CAN ONE PUSHFORWARD VECTOR FIELDS? Not necessarily.

We have seen that, for any smooth map $F: M \to N$ between smooth manifolds M and N, we have:

$$F_{*,p} : T_pM \longrightarrow T_{F(p)}N$$
$$X \longmapsto F_{*,p}(X)$$

with $F_{*,p}(X)(f) := X(f \circ F)$, $\forall f \in C^\infty(N)$.

If we start with a vector field Y on M, does $\{F_{*,p}(Y_p)\}_{p\in M}$ give a vector field on N?

→ NO: if F is not surjective.

Indeed, note that a vector field Z on N associates to any point $q \in N$ a tangent vector $Z_q \in T_qN$. So, if F is not surjective, there will be no $F_{*,p}(Y_p)$ associated to q's that are not in the image of F.

→ NO: if F is not injective.

If there exist $p_1, p_2 \in M$ such that
$$F(p_1) = F(p_2) = q \in N,$$
then $F_{*,p_1}(Y_{p_1})$ and $F_{*,p_2}(Y_{p_2})$ are BOTH tangent to N at q. But a vector field on N only associates ONE tangent vector to every point $q \in N$. So, $\{F_{*,p}(Y_p)\}_{p\in M}$ does not give a well-defined vector field on N in this case.

This motivates the following definition:

DEF: Let $F: M \to N$ be a smooth map, and $Y \in \mathfrak{X}(M)$. Suppose that there exists $Z \in \mathfrak{X}(N)$ such that

$$Z_{F(p)} = F_{*,p}(Y_p),$$

$\forall p \in M$. Then, Y and Z are said to <u>F-related</u>.

NOTE: Y and Z are F-related if and only if

$$Z(f) \circ F = Y(f \circ F)$$

for all $f \in C^\infty(N)$.

Pf: $\forall p \in M$,

$$Z(f) \circ F(p) = Z(f)(F(p)) = Z_{F(p)}(f) = F_{*,p}(Y_p)(f)$$

$$= Y_p(f \circ F) = Y(f \circ F)(p).$$

$$\Rightarrow Z(f) \circ F = Y(f \circ F). \quad \blacksquare$$

THM: If $F: M \to N$ is a diffeomorphism, then for any $Y \in \mathfrak{X}(M)$, there exists a <u>unique</u> $Z \in \mathfrak{X}(N)$ such that Y and Z are F-related.

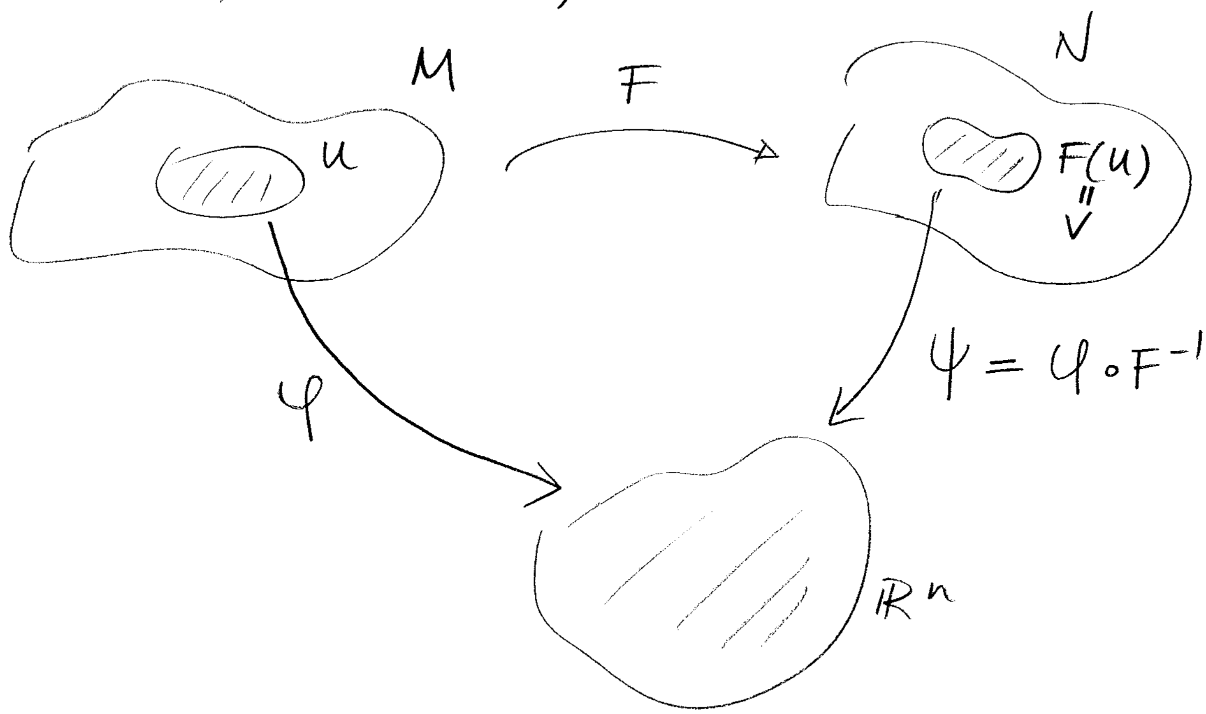
We denote this smooth vector field $F_* Y$ and call it the <u>push forward of Y by F</u>.

