The Typical Structure of Intersecting Families

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1 Typical Structure of Intersecting Families

This is joint work with Balogh, Das, Liu, and Sharifzadeh [2].

1.1 Introduction

The study of intersecting structures is central to extremal combinatorics. Let S_n denote the symmetric group on $[n]$. A family of permutations $\mathcal{F} \subset S_n$ is said to be *t-intersecting* if any two permutations in $\mathcal{F}$ agree on some t indices. In other words, for any $\sigma, \pi \in \mathcal{F}$,

$$|\sigma \cap \pi| = |\{i \in [n] : \sigma(i) = \pi(i)\}| \geq t.$$

When $t = 1$, we simply say that the family is *intersecting.*

Consider the following example. Fix a t-set, say $I \subseteq [n]$, and values $\{x_i : i \in I\}$. If for every $\sigma \in \mathcal{F}$ and $i \in I$ $\sigma(i) = x_i$, then $\mathcal{F}$ is clearly t-intersecting. Furthermore, we say that $\mathcal{F}$ is a *trivial t-intersecting family* of permutations. Note that the size of this family is at most $(n-t)!$. Ellis, Friedgut, and Pilpel [5] show that for n sufficiently large with respect to t a t-intersecting family of permutations, say $\mathcal{F}$, has size at most $(n-t)!$ with equality only if $\mathcal{F}$ is trivial.

We determine the *typical* structure of t-intersecting families, extending these results to show that almost all t-intersecting families are trivial. We also obtain sparse analogues of these extremal results, showing that they hold in random settings.

Theorem 1.1. *For any fixed $t \geq 1$ and n sufficiently large, almost all t-intersecting families of permutations in S_n are trivial, and there are $\left(\binom{n}{t}^2 t! + o(1)\right) 2^{(n-t)!}$ t-intersecting families.*

Additionally, we prove two extensions of Theorem 1.1 in the sparse setting. In the first we consider t-intersecting families of permutations of size m. Note that each trivial t-intersecting family contains $\binom{(n-t)!}{m}$ subfamilies of size m. The following result shows that, provided m is not too small, the number of non-trivial t-intersecting families of m permutations is a lower-order term.

Theorem 1.2. *For any fixed $t \geq 1$, n sufficiently large and $n2^{2n-2t+2} \log n \leq m \leq (n-t)!$, almost all t-intersecting families of m permutations in S_n are trivial.*

Secondly we obtain the following sparse extension of the result of Ellis, Friedgut and Pilpel [5]. Let $(S_n)_p$ denote the p-random subset of S_n, where each permutation in S_n is included independently with probability p. Provided p is not too small, we show that with high probability the largest t-intersecting family in $(S_n)_p$ is trivial. Note that the Ellis–Friedgut–Pilpel theorem corresponds to the case $p = 1$.

Theorem 1.3. *For fixed $t \geq 1$, n sufficiently large and $p = p(n) \geq \frac{800n2^{2n-2t}\log n}{(n-t)!}$, with high probability every largest t-intersecting family in $(S_n)_p$ is trivial.*

1.2 Proof Technique

We also obtain similar results for permutations, hypergraphs, and vector spaces, but the basic proof technique is the same. The key observations are the following two facts.

Observation 1.4. Any subset of a trivial t-intersecting family is itself trivial.

Observation 1.5. Any non-trivial t-intersecting family must be contained inside a maximal non-trivial t-intersecting family.

Suppose that we have a stability result that shows both that trivial t-intersecting families are the largest and bounds the size of largest non-trivial t-intersecting families. If we obtain strong bounds on the number of maximal t-intersecting families, then we can combine this with the stability result to obtain bounds on the total number of non-trivial t-intersecting families.

The following lemma, phrased in general terms that will be applicable in all of our settings, gives sufficient conditions for the trivial families to be typical.

Lemma 1.6. *Let N_0 denote the size of the largest trivial intersecting family, and let N_1 denote the size of the largest non-trivial intersecting family. Suppose further that there are at most M maximal intersecting families. Provided*

$$\log M + N_1 - N_0 \to -\infty, \tag{1}$$

almost all intersecting families are trivial. Moreover, if m is such that

$$\log M - m \log\left(\frac{N_0}{N_1}\right) \to -\infty, \tag{2}$$

then almost all intersecting families of size m are trivial.

