PMATH 465/665: Assignment 3

Due: Monday, 24 October, 2011

Suggested problems:

1. Problem 4-4, p.101.
2. Problem 4-5, p. 101.
3. Problem 4-8, p.101.
4. Problem 6-5, p. 151.
5. Problem 6-6, (a), (b), p. 152.

Problems to be handed in:

1. Problem 4-6, p.101.
2. *Parallelisable manifolds.* A smooth n-manifold M is called *parallelisable* if it admits n smooth vector fields $Y_1, \ldots, Y_n$ that are linearly independent at every point $p \in M$.

   (a) Show that if M is parallelisable, then its tangent bundle TM is diffeomorphic to the Cartesian product $M \times \mathbb{R}^n$.

   (b) Show that if $M_1, \ldots, M_k$ are parallellisable manifolds, then so is $M_1 \times \cdots \times M_k$.

   (c) Show that S^1, S^3, and n-tori $S^1 \times \cdots \times S^1$ are parallelisable.

   (d) Show that S^{2n} is not parallelisable for any integer $n \geq 1$.

   Note: The only spheres that are parallelisable are S^1, S^3, and S^7. Showing that odd-dimensional spheres S^{2n+1} with $2n+1 \neq 1, 3, 7$ are not parallelisable is, however, a highly non-trivial task, and was proved in 1958 independantly by Bott and Milnor (see `http://www.ams.org/journals/bull/1958-64-03/S0002-9904-1958-10166-4/S0002-9904-1958-10166-4.pdf`), and by Kervaire.

   (e) Show that M is parallelisable if and only if there exist n smooth 1-forms $\omega_1, \ldots, \omega_n$ on M that are linearly independent at every $p \in M$. Conclude that if M is parallelisable, then its cotangent bundle T^*M is diffeomorphic to the Cartesian product $M \times (\mathbb{R}^n)^*$.

3. Problem 4-11, p. 101.
4. Problem 6-5, p. 151.