq m$, we can continue inductively and replace all of the x_j's by w_i's to see that K is algebraic over $F(w_1, \dots, w_m)$, showing that $n = m$, as desired. □

A.0.14 Definition. Let K be a field, and let F be a subfield of K. The *transcendence degree* of K over F is the cardinality of any transcendence basis of K over F.