Intersecting Families of Permutations

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1. Definitions

- k-uniform hypergraphs

$\{1,2,3\}$
$\{1,4,5\}$
$\{1,6,7\}$
$\{2,4,6\}$

Definition

$\mathcal{F} \subseteq \binom{[n]}{k}$ *is* ***intersecting*** *if* $\forall F, G \in E(\mathcal{F}),\ |F \cap G| \geq 1$

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*intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is **trivial** if all edges share some vertex*

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2. Stability Results

Maximal intersecting families

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for fixed $i \in [n]$,

$$\left|\left\{F \in \binom{[n]}{k} : i \in F\right\}\right| = \binom{n-1}{k-1}$$

$$\binom{7-1}{3-1} = \binom{6}{2} = 15$$

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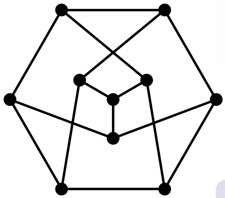
Theorem (Erdős-Ko-Rado 1961)

If $n \geq 2k$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting, then

$$e(\mathcal{F}) \leq \binom{n-1}{k-1}.$$

For $n > 2k$ we have equality only if $\mathcal{F}$ is trivial.



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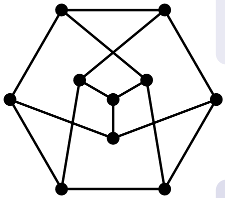
Kneser graph KG(n, k) on $\binom{[n]}{k}$ *has an edge if and only if the corresponding k-sets are disjoint*

independent sets in KG(n, k) correspond to intersecting families from $\binom{[n]}{k}$

Observation (Lovász 1979)

The minimum eigenvalue of KG(n, k) is $-\binom{n-k-1}{k-1}$.



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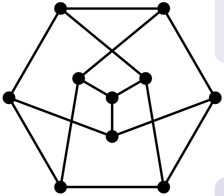

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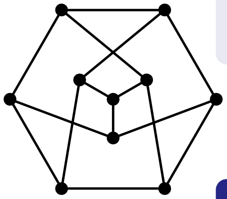

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Theorem (Erdős-Ko-Rado Restated)

For $n \geq 2k$, the independence number of $KG(n,k)$ is less than or equal to $\binom{n-1}{k-1}$.

Theorem (Hoffman's bound)

Let $G = (V,E)$ be a d-regular graph and λ be the smallest eigenvalue. If $I \subseteq V$ is an independent set, then

$$|I| \leq |V| \frac{-\lambda}{d-\lambda}.$$

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Proof.

Note that $KG(n,k)$ is a regular graph on $\binom{n}{k}$ vertices. Each vertex has degree $d = \binom{n-k}{k}$ (need $n \geq 2k$). The minimum eigenvalue of $KG(n,k)$ is $-\binom{n-k-1}{k-1}$. Using Hoffman's bound,

$$\alpha(KG(n,k)) \leq \binom{n}{k}\frac{-\lambda}{d-\lambda} = \binom{n}{k}\frac{\binom{n-k-1}{k-1}}{\binom{n-k}{k}+\binom{n-k-1}{k-1}} = \binom{n-1}{k-1}.$$

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What about non-trivial families?

Theorem (Hilton-Milner 1967)

For $n > 2k$, the largest non-trivial intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ have size

$$\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

3. Typical Structure

What does a "typical" family look like?

N_0 size of the largest trivial intersecting family
N_1 size of the largest non-trivial intersecting family
M upper bound on the number of maximal families

Observation

Any subset of a trivial intersecting family is itself trivial.

there are at least 2^{N_0} trivial families

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then trivial families are **typical**.

Bounding the number of **maximal** families is interesting.

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Set-pairs inequality

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ *is at most*

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

Corollary (BDDLS 2015)

Almost every intersecting family is trivially intersecting.

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We will use the skew-symmetric Bollobás set-pairs inequality

Theorem (Frankl 1982)

Let $A_1, \ldots, A_m$ be sets of size a and $B_1, \ldots, B_m$ be sets of size b such that

$$A_i \cap B_i = \emptyset \text{ and } A_i \cap B_j \neq \emptyset$$

for every $1 \leq i < j \leq m$. Then $m \leq \binom{a+b}{a}$.

