PMATH 465/665: Assignment 4

Due: Monday, 7 November, 2011

Suggested problems:

1. Let M be a smooth manifold.
   (a) Show that, for $\omega \in \Lambda^k(M)$ and $\eta \in \Lambda^l(M)$,
   $$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega.$$
   (b) Show that if $\omega_1, \ldots, \omega_k \in \Lambda^1(M)$, then for any $p \in M$ and $X_1, \ldots, X_k \in T_pM$,
   $$(\omega_1 \wedge \cdots \wedge \omega_k)_p(X_1, \ldots, X_p) = \det((\omega_j)_p(X_i)).$$
2. Exercise 12.5, p. 304.
3. Problem 12-6, p. 320.

Problems to be handed in:

1. *Orientable manifolds.* A smooth n-manifold is called *orientable* if it admits an atlas $\{(U_\alpha, \varphi_\alpha)\}$ whose transition functions have positive Jacobian:
   $$\det(Jac(\varphi_\alpha \circ \varphi_\beta^{-1})(x)) > 0$$
   for all $x \in \varphi_\beta(U_\alpha \cap U_\beta)$, for all α, β such that $U_\alpha \cap U_\beta \neq \emptyset$.
   (a) Show that S^n is orientable for all $n \geq 1$.
   (b) Show that $\mathbb{RP}^n$ is orientable if and only if n is odd.
   (c) Show that every parallelisable manifold is orientable.
   (d) Is the converse of (c) true? Justify your answer.
   (e) Show that the product of two orientable manifolds is orientable.
2. Problem 12-7, p. 320.
3. *Symplectic manifolds.* Let M be a $2n$-dimensional smooth manifold. A 2-form ω on M is called *symplectic* if it satisfies the following two conditions:
   - ω is closed;
   - ω is *non-degenerate*, i.e., for any $p \in M$ and $X \in T_pM$, if $\omega_p(X, Y) = 0$ for all $Y \in T_pM$, then $X = 0$.

   A *symplectic manifold* is a pair (M, ω), where M is smooth $2n$-dimensional manifold and ω is a symplectic form on M.

   Symplectic manifolds play a very important role in the Hamiltonian formulation of classical mechanics because, in the Hamiltonian formalism, the phase space of a mechanical system is identified with a (specific) symplectic manifold.

(a) Suppose that $M = \mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n)$. Show that the 2-form

$$\omega = \sum_{i=1}^{2n} dx_i \wedge dy_i$$

is symplectic, implying that $\mathbb{R}^{2n}$ is symplectic.

(b) Show that S^2 admits a symplectic form. (*Hint:* Consider the 2-form $\omega = \sin\phi d\theta \wedge d\phi$.)

(c) Let M be a symplectic manifold, and ω be a symplectic form on M. Show that

$$\omega^n := \omega \wedge \cdots \wedge \omega,$$

defined as the wedge product of n copies of ω, is a nowhere vanishing $2n$-form on M.

(d) Prove that any symplectic manifold is orientable.

(e) Is the converse of (d) true? Justify your answer.

(f) *(Optional)* Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. Show that $M_1 \times M_2$ is symplectic with respect to the 2-form $\omega = p_1^*\omega_1 - p_2^*\omega_2$, where $p_i : M_1 \times M_2 \to M_i$ denotes projection onto the i-th factor for $i = 1, 2$.

Bonus. *Complex manifolds.* A topological space M is called a *complex manifold* if it is an even dimensional topological manifold and admits an atlas $(U_\alpha, \varphi_\alpha)$ whose transition functions $\varphi_\alpha \circ \varphi_\beta^{-1}$ are complex analytic if we identify $\mathbb{R}^{2n}$, with coordinates $(x_1, y_1, \ldots, x_n, y_n)$, to $\mathbb{C}^n$, with coordinates $(z_1, \ldots, z_n)$, by setting $z_j = x_j + iy_j$ for all $j = 1, \ldots, n$.

(a) Show that $\mathbb{CP}^n$ is a complex manifold for all $n \geq 1$.

(b) Show that any complex manifold is orientable.