On Vehicle Routing Problems with Stochastic Demands — Scenario-Optimal Recourse Policies

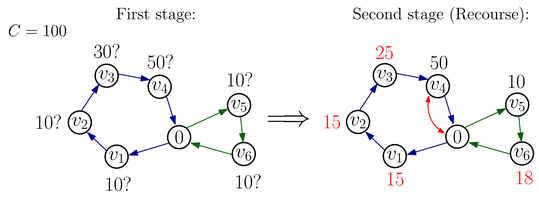
Matheus J. Ota and Ricardo Fukasawa

Institution: University of Waterloo, Department of Combinatorics and Optimization Email: mjota@uwaterloo.ca

UNIVERSITY OF WATERLOO

Introduction

Two-Stage VRPSDs: Input: graph $G = (V = \{0\} \dot\cup V_+, E)$; edge costs $c \in \mathbb{R}^E$; random demands d; vehicle capacity C.



Routes: $R_1 = (v_1, v_2, v_3, v_4)$ and $R_2 = (v_5, v_6)$.
Recourse function: $\mathcal{Q}(R)$ = expected **recourse cost** of route R.
Problem vrpsd($\mathcal{Q}$): find $\mathcal{R} = \{R_1, \dots, R_t\}$ (t is not fixed) such that $V_+ = \bigcup_{R\in\mathcal{R}} V_+(R)$ and $\sum_{R\in\mathcal{R}}[c(R) + \mathcal{Q}(R)]$ is minimum.
Scenarios: We have $d^1, \dots, d^N \in \mathbb{Q}_+^{V_+}$ such that $d = d^s$ with probability $1/N$, for all $s \in [N]$.

- **Remark:** Most works enforce $|\mathcal{R}| = k$ and $\mathbb{E}[d(R)] \le C$, but these assumptions were recently questioned by [1].

Literature Review:

- Investigated for **over 50 years** *[Gendreau et al., 2016]* with several works published in the **last decade.**
- Most works assume customer demands are **independent** and handling **correlations** is a recognized challenge.
- We recently addressed this gap with a **generic integer L-shaped (ILS) framework** for the VRPSD **with scenarios** [4].
- With scenarios, **pricing routes** in a branch-(cut-)and-price under any "reasonable" recourse policy is **strongly NP-hard** [3].

ILS Formulations:

Feasible solutions can be represented as

$$\mathcal{X} := \left\{ x \in [0,2]^E \cap \mathbb{Z}^E : \begin{array}{ll} x(\delta(v)) = 2, & \forall v \in V_+ \\ x(U) \le |U| - 1, & \forall \emptyset \subsetneq U \subseteq V_+ \end{array} \right\}.$$

Activation function: An **activation function** $W(x; \mathcal{X}')$, with $\mathcal{X}' \subseteq \mathcal{X}$, returns 1 if $x \in \mathcal{X}'$ and at most zero if $x \in \mathcal{X} \setminus \mathcal{X}'$.

- **Example 1:** If $\mathcal{X}(R)$ denotes the solutions containing route $R = (v_1, v_2)$, then $W(x; \mathcal{X}') = (x_{0v_1} - 1) + 2(x_{v_1v_2} - 1) + (x_{0v_2} - 1) + 1$.

Let variable θ_v represent $\mathcal{Q}(R, v)$, i.e., the **"recourse cost of a selected route R at customer $v \in V_+$"**. We formulate vrpsd($\mathcal{Q}$) as

$$\begin{aligned} \min\ & c^\intercal x + \mathbb{1}^\intercal \theta \\ & \theta(V_+(R)) \ge \mathcal{Q}(R) \cdot W(x; \mathcal{X}(R)), \quad \forall \text{ route } R, && \text{(1a)} \\ & (x, \theta) \in \mathcal{X} \times \mathbb{R}_+^{V_+}. && \text{(1b)} \end{aligned}$$

Inequalities (1a) are **ILS cuts**. In [4], we strengthen this model with ILS cuts based on **partial routes** and **customer sets**.

