

Discrete Bernoulli Convolutions

Taking the Convoluted out of Bernoulli Convolutions

Michelle Delcourt

University of Illinois, at Urbana-Champaign

July 12, 2013

A special thanks to:

Neil J. Calkin, Julia Davis, Zebediah Engberg, Jobby Jacob,
and Kevin James,

the National Science Foundation,

and

the organizers of the Seventh Czech-Slovak International
Symposium!

A special thanks to:

Neil J. Calkin, Julia Davis, Zebediah Engberg, Jobby Jacob,
and Kevin James,

the National Science Foundation,

and

the organizers of the Seventh Czech-Slovak International
Symposium!

A special thanks to:

Neil J. Calkin, Julia Davis, Zebediah Engberg, Jobby Jacob,
and Kevin James,

the National Science Foundation,

and

the organizers of the Seventh Czech-Slovak International
Symposium!

A special thanks to:

Neil J. Calkin, Julia Davis, Zebediah Engberg, Jobby Jacob,
and Kevin James,

the National Science Foundation,

and

the organizers of the Seventh Czech-Slovak International
Symposium!

Motivation

Motivation

A Bernoulli convolution for $0 < q < 1$ is the convolution

$$\mu_q(X) = b(X) * b(X/q) * b(X/q^2) * \dots$$

where b is the discrete Bernoulli measure concentrated at 1 and -1 each with weight $\frac{1}{2}$.

In 1935 Jessen and Wintner showed that μ_q is continuous for any q .

Motivation

A Bernoulli convolution for $0 < q < 1$ is the convolution

$$\mu_q(X) = b(X) * b(X/q) * b(X/q^2) * \dots$$

where b is the discrete Bernoulli measure concentrated at 1 and -1 each with weight $\frac{1}{2}$.

In 1935 Jessen and Wintner showed that μ_q is continuous for any q .

Motivation

Alternatively, for $0 < q < 1$, consider the functional equation

$$F(t) = \frac{1}{2}F\left(\frac{t-1}{q}\right) + \frac{1}{2}F\left(\frac{t+1}{q}\right)$$

for t on the interval $I_q := \left[\frac{-1}{1-q}, \frac{1}{1-q}\right]$.

There is a unique bounded solution $F_q(t)$, the distribution function of μ_q , $F_q(t) = \mu_q([-\infty, t])$.

Motivation

Alternatively, for $0 < q < 1$, consider the functional equation

$$F(t) = \frac{1}{2}F\left(\frac{t-1}{q}\right) + \frac{1}{2}F\left(\frac{t+1}{q}\right)$$

for t on the interval $I_q := \left[\frac{-1}{1-q}, \frac{1}{1-q}\right]$.

There is a unique bounded solution $F_q(t)$, the distribution function of μ_q , $F_q(t) = \mu_q([-\infty, t])$.

Motivation

Jessen and Wintner showed that $F_q(t)$ is either absolutely continuous or purely singular. The major question is:

Which values of q make the solution $F_q(t)$ absolutely continuous?

For $q = \frac{1}{2}$, $F_q(t)$ is absolutely continuous.

When $0 < q < \frac{1}{2}$, Kershner and Wintner have shown that $F_q(t)$ is always singular. For these values of q , the solution $F_q(t)$ is an example of a so called *Cantor function*, a function that is constant almost everywhere.

Motivation

Jessen and Wintner showed that $F_q(t)$ is either absolutely continuous or purely singular. The major question is:

Which values of q make the solution $F_q(t)$ absolutely continuous?

For $q = \frac{1}{2}$, $F_q(t)$ is absolutely continuous.

When $0 < q < \frac{1}{2}$, Kershner and Wintner have shown that $F_q(t)$ is always singular. For these values of q , the solution $F_q(t)$ is an example of a so called *Cantor function*, a function that is constant almost everywhere.

Motivation

Jessen and Wintner showed that $F_q(t)$ is either absolutely continuous or purely singular. The major question is:

Which values of q make the solution $F_q(t)$ absolutely continuous?

For $q = \frac{1}{2}$, $F_q(t)$ is absolutely continuous.

When $0 < q < \frac{1}{2}$, Kershner and Wintner have shown that $F_q(t)$ is always singular. For these values of q , the solution $F_q(t)$ is an example of a so called *Cantor function*, a function that is constant almost everywhere.

Motivation

Jessen and Wintner showed that $F_q(t)$ is either absolutely continuous or purely singular. The major question is:

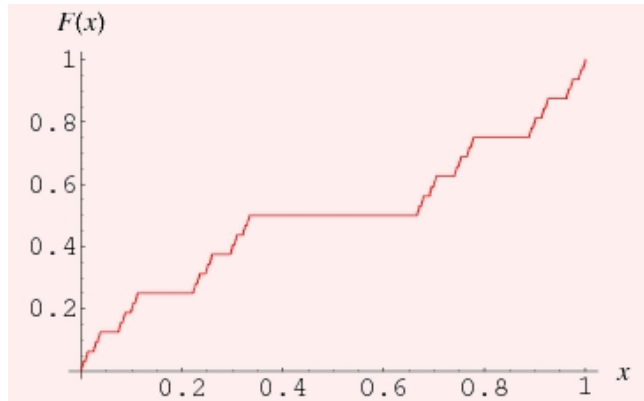
Which values of q make the solution $F_q(t)$ absolutely continuous?

For $q = \frac{1}{2}$, $F_q(t)$ is absolutely continuous.

When $0 < q < \frac{1}{2}$, Kershner and Wintner have shown that $F_q(t)$ is always singular. For these values of q , the solution $F_q(t)$ is an example of a so called *Cantor function*, a function that is constant almost everywhere.

Devil's staircase

From www.mathworld.com, a plot of the Devil's staircase:



Motivation

The case when $q > \frac{1}{2}$ is much harder and more interesting.

In 1939 Erdős showed that if q is of the form $q = \frac{1}{\theta}$ with θ a *Pisot number*, then $F_q(t)$ is again singular.