Pf: Since F is a diffeomorphism, $\forall q \in N$, there exists a unique $p \in M$ such that $F(p) = q$. We set

$$Z_q := F_{*,p}(Y_p).$$

Then, Z is a well-defined vector field on N that is F-related. Moreover, Z is, by construction, unique. The only thing left to check is that Z is smooth.

Let (U, φ) be a chart on M with $\varphi = (x_1, \dots, x_n)$. Note that since F is a diffeomorphism, then $(V := F(U), \psi := \varphi \circ F^{-1})$ is a chart of N:



Set $\psi = (y_1, \dots, y_n)$. Then, $Y_p = \sum_{i=1}^{n} Y^i(p) \frac{\partial}{\partial x_i}\Big|_p$

and, by $\mathbb{R}$-linearity of the push forward,

$$Z_p = \sum_{i=1}^{n} Y^i(p)\, F_{*,p}\left(\frac{\partial}{\partial x_i}\Big|_p \right).$$

Now, $F_{*,p}\left(\frac{\partial}{\partial x_i}\Big|_p\right) \in T_{F(p)}N$ and so

$$F_{*,p}\left(\frac{\partial}{\partial x_i}\Big|_p\right) = \sum_{j=1}^{n} b^j \frac{\partial}{\partial y_j}\Big|_{F(p)},$$

with
$$b^j = F_{*,p}\left(\frac{\partial}{\partial x_i}\Big|_p\right)(y_j)$$
$$= \frac{\partial}{\partial x_i}\Big|_p (y_j \circ F) = \frac{\partial}{\partial x_i}(y_j \circ F)(p)$$

$$\Rightarrow \mathcal{Z}_p = \sum_{j=1}^{n} \left[\sum_{i=1}^{n} Y^i(p) \frac{\partial}{\partial x_i}(y_j \circ F)(p)\right] \frac{\partial}{\partial y_j}\Big|_{F(p)}$$

with Y^i smooth, $\forall i$, since Y is smooth,

<u>AND</u> $\frac{\partial}{\partial x_i}(y_j \circ F)$ smooth, $\forall i, j$, since F is smooth and $y_j \circ F$ is the j-th component function of F.

$$\rightsquigarrow \left[\sum_{i=1}^{n} Y^i(p) \frac{\partial}{\partial x_i}(y_j \circ F)(p)\right]$$

is smooth for all j

$\Rightarrow \mathcal{Z}$ is smooth. $\boxtimes$

Lie brackets.