Our Goal

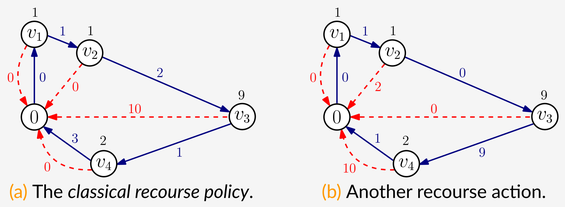
ILS cuts are **"big-M"-type** inequalities that are either **active** ($W(x; \mathcal{X}') = 1$) or **inactive** ($W(x; \mathcal{X}') \le 0$).

- *Can we design **stronger non-ILS cuts** valid for **any recourse function** $\mathcal{Q}$?*

We consider the notion of **"stronger cuts"** both theoretically and computationally.

What is a Recourse Policy?

Intuitively, a recourse policy specifies the actions a vehicle takes to **avoid capacity violations**.



(a) The *classical recourse policy*. (b) Another recourse action.

Figure 1. Illustration for $k = 1$ and $C = 10$. We show two recourse actions for a given scenario $s \in [N]$.

Recourse actions and policies: Let $R = (v_1, \dots, v_\ell)$ with $v_0 = v_{\ell+1} = 0$. A vector $y^s \in \mathbb{Z}_+^{V_+}$ is a **recourse action** for R and scenario $s \in [N]$ if there exists $f^s, g^s \ge 0$ such that:

$$\begin{aligned} f^s_{(v_{i-1},v_i)} + d^s(v_i) &= f^s_{(v_i,v_{i+1})} + g^s_{v_i}, && \forall i \in [\ell], && \text{(2a)} \\ f^s_{(v_{i-1},v_i)} &\le C, && \forall i \in [\ell+1] && \text{(2b)} \\ g^s_{v_i} &\le C \cdot y^s_{v_i}, && \forall i \in [\ell]. && \text{(2c)} \end{aligned}$$

- $\mathcal{Y}^s(R)$: set of all **recourse actions** y^s.
- $\Pi(R)$: set of all **recourse policies** $\pi = (y^1, \dots, y^N)$.

Scenario-optimal recourse policy: Given $w \in \mathbb{Q}_+^{V_+}$ and $b \in \mathbb{Z}_+^{V_+}$, the **scenario-optimal** recourse cost of route R is

$$\mathcal{Q}^*(R) := \min \left\{ \sum_{s\in[N]} \sum_{v\in V_+} w_v y^s_v : \pi = (y^1, \dots, y^N) \in \Pi(R) \right\}.$$

The Convex Hull of Recourse Policies

Theorem 1: $\text{conv}(\Pi(R) \cap [\mathbf{0}, b]^N)$ can be written as

$$\left\{ y \in [\mathbf{0}, b]^N : \ y^s(R') \ge \lceil d^s(R')/C \rceil - 1, \ \forall R' \subseteq R, s \in [N] \right\}.$$

Proof (sketch): Let A be the polytope in the statement.

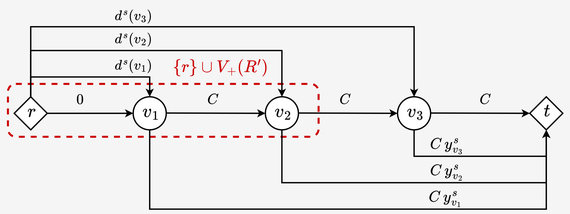


Figure 2. Formulation (2) is feasible $\iff$ max $r - t$ flow $= d^s(R)$.