Proof. Since a largest trivial intersecting family has size N_0, and all of its subfamilies are also trivial, there are at least 2^{N_0} trivial families. On the other hand, every non-trivial intersecting family is a subset of a maximal non-trivial intersecting family. Each maximal non-trivial family has size at most N_1, and thus at most 2^{N_1} subfamilies. Since there are at most M maximal families, the number of non-trivial families is at most $M2^{N_1}$. The proportion of non-trivial families is thus at most $M2^{N_1}/2^{N_0}$, which tends to 0 by (1). Hence, given (1), almost all intersecting families are trivial.

For the second claim, observe that the number of trivial subfamilies of size m is at least $\binom{N_0}{m}$ by considering subfamilies of one fixed trivial family. On the other hand, each non-trivial family has at most $\binom{N_1}{m}$ subfamilies of size m, and hence there are at most $M\binom{N_1}{m}$ non-trivial families of size m. We can thus bound the proportion of intersecting families of size m that are non-trivial by

$$M\binom{N_1}{m}/\binom{N_0}{m} \leq M\left(\frac{N_1}{N_0}\right)^m,$$

which tends to 0 by (2). Hence almost all intersecting families of size m are trivial as well. □

Lemma 1.7. *Let T denote the number of maximal trivial intersecting families, and suppose they all have the same size N_0. Suppose further that two distinct maximal families can have at most N_2 members in common. Provided*

$$2\log T + N_2 - N_0 \to -\infty, \tag{3}$$

the number of trivial intersecting families is $(T + o(1))\, 2^{N_0}$.

Proof. Suppose $\mathcal{F}_1, \dots, \mathcal{F}_T$ are the maximal trivial intersecting families. Every trivial family is a subset of some $\mathcal{F}_i$, and hence the collection of trivial families is given by $\cup_{i=1}^{T}\mathcal{P}(\mathcal{F}_i)$. The Bonferroni inequalities state that, for any sets $\mathcal{G}_1, \dots, \mathcal{G}_m$,

$$\sum_i |\mathcal{G}_i| - \sum_{i<j} |\mathcal{G}_i \cap \mathcal{G}_j| \leq |\cup_i \mathcal{G}_i| \leq \sum_i |\mathcal{G}_i|.$$

Applying this with $\mathcal{G}_i = \mathcal{P}(\mathcal{F}_i)$ for $1 \leq i \leq m = T$, we have $|\mathcal{G}_i| = |\mathcal{P}(\mathcal{F}_i)| = 2^{N_0}$ and $|\mathcal{G}_i \cap \mathcal{G}_j| = |\mathcal{P}(\mathcal{F}_i \cap \mathcal{F}_j)| \leq 2^{N_2}$. This gives

$$\sum_i |\mathcal{G}_i| = T \cdot 2^{N_0} \quad \text{and} \quad \sum_{i<j} |\mathcal{G}_i \cap \mathcal{G}_j| \leq 2^{N_2}\binom{T}{2} < 2^{2\log T + N_2 - N_0} \cdot 2^{N_0} = o\left(2^{N_0}\right),$$

from which the result follows. □

We make use of the following stability theorem obtained by Ellis, Friedgut and Pilpel [5] which was proved using representation theory and spectral methods.

Theorem 1.8. *For n sufficiently large with respect to t, the largest t-intersecting families in S_n are the trivial ones. Furthermore, for t fixed and $n \to \infty$, the largest non-trivial t-intersecting family has size*

$$\left(1 - \frac{1}{e} + o(1)\right)(n-t)!$$

Our proofs use the Bollobás set-pairs inequality [3] to bound the number of maximal intersecting families, which can then be combined with known stability theorems. In particular we use the following version, proven by Füredi [7].

Theorem 1.9 (Füredi)**.** *Let $A_1, \dots, A_m$ be sets of size a and $B_1, \dots, B_m$ be sets of size b such that we have $|A_i \cap B_i| < t$ and $|A_i \cap B_j| \geq t$ for $1 \leq i < j \leq m$. Then $m \leq \binom{a+b-2t+2}{a-t+1}$.*

1.3 Results

We first bound the number of maximal t-intersecting families of permutations, and then deduce from this Theorems 1.1, 1.2 and 1.3.