Given $V, W \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, we define

$$VW(f) := V(W(f)).$$

Clearly, $VW(f) \in C^\infty(M)$, so that VW takes smooth functions to smooth functions. It is therefore natural to ask whether VW is a smooth vector field. If it is, then VW would be a derivation at every point in M, and would in particular have to satisfy the product rule: $\forall f, g \in C^\infty(M)$,

$$VW(fg) \overset{?}{=} VW(f)g + f\,VW(g).$$

In fact,

$$\begin{aligned} VW(fg) &= V(W(fg)) \\ &= V(W(f)g + fW(g)) \\ &= V(W(f))g + W(f)V(g) + V(f)W(g) + fV(W(g)) \\ &= \left[VW(f)g + f\,VW(g)\right] + \underbrace{W(f)V(g) + V(f)W(g)}_{\text{extra terms!}} \end{aligned}$$

$\Rightarrow$ in general,

$$VW(fg) \neq VW(f)g + f\,VW(g)$$

and VW is <u>not</u> a vector field.

HOWEVER, what does give us a smooth vector field is the following:

DEF.: Given $V, W \in \mathfrak{X}(M)$, we define

$$[V, W] := VW - WV$$

$=$ <u>Lie bracket of V and W</u>.

LEMMA: For any $V, W \in \mathfrak{X}(M)$, we have $[V, W] \in \mathfrak{X}(M)$.

Pf: Let $f, g \in C^\infty(M)$. Then, from the above computation, we see that

$$[V, W](fg) = VW(fg) - WV(fg)$$
$$= [VW(f) - WV(f)]g + f[VW(g) - WV(g)]$$

$$\Rightarrow [V, W](fg) = [V, W](f)\, g + f\, [V, W](g)$$

$\Rightarrow [V, W]$ satisfies the product rule.

Since $[V, W]$ is clearly $\mathbb{R}$-linear, this tells us that $[V, W]$ is a derivation at every $p \in M$, and $[V, W]$ is a vector field on M.

To show that $[V, W]$ is smooth, let $f \in C^\infty(M)$. Then,

$$[V, W](f) = V(W(f)) - W(V(f)).$$

Now, $V(f), W(f) \in C^\infty(M)$ since $V, W \in \mathfrak{X}(M)$. Thus, $V(W(f)), W(V(f)) \in C^\infty(M)$.

$\Rightarrow [V, W](f) \in C^\infty(M)$ for all $f \in C^\infty(M)$, proving that $[V, W] \in \mathfrak{X}(M)$.

The Lie bracket in local coordinates:

Let (U,φ) be a chart with $\varphi=(x_1,\dots,x_n)$. If

$$V=\sum_{i=1}^n V^i\frac{\partial}{\partial x_i} \quad\text{and}\quad W=\sum_{i=1}^n W^i\frac{\partial}{\partial x_i},$$

then

$$\boxed{[V,W]=\sum_{j=1}^n\left(V(W^j)-W(V^j)\right)\frac{\partial}{\partial x_j},} \quad (*)$$

where $V(W^j)=\sum_{i=1}^n V^i\frac{\partial W^j}{\partial x_i}$ and $W(V^j)=\sum_{i=1}^n W^i\frac{\partial V^j}{\partial x_i}$.

Pf: $[V,W]=\sum_{j=1}^n b^j\frac{\partial}{\partial x_j}$ with

$$b^j=[V,W](x_j)=V(W(x_j))-W(V(x_j))$$
$$=V(W^j)-W(V^j).$$

$$\Rightarrow [V,W]=\sum_{j=1}^n\left(V(W^j)-W(V^j)\right)\frac{\partial}{\partial x_j}. \quad \blacksquare$$

Ex. 1) $\left[\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right]=0$, $\forall i,j$, since for all $f\in C^\infty(M)$

$$\left[\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right](f)=\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)-\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)=0$$

because the mixed partials of a smooth function are equal.