A *Pisot number* is an algebraic integer greater than 1 in absolute value, whose conjugates are all less than 1 in absolute value.

For example, the golden ratio $\tau = \frac{(1+\sqrt{5})}{2}$ is a Pisot number.

Motivation

The case when $q > \frac{1}{2}$ is much harder and more interesting.

In 1939 Erdős showed that if q is of the form $q = \frac{1}{\theta}$ with θ a *Pisot number*, then $F_q(t)$ is again singular.

A *Pisot number* is an algebraic integer greater than 1 in absolute value, whose conjugates are all less than 1 in absolute value.

For example, the golden ratio $\tau = \frac{(1+\sqrt{5})}{2}$ is a Pisot number.

Motivation

The case when $q > \frac{1}{2}$ is much harder and more interesting.

In 1939 Erdős showed that if q is of the form $q = \frac{1}{\theta}$ with θ a *Pisot number*, then $F_q(t)$ is again singular.

A *Pisot number* is an algebraic integer greater than 1 in absolute value, whose conjugates are all less than 1 in absolute value.

For example, the golden ratio $\tau = \frac{(1+\sqrt{5})}{2}$ is a Pisot number.

Motivation

The case when $q > \frac{1}{2}$ is much harder and more interesting.

In 1939 Erdős showed that if q is of the form $q = \frac{1}{\theta}$ with θ a *Pisot number*, then $F_q(t)$ is again singular.

A *Pisot number* is an algebraic integer greater than 1 in absolute value, whose conjugates are all less than 1 in absolute value.

For example, the golden ratio $\tau = \frac{(1+\sqrt{5})}{2}$ is a Pisot number.

Motivation

In 1995 Solomyak showed that almost every $q > \frac{1}{2}$ yields a solution $F_q(t)$ that is absolutely continuous.

Surprisingly, no actual example of such a q is known.

Specifically, $q = \frac{2}{3}$ is a mystery.

Motivation

In 1995 Solomyak showed that almost every $q > \frac{1}{2}$ yields a solution $F_q(t)$ that is absolutely continuous.

Surprisingly, no actual example of such a q is known.

Specifically, $q = \frac{2}{3}$ is a mystery.

Motivation

In 1995 Solomyak showed that almost every $q > \frac{1}{2}$ yields a solution $F_q(t)$ that is absolutely continuous.

Surprisingly, no actual example of such a q is known.

Specifically, $q = \frac{2}{3}$ is a mystery.

Motivation

If $F_q(t)$ is absolutely continuous, then one may consider its derivative $f_q(t) := F'_q(t)$,

$$f(t) = \frac{1}{2q} f\left(\frac{t-1}{q}\right) + \frac{1}{2q} f\left(\frac{t+1}{q}\right).$$

The existence of an absolutely continuous solution $F_q(t)$ is equivalent to the the existence of an $L^1(I_q)$ solution $f_q(t)$ to $f(t)$.

Motivation

If $F_q(t)$ is absolutely continuous, then one may consider its derivative $f_q(t) := F'_q(t)$,

$$f(t) = \frac{1}{2q} f\left(\frac{t-1}{q}\right) + \frac{1}{2q} f\left(\frac{t+1}{q}\right).$$

The existence of an absolutely continuous solution $F_q(t)$ is equivalent to the the existence of an $L^1(I_q)$ solution $f_q(t)$ to $f(t)$.

Motivation

Motivated by work of Girgensohn in 2007, for $q = \frac{2}{3}$, Calkin shifted the interval $I_q = [-3, 3]$ to $[0, 1]$ for simplicity and considered transform $T : L^1([0, 1]) \rightarrow L^1([0, 1])$ where

$$T : f(x) \mapsto \frac{3}{4}f\left(\frac{3x}{2}\right) + \frac{3}{4}f\left(\frac{3x-1}{2}\right).$$

Start with an arbitrary initial function $f^0(t) \in L^1(I_q)$ and iterate the transform T to gain a sequence of functions $f^0, f^1, f^2, \dots$

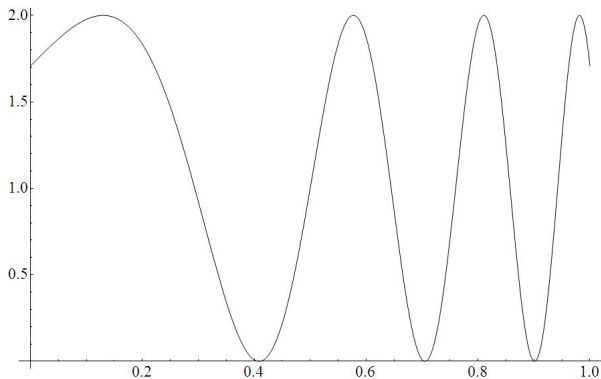
Motivation

Motivated by work of Girgensohn in 2007, for $q = \frac{2}{3}$, Calkin shifted the interval $I_q = [-3, 3]$ to $[0, 1]$ for simplicity and considered transform $T : L^1([0, 1]) \rightarrow L^1([0, 1])$ where

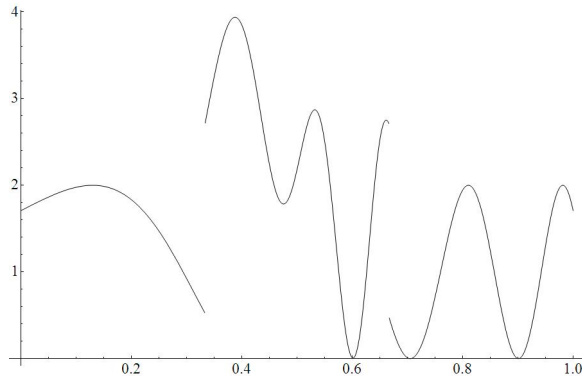
$$T : f(x) \mapsto \frac{3}{4}f\left(\frac{3x}{2}\right) + \frac{3}{4}f\left(\frac{3x-1}{2}\right).$$

