Preface

This book is designed as a first course in Markowitz Mean-Variance portfolio optimization. The sole prerequisite is elementary linear algebra: linear independence, matrix vector operations and inverse matrices. The presentation is quantitative and all material is developed precisely and concisely. Part of the author's motivation in writing this book was his realization that the mathematical tools of linear algebra and optimization can be used to formulate and present the key ideas of the subject very quickly and cleanly. A unique feature of the book is that it extends the key ideas of the Markowitz "budget constraint only" model to a model having linear inequality constraints. The linearly constrained model is what practitioners insist upon using.

The author is a researcher in, and a practitioner of, portfolio optimization. This book reflects that background.

Chapter 1 presents necessary and sufficient conditions for optimality for quadratic minimization subject to linear equality constraints. These results are a corner stone for the book. Optimality conditions (Karush-Kuhn-Tucker conditions) for linearly constrained nonlinear problems are also given as they are required in the development of linearly constrained Sharpe ratios and implied risk free rates. Extreme points are also developed as they play a key role in certain aspects of portfolio optimization.

Chapter 2 develops the key properties of the efficient frontier by posing the portfolio optimization problem as a parametric quadratic minimization problem and then solving it explicitly using results from Chapter 1. The equation of the efficient frontier follows quickly from this. Alternate derivations (maximization of expected return and minimization of risk) are formulated as exercises. Chapter 3 extends these results to problems having a risk free asset. This gives the Capital and Security Market lines, again by using the optimality conditions of Chapter 1. The tangency result for the Capital Market Line and the efficient frontier for risky assets is shown in quick and precise detail.

Chapter 4 develops Sharpe ratios and implied risk free rates in two ways. The first is by direct construction. The tangent to the efficient frontier is formulated and the implied risk free rate is calculated. The argument is reversible so that if the risk free rate is specified, the market portfolio can then be calculated. The second approach formulates each of these two problems as nonlinear optimization problems and the nonlinear optimization techniques of Chapter 1 are used to solve. The advantage of this approach is that it is easy to generalize to problems having linear constraints as is done in Chapter 9. Finally, it shown that any point on the efficient frontier gives an optimal solution to four portfolio optimization problems involving portfolio expected return, variance, risk free rates and market portfolios.

A knowledge of quadratic programming is essential for those who wish to solve practical portfolio optimization problems. Chapter 5 introduces the key concepts in a geometric way. The optimality conditions (Karush-Kuhn-Tucker conditions) are first formulated from graphical examples, then formulated algebraically and finally they are proven to be sufficient for optimality. A simple quadratic programming algorithm is formulated in Chapter 6. Its simplicity stems from leaving the methodology of solving the relevant linear equations unspecified and just focusses on determining the search direction and step size. The algorithm is implemented in a Matlab program called QPSolver and the details of the method are given at the end of the chapter.

Chapter 7 begins with an example of a problem with no short sales constraints and solves it step by step for all possible values of the risk aversion parameter. It is observed that the efficient portfolios are piecewise linear functions of the risk aversion parameter. This result is then shown to be true in general for a constrained portfolio optimization problem. The resulting efficient frontier is shown to be piecewise parabolic and (normally) differentiable. Chapter 8 extends these results into an algorithm which determines the entire (constrained) efficient frontier, its corner portfolios, the piecewise linear expected returns and the piecewise quadratic variances. This then is formulated as a Matlab program and is explained in detail at the end of Chapter 8.

Chapter 9 extends Sharpe ratios and implied risk free returns to portfolio optimization problems which have linear inequality constraints and shows how to determine them. Some of the results are perhaps surprising. For example, when a corner portfolio happens to be an extreme point (a simple example is given), the efficient frontier is not differentiable at this point. Further, if this portfolio is taken to be the market portfolio, the implied risk free return is no longer a unique point but is actually an infinite range of values (formulae are given).

The text shows clearly how to implement each technique by hand (for learning purposes). However, it is anticipated that the reader will want to apply these techniques to problems of their own choosing and solving by hand is prohibitive. Thus each chapter concludes with the presentation of several Matlab programs designed to implement the methods of the chapter. These programs are included on a CD which accompanies the book. Each program is explained, line by line, in the book itself. It is anticipated that a reader not familiar with Matlab (nor even computer programming) can easily use these programs to good advantage. One such program, upon being given the problem covariance matrix and expected return vector, computes the coefficients of the efficient frontier. For a portfolio optimization with arbitrary linear constraints, a second program computes any point on the efficient frontier for specified expected risk aversion parameter. A third program constructs the entire efficient frontier (corner points) for a linearly constrained problem. The book includes numerous exercises. Some of these are numerical and are designed to amplify concepts in the text with larger numerical examples. Others are more analytical and will exercise and enhance the reader's understanding of the textual material. The prerequisite information needed for some exercises is set up in the text and the exercises take it to its logical conclusion.