Proposition 1.10. For any $n \geq t \geq 1$, the number of maximal t-intersecting families in S_n is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2n-2t+2}{n-t+1}} \binom{n!}{i} < n^{n2^{2n-2t+1}}.$$

Proof. For a maximal t-intersecting family $\mathcal{F}$, we define $\mathcal{I}(\mathcal{F}) = \{\pi \in S_n : \forall \sigma \in \mathcal{F},\ |\pi \cap \sigma| \geq t\}$. Let $\mathcal{F}_0 = \{\sigma_1, \dots, \sigma_s\} \subset \mathcal{F}$ be a minimal generating set. By minimality, for each $1 \leq i \leq s$ we have some $\tau_i \in S_n$ such that $|\sigma_j \cap \tau_i| < t$ if and only if $i = j$.

To a permutation π we may assign the n-set of pairs $H_\pi = \{(1, \pi(1)), \dots, (n, \pi(n))\}$. Observe that for any two permutations π and π', $|H_\pi \cap H_{\pi'}| = |\pi \cap \pi'|$. Hence, if we denote $F_i = H_{\sigma_i}$ and $G_i = H_{\tau_i}$, we have $|F_i \cap G_j| < t$ if and only if $i = j$.

We apply the Bollobás set-pairs inequality to the sets $\{(A_i, B_i)\}_{i=1}^{2s}$, where for $1 \leq i \leq s$ we take $A_i = F_i$ and $B_i = G_i$, and for $s+1 \leq i \leq 2s$ we set $A_i = G_{i-s}$ and $B_i = F_{i-s}$. The conditions of Theorem 1.9 are clearly satisfied, and hence we deduce $s \leq \frac{1}{2}\binom{2n-2t+2}{n-t+1}$.

Thus, to every maximal family $\mathcal{F}$ we may assign a distinct generating set of at most $\frac{1}{2}\binom{2n-2t+2}{n-t+1}$ permutations, giving the above sum as a bound on the number of maximal families. The upper bound follows since $n! \leq n^n$ and $\binom{2n-2t+2}{n-t+1} \leq 2^{2n-2t+2}$. □

Proof of Theorem 1.1. We first apply Lemma 1.6 to show that almost all intersecting families are trivial. We have

$$\log M + N_1 - N_0 = n2^{2n-2t+1} \log n - (1/e + o(1))\,(n-t)! \to -\infty,$$

and so (1) is satisfied. This shows that the number of non-trivial t-intersecting families is $o\left(2^{(n-t)!}\right)$.

We use Lemma 1.7 to count the number of trivial families. We see that (3) holds, since

$$2\log T + N_2 - N_0 = 2\log\left(\binom{n}{t}^2 t!\right) + (n-t-1)! - (n-t)! \leq 4t\log(nt) - (n-t-1)(n-t-1)! \to -\infty.$$

Hence the number of trivial families is $\left(\binom{n}{t}^2 t! + o(1)\right) 2^{(n-t)!}$. As the non-trivial families constitute a lower-order term, this completes the proof. □

Proof of Theorem 1.2. To prove that almost every t-intersecting family of m permutations is trivial, we show that (2) is satisfied. Indeed, for $m \geq n2^{2n-2t+2}\log n$,

$$\log M - m\log\left(\frac{N_0}{N_1}\right) = n2^{2n-2t+1}\log n - m\log\left(\frac{(n-t)!}{(1-1/e+o(1))\,(n-t)!}\right)$$
$$\leq n2^{2n-2t+1}\log n - 0.6m \to -\infty.$$

□

Finally, we seek to prove Theorem 1.3, showing that when $p \geq \frac{800n2^{2n-2t}\log n}{(n-t)!}$, with high probability the largest t-intersecting family in the p-random set of permutations $(S_n)_p$ is trivial.