Start with an arbitrary initial function $f^0(t) \in L^1(I_q)$ and iterate the transform T to gain a sequence of functions $f^0, f^1, f^2, \dots$

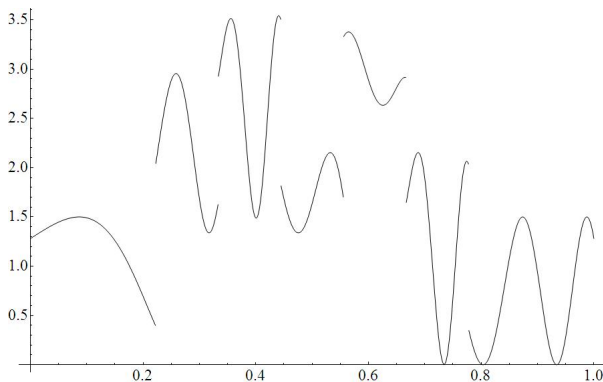
Motivation



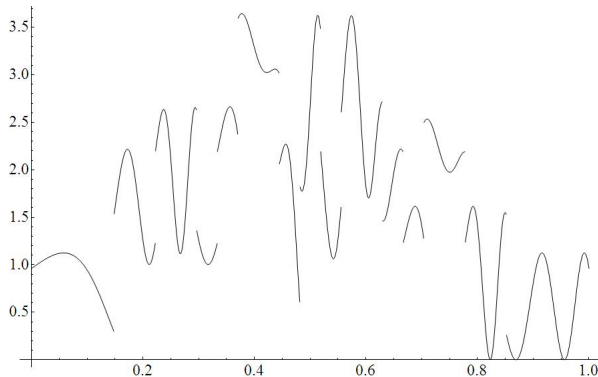
Motivation



Motivation



Motivation



Motivation

Intuitively, this transform takes two scaled copies of $f(x)$: one on the interval $[0, \frac{2}{3}]$ and the other on $[\frac{1}{3}, 1]$, and adds them.

The scaling factor of $\frac{1}{2q} = \frac{3}{4}$ gives us that

$$\int_0^1 f(x) dx = \int_0^1 Tf(x) dx.$$

In this setting, the question to be answered is: starting with the function $f^0(x) = 1$, does the iteration determined by this transform converge to a bounded function?

Motivation

Intuitively, this transform takes two scaled copies of $f(x)$: one on the interval $[0, \frac{2}{3}]$ and the other on $[\frac{1}{3}, 1]$, and adds them.

The scaling factor of $\frac{1}{2q} = \frac{3}{4}$ gives us that

$$\int_0^1 f(x) dx = \int_0^1 Tf(x) dx.$$

In this setting, the question to be answered is: starting with the function $f^0(x) = 1$, does the iteration determined by this transform converge to a bounded function?

Motivation

Intuitively, this transform takes two scaled copies of $f(x)$: one on the interval $[0, \frac{2}{3}]$ and the other on $[\frac{1}{3}, 1]$, and adds them.

The scaling factor of $\frac{1}{2q} = \frac{3}{4}$ gives us that

$$\int_0^1 f(x) dx = \int_0^1 Tf(x) dx.$$

In this setting, the question to be answered is: starting with the function $f^0(x) = 1$, does the iteration determined by this transform converge to a bounded function?

Duplicate, Shift, Add

Instead of viewing T as a transform on $[0, 1]$, we consider a combinatorial analogue.

Consider the two maps $\text{dup}_n, \text{shf}_n : \mathbb{R}^n \longrightarrow \mathbb{R}^{3n}$ defined by

$$\text{dup}_n : (a_1, a_2, \dots, a_{n-1}, a_n) \longmapsto (a_1, a_1, a_2, a_2, \dots, a_{n-1}, a_{n-1}, a_n, \overbrace{0, \dots, 0}^{n \text{ times}})$$

$$\text{shf}_n : (a_1, a_2, \dots, a_{n-1}, a_n) \longmapsto (\overbrace{0, \dots, 0}^{n \text{ times}}, a_1, a_1, a_2, a_2, \dots, a_{n-1}, a_{n-1}, a_n, a_n).$$

Duplicate, Shift, Add

Instead of viewing T as a transform on $[0, 1]$, we consider a combinatorial analogue.

Consider the two maps $\text{dup}_n, \text{shf}_n : \mathbb{R}^n \longrightarrow \mathbb{R}^{3n}$ defined by

$$\text{dup}_n : (a_1, a_2, \dots, a_{n-1}, a_n) \longmapsto (a_1, a_1, a_2, a_2, \dots, a_{n-1}, a_{n-1}, a_n, \overbrace{0, \dots, 0}^{n \text{ times}})$$

$$\text{shf}_n : (a_1, a_2, \dots, a_{n-1}, a_n) \longmapsto (\overbrace{0, \dots, 0}^{n \text{ times}}, a_1, a_1, a_2, a_2, \dots, a_{n-1}, a_{n-1}, a_n, a_n).$$

Duplicate, Shift, Add

Instead of viewing T as a transform on $[0, 1]$, we consider a combinatorial analogue.

Consider the two maps $\text{dup}_n, \text{shf}_n : \mathbb{R}^n \longrightarrow \mathbb{R}^{3n}$ defined by

$$\text{dup}_n : (a_1, a_2, \dots, a_{n-1}, a_n) \longmapsto (a_1, a_1, a_2, a_2, \dots, a_{n-1}, a_{n-1}, a_n, \overbrace{0, \dots, 0}^{n \text{ times}})$$

$$\text{shf}_n : (a_1, a_2, \dots, a_{n-1}, a_n) \longmapsto (\overbrace{0, \dots, 0}^{n \text{ times}}, a_1, a_1, a_2, a_2, \dots, a_{n-1}, a_{n-1}, a_n, a_n).$$

Duplicate, Shift, Add

Instead of viewing T as a transform on $[0, 1]$, we consider a combinatorial analogue.

