

Infinite Families of Ramanujan Hypergraphs

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Ramanujan Graphs

Brief Review of Graph Expansion

Three classical views of graph expansion:

- **Combinatorial**—Isoperimetric inequalities
- **Linear Algebraic**—Spectral gap
- **Probabilistic**—Rapid convergence of the random walk

View from Linear Algebra

Definition

The **adjacency matrix** A of simple graph $G = (V, E)$ is the $|V| \times |V|$ matrix with entries

$$A_{x,y} = \begin{cases} 1, & \text{if } xy \in E \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, A has n real eigenvalues (including multiplicities),

$$\lambda_1(G) \geq \dots \geq \lambda_n(G).$$

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Simple Observations

- If G is d -regular, then $\lambda_1(G) = d$.
- G is connected if and only if $\lambda_1(G) > \lambda_2(G)$. The quantity $\lambda_1(G) - \lambda_2(G)$ is known as the **spectral gap**.
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Combinatorial Definition

Definition

A graph $G = (V, E)$ is said to be a ***c-expander*** if for every partition $S \subseteq V$ and $\bar{S} = V \setminus S$ with $|S| \leq \frac{|V|}{2}$

$$e(S, \bar{S}) \geq c|S|.$$

Spectral Gap

Lemma

Let $G = (V, E)$ be a d -regular graph on n vertices. Then for $S \subseteq V$ with $|S| \leq \frac{n}{2}$

$$\frac{e(S, \bar{S})}{|S|} \geq \frac{n - |S|}{n} (d - \lambda_2(G)) \geq \frac{1}{2} (d - \lambda_2(G)).$$

Thus, for d -regular G

$$e(S, \bar{S}) \geq \frac{1}{2} (\lambda_1(G) - \lambda_2(G)) |S|,$$

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Alon-Boppana

A natural question is:
how large can the spectral gap of a d -regular graph be?

In other words, how small can $\lambda_2(G)$ be?

Theorem (Alon-Boppana 1984)

Let $\{G_m\}_{m \geq 1}$ be a family of finite, connected, d -regular graphs with $|V_m| \rightarrow +\infty$ as $m \rightarrow +\infty$. Then

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Ramanujan Graphs

Definition

A finite, connected, d -regular graph G is said to be Ramanujan if for every eigenvalue $|\lambda| \neq d$ satisfies

$$|\lambda| \leq 2\sqrt{d-1}.$$

Examples

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$K_{d+1} \\ \{d, -1, \dots, -1\}$$

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$$K_{d,d} \\ \{d, 0, \dots, 0, -d\}$$

Infinite Families

Conjecture (Lubotzky, Phillips, and Sarnak; Margulis 1988)

For all integers $d \geq 3$, there exists an infinite family of d -regular Ramanujan graphs.

Results

For the following values of d , there exist infinite families of d -regular (non-bipartite) Ramanujan graphs:

- 1 $d = p + 1$, p an odd prime (1988 LPS and 1988 Margulis),
- 2 $d = 2 + 1 = 3$ (1992 Chiu),
- 3 $d = q + 1$, q a prime power (1994 Morgenstern).

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Applications

Infinite family of d -regular Ramanujan graphs are expanders which are “optimal” from a spectral point of view.

Ramanujan graphs have many applications.

Explicit constructions of the error correcting codes of Sipser and Spielman require non-bipartite Ramanujan expanders whereas improvements of this construction require bipartite Ramanujan expanders.

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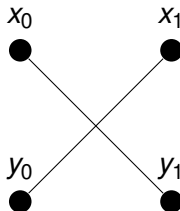
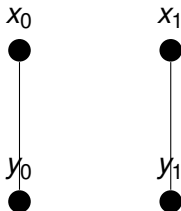
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Graph 2-lifts

What are 2-lifts?

Take two copies of the vertex set of a base graph $G = (V, E)$.

For each edge $xy \in E$, either add a pair of parallel edges or crossing edges:

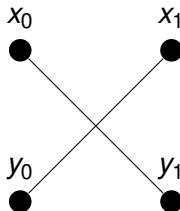
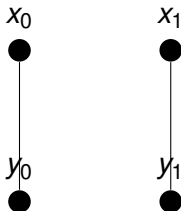


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“Old Eigenvalues” of \tilde{G}

Lemma (Bilu and Linial 2006)

The lifted graph inherits every eigenvalue of the base graph.

Proof.

Take any eigenfunction f of G , and assign the value $f(x)$ to every vertex in \tilde{G} in the preimage of x . This is an eigenfunction of \tilde{G} with the same eigenvalue as f in G . □

Bilu and Linial suggested trying to construct Ramanujan graphs by iteratively applying 2-lifts to a base graph, for example K_{d+1} .