Let $\mathcal{T} \subset S_n$ be a fixed maximal trivial family, and let $\mathcal{F}_1, \dots, \mathcal{F}_M$ be the maximal non-trivial families. Then the largest trivial family in $(S_n)_p$ has size at least $|(\mathcal{T})_p|$, while the largest non-trivial family has size $\max_i |(\mathcal{F}_i)_p|$. In expectation, $\mathbb{E}[|(\mathcal{T})_p|] = p|\mathcal{T}| > p|\mathcal{F}_i| = \mathbb{E}[|(\mathcal{F}_i)_p|]$, and our bound on M is strong enough for a union bound calculation to go through. We require the following version of Hoeffding's Inequality that is derived from [**?**, Theorem 2.3].

Theorem 1.11 (Hoeffding)**.** *Let the random variables $X_1, X_2, \dots, X_n$ be independent, with $0 \leq X_k \leq 1$ for each k. Let $X = \sum_{k=1}^n X_k$, let $\mu = \mathbb{E}[X]$. Then, for any $\varepsilon > 0$,*

$$\mathbb{P}\left(X \geq (1+\varepsilon)\mu\right) \leq \exp\left(-\frac{1}{2}\varepsilon^2\mu\right) \quad \textit{and} \quad \mathbb{P}\left(X \leq (1-\varepsilon)\mu\right) \leq \exp\left(-\frac{1}{2}\varepsilon^2\mu\right).$$

Proof of Theorem 1.3. Let $(\mathcal{T})_p = \mathcal{T} \cap (S_n)_p$, let $(\mathcal{F}_i)_p = \mathcal{F}_i \cap (S_n)_p$, and set $\varepsilon = 1/10$. Let E_0 be the event that $|(\mathcal{T})_p| < (1-\varepsilon)p|\mathcal{T}| = (1-\varepsilon)pN_0$, and let E_i be the event that $|(\mathcal{F}_i)_p| > (1+\varepsilon)pN_1$. Since $N_0 = (n-t)!$ and $N_1 = (1 - 1/e + o(1))\,(n-t)!$, we have $(1+\varepsilon)pN_1 < (1-\varepsilon)pN_0$. If there is a non-trivial largest t-intersecting family in $(S_n)_p$, we must have $\max_i |(\mathcal{F}_i)_p| \geq |(\mathcal{T})_p|$, and so at least one of the events E_j, $0 \leq j \leq M$, must hold.

Now $|(\mathcal{T})_p| \sim \mathrm{Bin}(N_0, p)$, and so applying Theorem 1.11 with $\mu = pN_0$, we have $\mathbb{P}(E_0) \leq \exp\left(-\frac{pN_0}{200}\right)$. Similarly, for $1 \leq i \leq M$, $|(\mathcal{F}_i)_p| \sim \mathrm{Bin}(|\mathcal{F}_i|, p)$, where $|\mathcal{F}_i| \leq N_1$. Let $X \sim \mathrm{Bin}(N_1, p)$. Applying Theorem 1.11 to X with $\mu = pN_1$, we have

$$\mathbb{P}(E_i) = \mathbb{P}(|(\mathcal{F}_i)_p| \geq (1+\varepsilon)pN_1) \leq \mathbb{P}(X \geq (1+\varepsilon)pN_1) \leq \exp\left(-\frac{pN_1}{200}\right).$$

Hence, by the union bound,

$$\mathbb{P}\left(\cup_{i=0}^M E_i\right) = \exp\left(-\frac{pN_0}{200}\right) + M\exp\left(-\frac{pN_1}{200}\right) \leq \left(n^{n2^{2n-2t+1}} + 1\right) \cdot \exp\left(-\frac{pN_1}{200}\right) = o(1)$$

when $p \geq \frac{800n2^{2n-2t}\log n}{(n-t)!} \geq \frac{200}{N_1} n2^{2n-2t+1}\log n$. Thus, for such p, the largest t-intersecting families in $(S_n)_p$ are trivial with high probability. □

1.4 Further Directions

A number of related open problems remain which I would like to explore. These ideas could be applied to a number of extremal problems in discrete mathematics.

For example, we say that a family of permutations of $[n]$ is *t-set-intersecting* if for every pair of permutations σ, π there is some t-set $X \subset [n]$ such that $\sigma(X) = \pi(X)$. Ellis [4] proved that for n sufficiently large, the biggest t-set-intersecting families are trivial; namely, they send a fixed set of t indices to a fixed set of t images.

Question 1.12. Can we show that these trivial families are also typical?

References

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