Consider the two maps $\text{dup}_n, \text{shf}_n : \mathbb{R}^n \longrightarrow \mathbb{R}^{3n}$ defined by

$$\text{dup}_n : (a_1, a_2, \dots, a_{n-1}, a_n) \longmapsto (a_1, a_1, a_2, a_2, \dots, a_{n-1}, a_{n-1}, a_n, \overbrace{0, \dots, 0}^{n \text{ times}})$$

$$\text{shf}_n : (a_1, a_2, \dots, a_{n-1}, a_n) \longmapsto (\overbrace{0, \dots, 0}^{n \text{ times}}, a_1, a_1, a_2, a_2, \dots, a_{n-1}, a_{n-1}, a_n, a_n).$$

Duplicate, Shift, Add

The combinatorial analogue of T on $[0, 1]$ with $f^0(x) = 1$ is provided by the sequences

$$B_0 = (1) \text{ and}$$

$$B_{n+1} = \text{dup}_n(B_n) + \text{shf}_n(B_n).$$

$$\begin{array}{r}
 1 \longrightarrow \begin{array}{ccc} 1 & 1 & \\ & 1 & 1 \\ \hline & 1 & 2 & 1 \end{array} \\
 \\
 \begin{array}{ccccccc} 1 & 2 & 1 & \longrightarrow & 1 & 1 & 2 & 2 & 1 & 1 \\ & & & & & 1 & 1 & 2 & 2 & 1 & 1 \\ \hline & & & & 1 & 1 & 2 & 3 & 2 & 3 & 2 & 1 & 1 \end{array} \\
 \\
 \begin{array}{cc} 1 & 1 & 2 & 3 & 2 & 3 & 2 & 1 & 1 & \longrightarrow & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 \\ & & & & & & & & & & & & & & & & & & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 \\ \hline & & & & & & & & & & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 2 & 3 & 4 & 3 & 4 & 3 & 4 & 4 & 3 & 2 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 \end{array}
 \end{array}$$

A Useful Property

The fact that B_n has a total of 3^n terms follows directly from the definition of dup_n and shf_n .

The average value of B_n , $\mu(B_n) = \left(\frac{4}{3}\right)^n$.

The first few maximum values of B_n , m_n , are
 1, 2, 3, 4, 6, 8, 11, 14, 18, 25, 33, 43, 56, 75, 99, 131, 176, 232, ...

Does m_n also grow like $\left(\frac{4}{3}\right)^n$?

If $m_n = O\left(\left(\frac{4}{3}\right)^n\right)$, then $F_q(t)$ is absolutely continuous at $q = \frac{2}{3}$.

A Useful Property

The fact that B_n has a total of 3^n terms follows directly from the definition of dup_n and shf_n .

The average value of B_n , $\mu(B_n) = \left(\frac{4}{3}\right)^n$.

The first few maximum values of B_n , m_n , are
 1, 2, 3, 4, 6, 8, 11, 14, 18, 25, 33, 43, 56, 75, 99, 131, 176, 232, ...

Does m_n also grow like $\left(\frac{4}{3}\right)^n$?

If $m_n = O\left(\left(\frac{4}{3}\right)^n\right)$, then $F_q(t)$ is absolutely continuous at $q = \frac{2}{3}$.

A Useful Property

The fact that B_n has a total of 3^n terms follows directly from the definition of dup_n and shf_n .

The average value of B_n , $\mu(B_n) = \left(\frac{4}{3}\right)^n$.

The first few maximum values of B_n , m_n , are
 1, 2, 3, 4, 6, 8, 11, 14, 18, 25, 33, 43, 56, 75, 99, 131, 176, 232, ...

Does m_n also grow like $\left(\frac{4}{3}\right)^n$?

If $m_n = O\left(\left(\frac{4}{3}\right)^n\right)$, then $F_q(t)$ is absolutely continuous at $q = \frac{2}{3}$.

A Useful Property

The fact that B_n has a total of 3^n terms follows directly from the definition of dup_n and shf_n .

The average value of B_n , $\mu(B_n) = \left(\frac{4}{3}\right)^n$.

The first few maximum values of B_n , m_n , are
 1, 2, 3, 4, 6, 8, 11, 14, 18, 25, 33, 43, 56, 75, 99, 131, 176, 232, ...

Does m_n also grow like $\left(\frac{4}{3}\right)^n$?

If $m_n = O\left(\left(\frac{4}{3}\right)^n\right)$, then $F_q(t)$ is absolutely continuous at $q = \frac{2}{3}$.

A Useful Property

The fact that B_n has a total of 3^n terms follows directly from the definition of dup_n and shf_n .

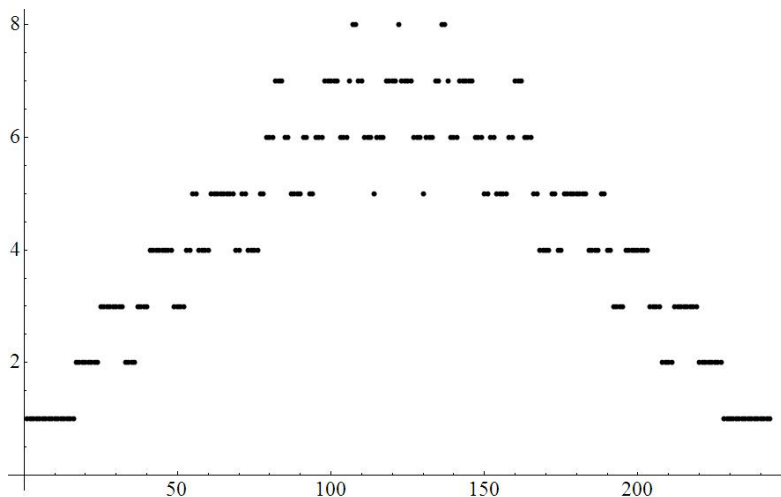
The average value of B_n , $\mu(B_n) = \left(\frac{4}{3}\right)^n$.