CONV$(\Pi(R) \cap [\mathbf{0}, b]^N) \subseteq A$: Given $\bar y^s \in \mathbb{R}_+^{V_+}$, by **max-flow/min-cut theorem**, Formulation (2) is feasible if and only if, for every subroute R' of R,

$$C + C\,\bar y^s(R') + d^s(R \setminus R') \ge d^s(R) \iff \bar y^s(R') \ge d^s(R')/C - 1.$$

$A \subseteq$ CONV$(\Pi(R) \cap [\mathbf{0}, b]^N)$:

$\Longrightarrow$ Let $\hat\pi = (\hat y^1, \dots, \hat y^N) \in [\mathbf{0}, b]^N$ be an **extreme point** of A.
$\Longrightarrow$ The matrix M that defines A has **consecutive ones**. Hence, M is totally unimodular and $\hat\pi$ is **integral**.
$\Longrightarrow$ $\hat\pi$ satisfies the **cut conditions** for every subroute R' of R and scenario $s \in [N]$, so $\hat\pi \in \Pi(R)$. $\square$

Mathematical Formulations

Theorem 2: Problem vrpsd($\mathcal{Q}^*$) is equivalent to

$$\begin{aligned} \min\ & c^\intercal x + \sum_{v\in V_+} \sum_{s\in[N]} (w_v/N)\, y^s_v, && \text{(3a)} \\ \text{s.t. } & y^s(U) \ge \lceil d^s(U)/C \rceil + x(E(U)) - |U|, \ \ \forall \emptyset \subsetneq U \subseteq V_+, \ s \in [N], && \text{(3b)} \\ & x \in \mathcal{X}, \ y \in [\mathbf{0}, b]^N. && \text{(3c)} \end{aligned}$$

- Inequalities (3b) are **scenario-recourse inequalities (SRIs)**.
- SRIs are valid for any recourse function $\mathcal{Q}$ satisfying:

Assumption 1: For every route R, there exists $\bar y_R \in \Pi(R) \cap [\mathbf{0}, b]^N$ such that $\mathcal{Q}(R, v) \ge (w_v/N) \sum_{s\in[N]} (\bar y_R)^s_v$, for all $v \in V_+(R)$.

Projecting Out the Scenario Variables

Form. (3) is large due to its $N \cdot |V_+|$ y-variables. $\implies$ We **project** it onto the (x, θ)-space of Form. (1). Let $\mathcal{P}$ be the LP relax. of (3).

$$\text{map}(\mathcal{P}) := \left\{ (x, \theta) : \exists (x, y) \in \mathcal{P}, \ \theta_v \ge \frac{w_v}{N} \sum_{s\in[N]} y^s_v, \ \forall v \in V_+ \right\}.$$

If **Assumption 1** holds, then map($\mathcal{P}$) yields a **relaxation** of (1).

Example 2: For $S' \subseteq [N]$, the inequality below is valid for map($\mathcal{P}$):

$$\sum_{v\in U} \frac{N}{w_v} \theta_v \ge \sum_{s\in S'} y^s(U) \ge \sum_{s\in S'} \left(\lceil d^s(U)/C \rceil + x(E(U)) - |U| \right).$$

Projected SRIs: Generalization of **Example 2** using **conic combinations** of SRIs and bound constraints $y^s_v \le b_v$.

Theorem 3 (informal): The optimal duals (α, β) of Form. (3) gives a **projected SRI** that recovers $z_{LP} = \min_{(x,\theta)\in\text{map}(\mathcal{P})} \{c^\intercal x + \mathbb{1}^\intercal \theta\}$.

Theoretical Comparisons with ILS Cuts

Partial route: tuple of customers $H = (U_1, \dots, U_t)$. Define $h_i := \sum_{j\in[i]} |U_j|$. A route $R = (v_1, \dots, v_\ell)$ **adheres** to H if

$$R = (\underbrace{v_1, \dots, v_{h_1}}_{U_1}, \dots, \underbrace{v_{h_{i-1}+1}, \dots, v_{h_i}}_{U_i}, \dots, \underbrace{v_{h_{t-1}+1}, \dots, v_{h_t}}_{U_t}).$$

