Infinite Families of Ramanujan Hypergraphs

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Ramanujan Graphs

Brief Review of Graph Expansion

Three classical views of graph expansion:

- **Combinatorial**–Isoperimetric inequalities
- **Linear Algebraic**–Spectral gap
- **Probabilistic**–Rapid convergence of the random walk

View from Linear Algebra

Definition

The **adjacency matrix** *A of simple graph $G = (V, E)$ is the $|V| \times |V|$ matrix with entries*

$$A_{x,y} = \begin{cases} 1, & \text{if } xy \in E \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, A has n real eigenvalues (including multiplicities),

$$\lambda_1(G) \geq \ldots \geq \lambda_n(G).$$

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Simple Observations

- If G is d-regular, then $\lambda_1(G) = d$.
- G is connected if and only if $\lambda_1(G) > \lambda_2(G)$. The quantity $\lambda_1(G) - \lambda_2(G)$ is known as the **spectral gap.**
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Combinatorial Definition

Definition

*A graph $G = (V, E)$ is said to be a c-**expander** if for every partition $S \subseteq V$ and $\overline{S} = V \backslash S$ with $|S| \leq \frac{|V|}{2}$*

$$e(S, \overline{S}) \geq c|S|.$$

Spectral Gap

Lemma

Let $G = (V, E)$ be a d-regular graph on n vertices. Then for $S \subseteq V$ with $|S| \leq \frac{n}{2}$

$$\frac{e(S, \overline{S})}{|S|} \geq \frac{n - |S|}{n}(d - \lambda_2(G)) \geq \frac{1}{2}(d - \lambda_2(G)).$$

Thus, for d-regular G

$$e(S, \overline{S}) \geq \frac{1}{2}(\lambda_1(G) - \lambda_2(G))|S|,$$

and a large spectral gap implies good expansion.

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Alon-Boppana

A natural question is:
how large can the spectral gap of a *d*-regular graph be?

In other words, how small can $\lambda_2(G)$ be?

Theorem (Alon-Boppana 1984)

Let $\{G_m\}_{m\geq 1}$ be a family of finite, connected, d-regular graphs with $|V_m| \to +\infty$ as $m \to +\infty$. Then

$$\liminf_{m\to+\infty} \lambda_2(G_m) \geq 2\sqrt{d-1}.$$

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Ramanujan Graphs

Definition

A finite, connected, d-regular graph G is said to be Ramanujan if for every eigenvalue $|\lambda| \neq d$ satisfies

$$|\lambda| \leq 2\sqrt{d-1}.$$

Examples

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

K_{d+1}

$\{d, -1, \ldots, -1\}$

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$K_{d,d}$

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Infinite Families

Conjecture (Lubotzky, Phillips, and Sarnak; Margulis 1988)

For all integers $d \geq 3$, there exists an infinite family of d-regular Ramanujan graphs.

Results

For the following values of d, there exist infinite families of d-regular (non-bipartite) Ramanujan graphs:

1. $d = p + 1$, p an odd prime (1988 LPS and 1988 Margulis),
2. $d = 2 + 1 = 3$ (1992 Chiu),
3. $d = q + 1$, q a prime power (1994 Morgenstern).

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Applications

Infinite family of d-regular Ramanujan graphs are expanders which are "optimal" from a spectral point of view.

Ramanujan graphs have many applications.

Explicit constructions of the error correcting codes of Sipser and Spielman require non-bipartite Ramanujan expanders whereas improvements of this construction require bipartite Ramanujan expanders.

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Graph 2-lifts

What are 2-lifts?

Take two copies of the vertex set of a base graph $G = (V, E)$.

For each edge $xy \in E$, either add a pair of parallel edges or crossing edges:

We can think of a 2-lift $\widetilde{G}$ as a covering space of G.

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“Old Eigenvalues” of $\widetilde{G}$

Lemma (Bilu and Linial 2006)

The lifted graph inherits every eigenvalue of the base graph.

Proof.

Take any eigenfunction f of G, and assign the value $f(x)$ to every vertex in $\widetilde{G}$ in the preimage of x. This is an eigenfunction of $\widetilde{G}$ with the same eigenvalue as f in G. □

Bilu and Linial suggested trying to construct Ramanujan graphs by iteratively applying 2-lifts to a base graph, for example K_{d+1}.

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2-lifts

Conjecture (Bilu-Linial 2006)

For integer valued $d \geq 3$, every d-regular graph G has a 2-lift $\widetilde{G}$ where all "new" eigenvalues of $\widetilde{G}$, say λ, satisfy

$$|\lambda| \leq 2\sqrt{d-1}.$$

2-lifts

Marcus, Spielman, and Srivastava take appropriate 2-lifts of $K_{d,d}$ using the "method of interlacing polynomials".