The first few maximum values of B_n , m_n , are
 1, 2, 3, 4, 6, 8, 11, 14, 18, 25, 33, 43, 56, 75, 99, 131, 176, 232, ...

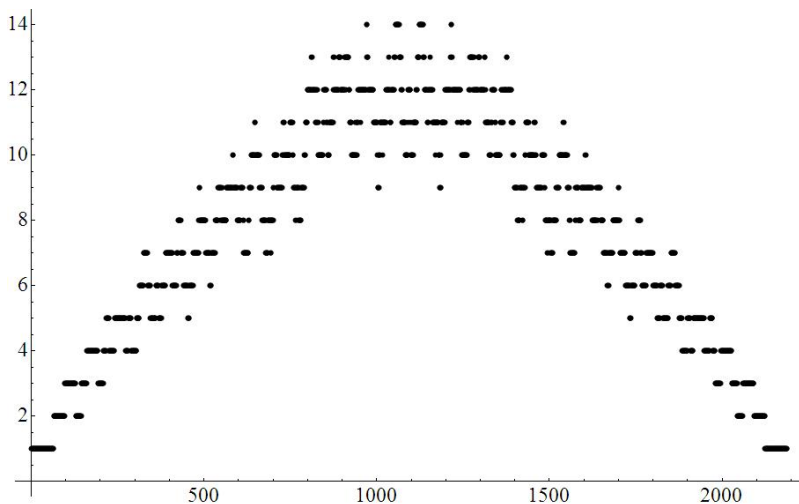
Does m_n also grow like $\left(\frac{4}{3}\right)^n$?

If $m_n = O\left(\left(\frac{4}{3}\right)^n\right)$, then $F_q(t)$ is absolutely continuous at $q = \frac{2}{3}$.

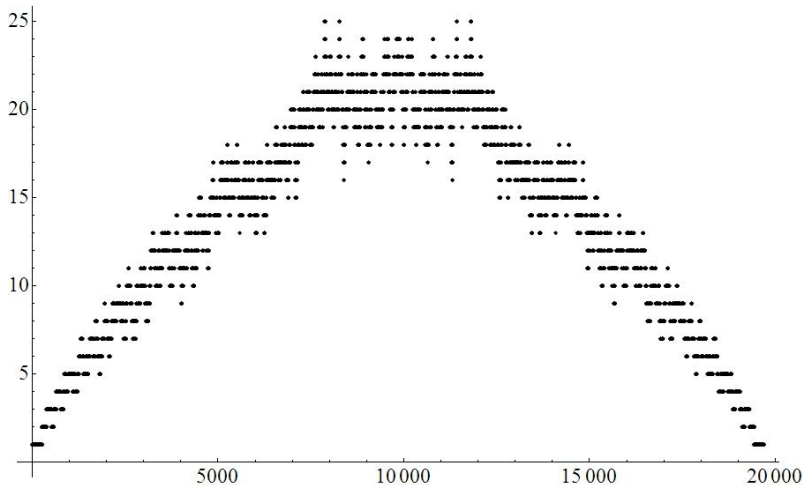
Index versus B_5 entry



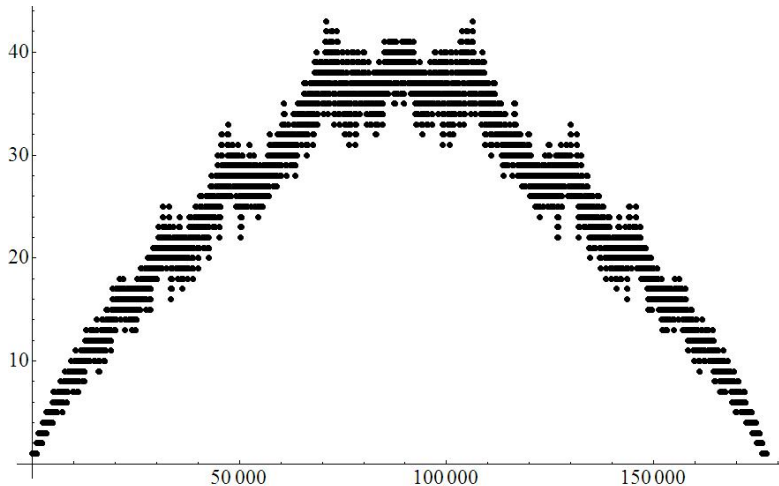
Index versus B_7 entry



Index versus B_q entry



Index versus B_{11} entry



Polynomial Approach

Translating DSA as a Polynomial Recursion

Consider the polynomial $p_n(x) := b_0 + b_1x + \dots + b_tx^t$ where $B_n = (b_0, b_1, \dots, b_t)$ is the Bernoulli sequence on level n where $t = 3^n - 1$.

We see that the duplication $b_0, b_0, b_1, b_1, \dots, b_r, b_r$ corresponds to the polynomial $(1 + x)p_n(x^2)$. Shifting the sequence 3^n places to the right corresponds to multiplication by x^{3^n} .

Thus, for $p_0(x) = 1$ we have the recurrence

$$p_{n+1}(x) = (1 + x)p_n(x^2) (1 + x^{3^n}).$$

Translating DSA as a Polynomial Recursion

Consider the polynomial $p_n(x) := b_0 + b_1x + \dots + b_tx^t$ where $B_n = (b_0, b_1, \dots, b_t)$ is the Bernoulli sequence on level n where $t = 3^n - 1$.

We see that the duplication $b_0, b_0, b_1, b_1, \dots, b_r, b_r$ corresponds to the polynomial $(1 + x)p_n(x^2)$. Shifting the sequence 3^n places to the right corresponds to multiplication by x^{3^n} .

Thus, for $p_0(x) = 1$ we have the recurrence

$$p_{n+1}(x) = (1 + x)p_n(x^2) \left(1 + x^{3^n}\right).$$

Translating DSA as a Polynomial Recursion

Consider the polynomial $p_n(x) := b_0 + b_1x + \dots + b_tx^t$ where $B_n = (b_0, b_1, \dots, b_t)$ is the Bernoulli sequence on level n where $t = 3^n - 1$.