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2-lifts

Conjecture (Bilu-Linial 2006)

For integer valued $d \geq 3$, every d -regular graph G has a 2-lift \tilde{G} where all “new” eigenvalues of \tilde{G} , say λ , satisfy

$$|\lambda| \leq 2\sqrt{d-1}.$$

2-lifts

Marcus, Spielman, and Srivastava take appropriate 2-lifts of $K_{d,d}$ using the “method of interlacing polynomials”.

Lemma (Marcus, Spielman, and Srivastava 2015+)

For all $d \geq 3$, there are infinitely many d -regular, bipartite Ramanujan graphs.

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2-lifts

Definition

For all $c, d \geq 3$, a (c, d) -biregular bipartite graph G is said to be Ramanujan if for every eigenvalue $|\lambda| \neq \sqrt{cd}$ satisfies

$$|\lambda| \leq \sqrt{c-1} + \sqrt{d-1}.$$

They also take appropriate 2-lifts of $K_{c,d}$ using the “method of interlacing polynomials”.

Lemma (Marcus, Spielman, and Srivastava 2015+)

For every integer valued $c, d \geq 3$, there is an infinite sequence of (c, d) -biregular, bipartite Ramanujan graphs.

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Hypergraph Setting

Hypergraphs

Definition

A **hypergraph** \mathcal{H} has vertex set V and hyperedge set E consisting of nonempty subsets of V .

Definition

\mathcal{H} is **c -uniform** if each hyperedge is a c -element subset of V .

Definition

\mathcal{H} is **d -regular** if every vertex is contained in d hyperedges.

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Hypergraph Eigenvalues

Definition

The **adjacency matrix** A of hypergraph $\mathcal{H} = (V, E)$ is the $|V| \times |V|$ matrix with entries

$$A_{x,y} = \begin{cases} |\{e \in E : \{x, y\} \subseteq e\}|, & \text{if } x \neq y, \text{ and } x, y \in V, \\ 0, & \text{if } x = y \in V. \end{cases}$$

Analogue of Alon-Boppana

Theorem (Li and Solé 1996)

For c -uniform, d -regular hypergraphs \mathcal{H} , one has

$$\liminf \lambda_2(\mathcal{H}) \geq c - 2 + 2\sqrt{(d-1)(c-1)}$$

as the number of vertices in \mathcal{H} tends to infinity.

Ramanujan Hypergraphs

Definition

A c -uniform, d -regular hypergraph \mathcal{H} is said to be **Ramanujan** if any eigenvalue $\lambda \neq d(c-1)$, $\lambda \neq -d$ satisfies

$$|\lambda(\mathcal{H}) - c + 2| \leq 2\sqrt{(d-1)(c-1)}.$$

Infinite Families

Conjecture

For integers $c, d \geq 3$, there exists an infinite family of c -uniform, d -regular Ramanujan hypergraphs.

Theorem (D. 2015+)

For integer valued $d \geq 3$, there exists an infinite family of d -uniform, d -regular Ramanujan hypergraphs.

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Incidence Graphs

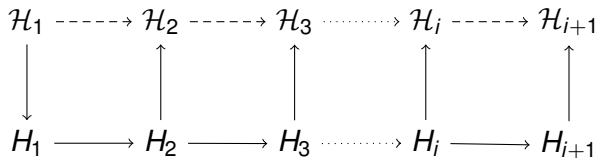
\mathcal{H} can be represented by a (c, d) -biregular, bipartite graph.

Definition

*The **incidence graph** H of hypergraph \mathcal{H} has partitions V and E with $v \in V$ and $e \in E$ adjacent if and only if $\{v\} \subseteq e$ in \mathcal{H} .*

Proof Sketch

For \mathcal{H}_1 a d -uniform, d -regular Ramanujan hypergraph, we can construct an infinite family of Ramanujan hypergraphs:



2-lifts of Hypergraphs

Definition

A **2-lift** of a c -uniform hypergraph \mathcal{H} is a c -uniform hypergraph $\tilde{\mathcal{H}}$ obtained by replacing each vertex of \mathcal{H} with a pair of vertices and each edge with two disjoint edges among the new corresponding vertices.

The process before is actually equivalent to taking good 2-lifts of Ramanujan hypergraphs.

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Infinite Families

In particular, consider the d -uniform, d -regular hypergraph on $d + 1$ vertices where every set of d vertices forms a hyperedge.

The eigenvalues are $d^2 - d$ with multiplicity 1 and $1 - d$ with multiplicity d .

Further Directions

Question

Can we apply the “method of interlacing polynomials” directly to the hypergraphs?

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These results show existence. Can we explicitly construct infinite families of graphs/hypergraphs?

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Thank you for listening!