Lemma (Marcus, Spielman, and Srivastava 2015+)

For all $d \geq 3$, there are infinitely many d-regular, bipartite Ramanujan graphs.

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Definition

For all $c, d \geq 3$, a (c, d)-biregular bipartite graph G is said to be Ramanujan if for every eigenvalue $|\lambda| \neq \sqrt{cd}$ satisfies

$$|\lambda| \leq \sqrt{c-1} + \sqrt{d-1}.$$

They also take appropriate 2-lifts of $K_{c,d}$ using the "method of interlacing polynomials".

Lemma (Marcus, Spielman, and Srivastava 2015+)

For every integer valued $c, d \geq 3$, there is an infinite sequence of (c, d)-biregular, bipartite Ramanujan graphs.

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Hypergraph Setting

Hypergraphs

Definition

A ***hypergraph*** $\mathcal{H}$ *has vertex set* V *and hyperedge set* E *consisting of nonempty subsets of* V.

Definition

$\mathcal{H}$ *is* c***-uniform*** *if each hyperedge is a* c*-element subset of* V.

Definition

$\mathcal{H}$ *is* d***-regular*** *if every vertex is contained in* d *hyperedges.*

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Hypergraph Eigenvalues

Definition

The **adjacency matrix** *A of hypergraph $\mathcal{H} = (V, E)$ is the $|V| \times |V|$ matrix with entries*

$$A_{x,y} = \begin{cases} |\{e \in E : \{x, y\} \subseteq e\}|, & \text{if } x \neq y, \text{ and } x, y \in V, \\ 0, & \text{if } x = y \in V. \end{cases}$$

Analogue of Alon-Boppana

Theorem (Li and Solé 1996)

For c-uniform, d-regular hypergraphs $\mathcal{H}$, one has

$$\liminf \lambda_2(\mathcal{H}) \geq c - 2 + 2\sqrt{(d-1)(c-1)}$$

as the number of vertices in $\mathcal{H}$ tends to infinity.

Ramanujan Hypergraphs

Definition

A c-uniform, d-regular hypergraph $\mathcal{H}$ is said to be ***Ramanujan*** *if any eigenvalue $\lambda \neq d(c-1)$, $\lambda \neq -d$ satisfies*

$$|\lambda(\mathcal{H}) - c + 2| \leq 2\sqrt{(d-1)(c-1)}.$$

Infinite Families

Conjecture

For integers $c, d \geq 3$, there exists an infinite family of c-uniform, d-regular Ramanujan hypergraphs.

Theorem (D. 2015+)

For integer valued $d \geq 3$, there exists an infinite family of d-uniform, d-regular Ramanujan hypergraphs.

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Incidence Graphs

$\mathcal{H}$ can be represented by a (c, d)-biregular, bipartite graph.

Definition

The **incidence graph** *H of hypergraph $\mathcal{H}$ has partitions V and E with $v \in V$ and $e \in E$ adjacent if and only if $\{v\} \subseteq e$ in $\mathcal{H}$.*

Proof Sketch

For $\mathcal{H}_1$ a d-uniform, d-regular Ramanujan hypergraph, we can construct an infinite family of Ramanujan hypergraphs:

$$\begin{array}{ccccccccc}
\mathcal{H}_1 & \dashrightarrow & \mathcal{H}_2 & \dashrightarrow & \mathcal{H}_3 & \cdots\rightarrow & \mathcal{H}_i & \dashrightarrow & \mathcal{H}_{i+1} \\
\downarrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
H_1 & \longrightarrow & H_2 & \longrightarrow & H_3 & \cdots\rightarrow & H_i & \longrightarrow & H_{i+1}
\end{array}$$

2-lifts of Hypergraphs

Definition

A ***2-lift*** *of a c-uniform hypergraph $\mathcal{H}$ is a c-uniform hypergraph $\widetilde{\mathcal{H}}$ obtained by replacing each vertex of $\mathcal{H}$ with a pair of vertices and each edge with two disjoint edges among the new corresponding vertices.*

The process before is actually equivalent to taking good 2-lifts of Ramanujan hypergraphs.

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Infinite Families

In particular, consider the d-uniform, d-regular hypergraph on $d+1$ vertices where every set of d vertices forms a hyperedge.

The eigenvalues are $d^2 - d$ with multiplicity 1 and $1 - d$ with multiplicity d.

Further Directions

Question

Can we apply the "method of interlacing polynomials" directly to the hypergraphs?

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These results show existence. Can we explicitly construct infinite families of graphs/hypergraphs?

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Thank you for listening!