We see that the duplication $b_0, b_0, b_1, b_1, \dots, b_r, b_r$ corresponds to the polynomial $(1 + x)p_n(x^2)$. Shifting the sequence 3^n places to the right corresponds to multiplication by x^{3^n} .

Thus, for $p_0(x) = 1$ we have the recurrence

$$p_{n+1}(x) = (1 + x)p_n(x^2) (1 + x^{3^n}).$$

Translating DSA as a Polynomial Recursion

This formula allows us to explicitly solve for $p_n(x)$.

Theorem

The polynomials $p_n(x)$ satisfy

$$p_n(x) = \prod_{i=0}^{n-1} (1 + x^{2^i}) \prod_{j=0}^{n-1} (1 + x^{2^{n-1}(3/2)^j}).$$

This follows by induction.

Translating DSA as a Polynomial Recursion

This formula allows us to explicitly solve for $p_n(x)$.

Theorem

The polynomials $p_n(x)$ satisfy

$$p_n(x) = \prod_{i=0}^{n-1} \left(1 + x^{2^i}\right) \prod_{j=0}^{n-1} \left(1 + x^{2^{n-1}(3/2)^j}\right).$$

This follows by induction.

A Bound on the Coefficients

By factoring p_n in a clever way, we can put a bound on how fast the coefficients grow with the level n .

Theorem

The maximum values obeys $m_n = O((\sqrt{2})^n)$.

A Bound on the Coefficients

By factoring p_n in a clever way, we can put a bound on how fast the coefficients grow with the level n .

Theorem

The maximum values obeys $m_n = O((\sqrt{2})^n)$.

Clever Factoring

To start, define polynomials q_n, r_n, s_n by

$$q_n(x) = \prod_{i=0}^{n-1} (1 + x^{2^i}) \qquad s_n(x) = \prod_{\substack{1 \leq j \leq n-1 \\ j \text{ odd}}} (1 + x^{2^{n-1}(3/2)^j})$$

$$r_n(x) = \prod_{\substack{1 \leq j \leq n-1 \\ j \text{ even}}} (1 + x^{2^{n-1}(3/2)^j}) = \prod_{j=1}^{\lfloor (n-1)/2 \rfloor} (1 + x^{2^{n-1}(9/4)^j}).$$

Clever Factoring

To start, define polynomials q_n, r_n, s_n by

$$q_n(x) = \prod_{i=0}^{n-1} (1 + x^{2^i}) \quad s_n(x) = \prod_{\substack{1 \leq j \leq n-1 \\ j \text{ odd}}} (1 + x^{2^{n-1}(3/2)^j})$$

$$r_n(x) = \prod_{\substack{1 \leq j \leq n-1 \\ j \text{ even}}} (1 + x^{2^{n-1}(3/2)^j}) = \prod_{j=1}^{\lfloor (n-1)/2 \rfloor} (1 + x^{2^{n-1}(9/4)^j}).$$

Clever Factoring

We see that

$$p_n(x) = q_n(x) \left(1 + x^{2^{n-1}}\right) r_n(x) s_n(x).$$

Consider the polynomial $q_n(x)r_n(x)$. Because $9/4 > 2$, we have distinct powers of x when we expand $q_n(x)r_n(x)$.

In other words, the coefficients are all either 0 or 1. Hence the coefficients of $q_n(x)r_n(x)(1 + x^{2^{n-1}})$ are all either 0, 1, or 2.

Clever Factoring

We see that

$$p_n(x) = q_n(x) \left(1 + x^{2^{n-1}}\right) r_n(x) s_n(x).$$

Consider the polynomial $q_n(x)r_n(x)$. Because $9/4 > 2$, we have distinct powers of x when we expand $q_n(x)r_n(x)$.

In other words, the coefficients are all either 0 or 1. Hence the coefficients of $q_n(x)r_n(x)(1 + x^{2^{n-1}})$ are all either 0, 1, or 2.

Clever Factoring

We see that

$$p_n(x) = q_n(x) \left(1 + x^{2^{n-1}}\right) r_n(x) s_n(x).$$

Consider the polynomial $q_n(x)r_n(x)$. Because $9/4 > 2$, we have distinct powers of x when we expand $q_n(x)r_n(x)$.

In other words, the coefficients are all either 0 or 1. Hence the coefficients of $q_n(x)r_n(x)(1 + x^{2^{n-1}})$ are all either 0, 1, or 2.

Clever Factoring

In particular, the coefficients are bounded. On the other hand, there are at most $n/2$ terms in the product defining $s_n(x)$.

Hence there are at most $2^{n/2}$ nonzero terms in the polynomial $s_n(x)$ since we have 2 choices from each term in the product.

Therefore the coefficients of $p_n(x)$ are all $O(2^{n/2}) = O((\sqrt{2})^n)$.

Clever Factoring

In particular, the coefficients are bounded. On the other hand, there are at most $n/2$ terms in the product defining $s_n(x)$.

Hence there are at most $2^{n/2}$ nonzero terms in the polynomial $s_n(x)$ since we have 2 choices from each term in the product.

Therefore the coefficients of $p_n(x)$ are all $O(2^{n/2}) = O((\sqrt{2})^n)$.

Clever Factoring

In particular, the coefficients are bounded. On the other hand, there are at most $n/2$ terms in the product defining $s_n(x)$.

Hence there are at most $2^{n/2}$ nonzero terms in the polynomial $s_n(x)$ since we have 2 choices from each term in the product.

Therefore the coefficients of $p_n(x)$ are all $O(2^{n/2}) = O((\sqrt{2})^n)$.

Better Bound

Normalize

Seeing that our sequence on level n has length 3^n , we naturally index it by the first 3^n nonnegative integers.

In certain circumstances, it is advantageous to normalize the indexing in such a way that each index is on the interval $[0, 1]$.