Partial route inequality (ILS cut):

$$\theta(V_+(H)) \ge \mathcal{L}(H) \cdot W(x; \mathcal{X}(H)), \qquad \text{(4)}$$

where $\mathcal{X}(H)$ are the solutions containing **routes adhering to H**. By **Theorem 1**, we may set $\mathcal{L}(H) = \sum_{s\in[N]} \mathcal{L}^s(H)$, where

$$\begin{aligned} \mathcal{L}^s(H) = \min\ & \sum_{v\in H} w_v y^s_v, && && \text{(5a)} \\ \text{s.t. } & y^s(H') \ge \lceil d^s(H')/C \rceil - 1, && \forall H' \subseteq H, && (\alpha^s_{H'}) \ \text{(5b)} \\ & y^s_v \le b_v, && \forall v \in V_+, && (\beta^s_v) \ \text{(5c)} \\ & y^s_v \ge 0, && \forall v \in V_+. && \text{(5d)} \end{aligned}$$

Theorem 4: If $W(x; \mathcal{X}(H))$ is the activation function in [4], the projected SRI defined by (α, β) **dominates** (4).

Corollary 1: For $\mathcal{Q}^*$, the following are **valid for** map($\mathcal{P}$):

(i) inequality (4) (with the activation function of [2]);
(ii) **ILS path cuts** of [5]; and
(iii) **ILS cuts** of *[Gendreau et al., 1995]*.

- Similar argument shows that the **ILS set cuts** used in [4] (based on [5]) are also valid for map($\mathcal{P}$).

Computational Experiments

A Two-Step Algorithm

- Step 1: solve the LP relaxation of formulation (3)

- Separate SRIs using a heuristic based on the **CVRPSEP** package and a **MIP model**.
- Stop after **60 seconds** with a bound of $\hat z$ and a fractional solution $(\bar x, \bar y)$.

- Step 2: add projected SRIs to ILS formulation

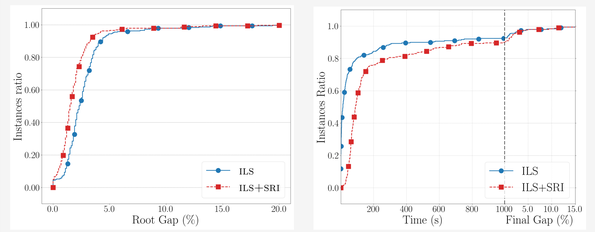
Initialize the algorithm in [4] as follows.

- First, we add the inequalities tight at $\bar x$.
- For each SRI tight at $(\bar x, \bar y)$, we add the **projected SRI** in **Example 2** by choosing the scenarios in S'.
- If necessary, apply **Theorem 3** to recover $\hat z$.

Results

- Implemented in C++/Gurobi 12 with a **1000s time limit**.
- **290 instances**: 270 from *[Jabali et al., 2014]* and 20 from *[Dinh et al., 2018]*.
- **Classical recourse** policy (projected SRIs are valid).
- VRPSD-Basic uses $\mathcal{X}$, VRPSD uses

$$\mathcal{X}_{\text{cvrp}} := \left\{ x \in \mathcal{X} : \begin{array}{l} x(\delta(0)) = 2k, \\ x(U) \le |U| - \lceil \mathbb{E}[d(U)]/C \rceil, \ \forall \emptyset \subsetneq S \subseteq V_+ \end{array} \right\}$$



(a) VRPSD — Root gap. (b) VRPSD — Execution time.
(c) VRPSD-Basic — Root gap. (d) VRPSD-Basic — Execution time.

Figure 3. Computational results.

References

[1] Hoogendoorn, Y. N.; Spliet, R. An evaluation of common modeling choices for the vehicle routing problem with stochastic demands. European Journal of Operational Research (2025).

[2] Hoogendoorn, Y. N.; Spliet, R. An improved integer L-shaped method for the vehicle routing problem with stochastic demands. INFORMS Journal on Computing (2023).

[3] Ota, M.J.; Fukasawa, R. Hardness of pricing routes for two-stage stochastic vehicle routing problems with scenarios. Operations Research (2024).

[4] Ota, M.J.; Fukasawa, R. On VRPs with stochastic demands: Generic integer L-shaped formulations. Submitted (2025). Preprint: https://arxiv.org/abs/2510.04043.

[5] Parada, L.; Legault, R.; Côté, J.-F.; Gendreau, M. A disaggregated integer L-shaped method for stochastic vehicle routing problems with monotonic recourse. European Journal of Operational Research (2024).