2) Suppose that $M = \mathbb{R}^3$ with coordinates (x, y, z), and consider $V = (x+2y)\frac{\partial}{\partial x} - x^2 \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$

and

$$W = z\frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

Then,

$$\begin{aligned}[V, W] &= \big(V(z) - W(x+2y)\big)\frac{\partial}{\partial x} \\ &\quad + \big(V(1) - W(-x^2)\big)\frac{\partial}{\partial y} \\ &\quad + \big(V(0) - W(1)\big)\frac{\partial}{\partial z} \\ &= \big(1 - (z\cdot 1 + 2)\big)\frac{\partial}{\partial x} - \big(z\cdot(-2x) + 0\big)\frac{\partial}{\partial y} \\ &= -(z+1)\frac{\partial}{\partial x} + 2xz\frac{\partial}{\partial y}.\end{aligned}$$

PROPERTIES of the LIE BRACKET:

(i) Bilinearity: $[a_1 V_1 + a_2 V_2, W] = a_1 [V_1, W] + a_2 [V_2, W]$.

$[V, b_1 W_1 + b_2 W_2] = b_1 [V, W_1] + b_2 [V, W_2]$.

(ii) Antisymmetry: $[W, V] = -[V, W]$.

(iii) Jacobi Identity:

$$[[V, W], Z] + [[W, Z], V] + [[Z, V], W] = 0.$$

(iv) For $f, g \in C^\infty(M)$,

$$[fV, gW] = fg[V, W] + (fV(g))W - (gW(f))V.$$

(v) Naturality: Let $F: M \to N$ be a smooth map. Suppose that $V_1, V_2 \in \mathfrak{X}(M)$ and $W_1, W_2 \in \mathfrak{X}(M)$ are such that V_i and W_i are F-related, for $i = 1, 2$. Then, $[V_1, V_2]$ and $[W_1, W_2]$ are F-related.

(vi) If $F: M \to N$ is a diffeomorphism, then

$$F_*[V, W] = [F_* V, F_* W].$$

Pf: Properties (i) and (ii) follow straight from the definition of the Lie bracket. The Jacobi Identity is a direct computation: see Lee p. 92 for details.

Let us check properties (iv) to (vi).

(iv): Let $h \in C^\infty(M)$. Then,

$$[fV, gW](h) = fV(gW(h)) - gW(fV(h))$$

$$= f\left(V(g)W(h) + g\,VW(h)\right) - g\left(W(f)V(h) + f\,WV(h)\right)$$

$$= fg[VW(h) - WV(h)] + f\,V(g)W(h) - g\,W(f)V(h)$$

$$= fg[V,W](h) + fV(g)W(h) - gW(f)V(h).$$

$$\Rightarrow [fV, gW] = fg[V,W] + fV(g)W - gW(f)V.$$

(V) Recall that $Y \in \mathfrak{X}(M)$ and $Z \in \mathfrak{X}(N)$ are F-related if and only if

$$Z(f) \circ F = Y(f \circ F)$$

for all for $f \in C^\infty(M)$. Hence,

$$W_i(f) \circ F = V_i(f \circ F)$$

for $i = 1, 2$, and we need to check that

$$[W_1, W_2](f) \circ F = [V_1, V_2](f \circ F).$$

Now,

$$[W_1, W_2](f) \circ F = \left(W_1(W_2(f)) - W_2(W_1(f))\right) \circ F$$

$$= W_1(\underbrace{W_2(f)}_{\in C^\infty(M)}) \circ F - W_2(\underbrace{W_1(f)}_{\in C^\infty(M)}) \circ F$$

$$= V_1\left(W_2(f) \circ F\right) - V_2\left(W_1(f) \circ F\right)$$

$$= V_1\left(V_2(f \circ F)\right) - V_2\left(V_1(f \circ F)\right)$$

$$= [V_1, V_2](f \circ F).$$

(vi) Since F is a diffeomorphism, $F_* V_1$ and $F_* V_2$ exist. Moreover, V_i and $F_* V_i$ are F-related by the definition of the pushforward, for $i=1,2$. So, by (v), $[V_1, V_2]$ and $[F_* V_1, F_* V_2]$ are F-related. BUT, since F is a diffeomorphism, $F_*[V_1, V_2]$ is the <u>ONLY</u> smooth vector field on N that's F-related to $[V_1, V_2]$.

$$\Longrightarrow [F_* V_1, F_* V_2] = F_* [V_1, V_2].$$

$\boxtimes$