To this end, we can simply take the image of $k \in \{0, 1, 2, \dots, 3^n - 1\}$ under the map $k \mapsto k/3^n$.

Normalize

Seeing that our sequence on level n has length 3^n , we naturally index it by the first 3^n nonnegative integers.

In certain circumstances, it is advantageous to normalize the indexing in such a way that each index is on the interval $[0, 1]$.

To this end, we can simply take the image of $k \in \{0, 1, 2, \dots, 3^n - 1\}$ under the map $k \mapsto k/3^n$.

Normalize

Seeing that our sequence on level n has length 3^n , we naturally index it by the first 3^n nonnegative integers.

In certain circumstances, it is advantageous to normalize the indexing in such a way that each index is on the interval $[0, 1]$.

To this end, we can simply take the image of $k \in \{0, 1, 2, \dots, 3^n - 1\}$ under the map $k \mapsto k/3^n$.

Notation

Let $g_n(x)$ denote the n^{th} level Bernoulli sequence where now $x \in [0, 1]$. In other words,

$$g_n\left(\frac{k}{3^n}\right) = b_k \quad \text{for } k = 0, 1, \dots, 3^n - 1.$$

For a subset $S \subset [0, 1]$, we define

$$\Gamma_n(S) = \max_{x \in \bar{S}} g_n(x)$$

where $\bar{S} = S \cap \left\{0, \frac{1}{3^n}, \frac{2}{3^n}, \dots, \frac{3^n-1}{3^n}\right\}$.

Notation

Let $g_n(x)$ denote the n^{th} level Bernoulli sequence where now $x \in [0, 1]$. In other words,

$$g_n\left(\frac{k}{3^n}\right) = b_k \quad \text{for } k = 0, 1, \dots, 3^n - 1.$$

For a subset $S \subset [0, 1]$, we define

$$\Gamma_n(S) = \max_{x \in \bar{S}} g_n(x)$$

where $\bar{S} = S \cap \left\{0, \frac{1}{3^n}, \frac{2}{3^n}, \dots, \frac{3^n-1}{3^n}\right\}$.

An Example

We now walk through an example to demonstrate our algorithm.

Each entry on level n can be written as a sum of entries of previous levels. In this particular example we write each entry on level n as a sum of entries on level $n - 3$.

We break up the interval $[0, 1]$ into subintervals of length $1/81$. Let's see what we get.

An Example

We now walk through an example to demonstrate our algorithm.

Each entry on level n can be written as a sum of entries of previous levels. In this particular example we write each entry on level n as a sum of entries on level $n - 3$.

We break up the interval $[0, 1]$ into subintervals of length $1/81$.
 Let's see what we get.

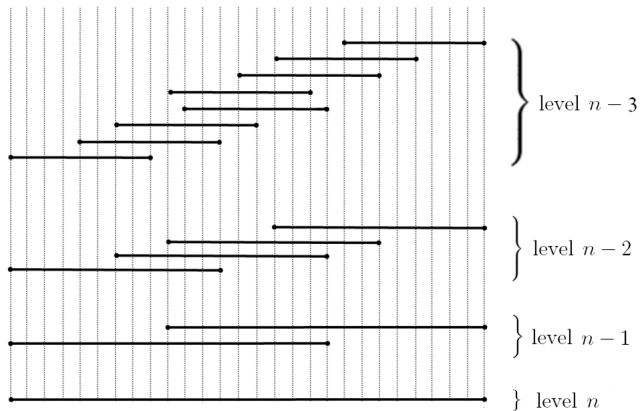
An Example

We now walk through an example to demonstrate our algorithm.

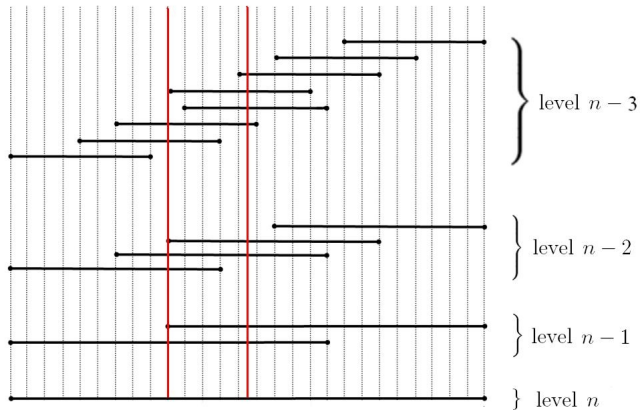
Each entry on level n can be written as a sum of entries of previous levels. In this particular example we write each entry on level n as a sum of entries on level $n - 3$.

We break up the interval $[0, 1]$ into subintervals of length $1/81$. Let's see what we get.

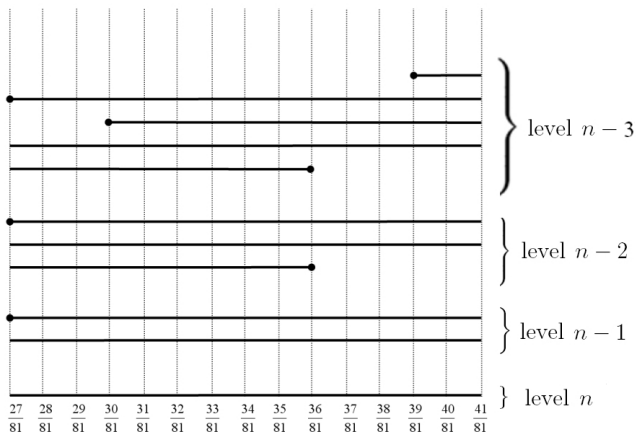
Pullback diagram



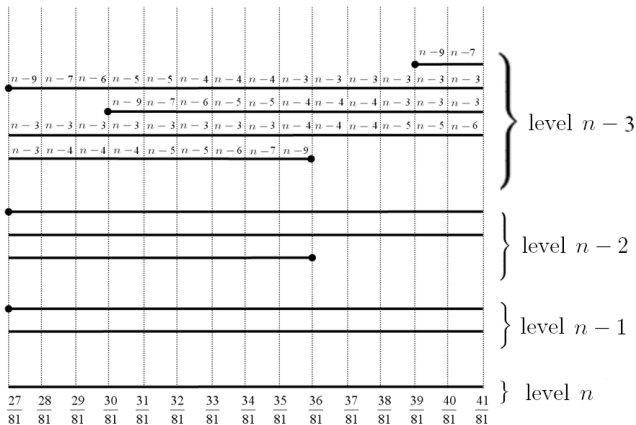
Pullback diagram



Pullback diagram



Pullback diagram



Largest real root

Interval	Polynomial	Largest real root
1	$x^n - 2x^{n-3} - x^{n-9}$	1.301688030...
2	$x^n - x^{n-3} - x^{n-4} - x^{n-7}$	1.288452726...
3	$x^n - x^{n-3} - x^{n-4} - x^{n-6}$	1.304077155...
4	$x^n - x^{n-3} - x^{n-4} - x^{n-5} - x^{n-9}$	1.349240712...
5	$x^n - x^{n-3} - x^{n-7} - 2x^{n-5}$	1.342242489...
6	$x^n - x^{n-3} - x^{n-4} - x^{n-5} - x^{n-6}$	1.380277569...
7	$x^n - x^{n-3} - x^{n-4} - x^{n-5} - x^{n-6}$	1.380277569...
8	$x^n - x^{n-3} - x^{n-4} - x^{n-5} - x^{n-7}$	1.366811194...
9	$x^n - x^{n-3} - 2x^{n-4} - x^{n-9}$	1.375394454...
10	$x^n - x^{n-3} - 2x^{n-4}$	1.353209964...
11	$x^n - x^{n-3} - 2x^{n-4}$	1.353209964...
12	$x^n - 2x^{n-3} - x^{n-5}$	1.363964602...
13	$x^n - 2x^{n-3} - x^{n-5} - x^{n-9}$	1.385877646...
14	$x^n - 2x^{n-3} - x^{n-6} - x^{n-7}$	1.383834352...

1.33997599527 ...

This example gives us the bound $m_n = O((1.385877646 \dots)^n)$.

Continuing this process by pulling back 25 levels for $n = 33$, we see that 1.33997599527 ... is the largest real root of the polynomial

$$X^{33} - 752X^8 - 520X^7 - 319X^6 - 231X^5 - 141X^4 - 101X^3 - 54X^2 - 50X - 83,$$

Therefore $m_n = O((1.33997599527 \dots)^n)$.

1.33997599527 ...

This example gives us the bound $m_n = O((1.385877646 \dots)^n)$.

Continuing this process by pulling back 25 levels for $n = 33$, we see that 1.33997599527 ... is the largest real root of the polynomial

$$X^{33} - 752X^8 - 520X^7 - 319X^6 - 231X^5 - 141X^4 - 101X^3 - 54X^2 - 50X - 83,$$

Therefore $m_n = O((1.33997599527 \dots)^n)$.

Conclusion

Upper Bound

The maximums satisfy $m_n = O((1.33997599527\dots)^n)$.

We conjecture that $m_n = O\left(\left(\frac{4}{3}\right)^n\right)$, and therefore $F_q(t)$ is absolutely continuous at $q = \frac{2}{3}$.

Upper Bound

The maximums satisfy $m_n = O((1.33997599527\dots)^n)$.

We conjecture that $m_n = O\left(\left(\frac{4}{3}\right)^n\right)$, and therefore $F_q(t)$ is absolutely continuous at $q = \frac{2}{3}$.

Upper Bound

The maximums satisfy $m_n = O((1.33997599527\dots)^n)$.

We conjecture that $m_n = O\left(\left(\frac{4}{3}\right)^n\right)$, and therefore $F_q(t)$ is absolutely continuous at $q = \frac{2}{3}$.

Further Questions

- 1 Can our bound improvement algorithm be pushed to further lower the bound given more computational power?
- 2 Is it possible to conclusively prove our conjecture?
- 3 Furthermore, is there an explicit formula to describe m_n for any arbitrary level?
- 4 What else can be said regarding the global behavior of the Bernoulli sequence B_n ?

Further Questions

- 1 Can our bound improvement algorithm be pushed to further lower the bound given more computational power?
- 2 Is it possible to conclusively prove our conjecture?
- 3 Furthermore, is there an explicit formula to describe m_n for any arbitrary level?
- 4 What else can be said regarding the global behavior of the Bernoulli sequence B_n ?

Further Questions

- 1 Can our bound improvement algorithm be pushed to further lower the bound given more computational power?
- 2 Is it possible to conclusively prove our conjecture?
- 3 Furthermore, is there an explicit formula to describe m_n for any arbitrary level?
- 4 What else can be said regarding the global behavior of the Bernoulli sequence B_n ?

Further Questions

- 1 Can our bound improvement algorithm be pushed to further lower the bound given more computational power?
- 2 Is it possible to conclusively prove our conjecture?
- 3 Furthermore, is there an explicit formula to describe m_n for any arbitrary level?
- 4 What else can be said regarding the global behavior of the Bernoulli sequence B_n ?

References I

-  D. H. Bailey, J. M. Borwein, N. J. Calkin, R. Girgensohn, D. R. Luke, and V. H. Moll. *Experimental mathematics in action*. A K Peters Ltd., Wellesley, MA, 2007.
-  N. Calkin, J. Davis, M. Delcourt, Z. Engberg, J. Jacob, and K. James. Discrete Bernoulli convolutions: an algorithmic approach toward bound improvement. *Proc. Amer. Math. Soc.* 139 no. 5, 1579–1584, 2011.
-  P. Erdős. On a family of symmetric Bernoulli convolutions. *Amer. J. Math.*, 61:974–976, 1939.
-  B. Jessen and A. Wintner. Distribution functions and the Riemann zeta function. *Trans. Amer. Math. Soc.*, 38(1):48–88, 1935.

Thank you for listening!