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$\mathcal{F} = \mathcal{I}(\mathcal{G})$

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- $\mathcal{F} \subseteq \binom{[n]}{k}$ maximal intersecting
- $\mathcal{G} \subseteq \mathcal{F}$ is a **generating set** if $\mathcal{F} = \mathcal{I}(\mathcal{G})$

Maximal intersecting families

Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2k}{k}} \binom{\binom{n}{k}}{i} \leq \binom{n}{k}^{\frac{1}{2}\binom{2k}{k}}.$$

$\mathcal{F} = \mathcal{I}(\mathcal{G})$ $\mathcal{G}$

$\{1,2,3\}\{1,3,7\}$ $\{1,2,3\}$
$\{1,2,4\}\{1,4,5\}$ $\{1,2,4\}$
$\{1,2,5\}\{1,4,6\}$ $\{1,3,4\}$
$\{1,2,6\}\{1,4,7\}$ $\{1,5,6\}$
$\{1,2,7\}\{1,5,6\}$ $\{1,5,7\}$
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- $\mathcal{F} \subseteq \binom{[n]}{k}$ maximal intersecting
- $\mathcal{G} \subseteq \mathcal{F}$ is a **generating set** if $\mathcal{F} = \mathcal{I}(\mathcal{G})$

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$\mathcal{G}$

$\{1,2,3\}$
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$\mathcal{F}_0$

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$\mathcal{I}(\mathcal{F}_0 \setminus \{F_1\})$	
$\{1,2,3\}\{1,3,7\}$	F_1
$\{1,2,4\}\{1,4,5\}$	$\{1,2,3\}$
$\{1,2,5\}\{1,4,6\}$	$\mathcal{F}_0 \setminus \{F_1\}$
$\{1,2,6\}\{1,4,7\}$	$\{1,2,4\}$
$\{1,2,7\}\{1,5,6\}$	$\{1,3,4\}$
$\{1,3,4\}\{1,5,7\}$	$\{1,5,6\}$
$\{1,3,5\}\{1,6,7\}$	$\{1,5,7\}$
$\{1,3,6\}\{4,5,6\}$	$\{1,6,7\}$
$\{4,5,6\}\{4,6,7\}$	

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- $\forall i \, \exists G_i \in \mathcal{I}(\mathcal{F}_0 \setminus \{F_i\}) \setminus \mathcal{F}$

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	$\{1,2,3\}$
$\{1,2,3\}\{1,3,7\}$	
$\{1,2,4\}\{1,4,5\}$	$\mathcal{F}_0 \setminus \{F_1\}$
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$\{1,2,6\}\{1,4,7\}$	$\{1,3,4\}$
$\{1,2,7\}\{1,5,6\}$	$\{1,5,6\}$
$\{1,3,4\}\{1,5,7\}$	$\{1,5,7\}$
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F_1

$\{1,2,3\}$

$\mathcal{F}_0 \setminus \{F_1\}$

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- $G_i \notin \mathcal{F} = \mathcal{I}(\mathcal{F}_0)$
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- For $1 \leq i \leq s$, $A_i = F_i$ and $B_i = G_i$
- Then $|\mathcal{F}_0| \leq \binom{2k}{k}$

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$\mathcal{F}_0$	$\mathcal{F}_0'$
$\{1,2,3\}$	$\{1,3,5\}$
$\{1,2,4\}$	$\{1,3,7\}$
$\{1,3,4\}$	$\{1,5,7\}$
$\{1,5,6\}$	$\{1,2,4\}$
$\{1,5,7\}$	$\{1,2,6\}$
$\{1,6,7\}$	$\{1,4,6\}$

- $\mathcal{F}_0$ is not necessarily unique
- because $\mathcal{F} = \mathcal{I}(\mathcal{F}_0)$, $\mathcal{F} \mapsto \mathcal{F}_0$ is an injection
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- the number of maximal intersecting hypergraphs is bounded by the number of sets of at most $\binom{2k}{k}$ edges
- being more clever we can get $\frac{1}{2}\binom{2k}{k}$ instead of $\binom{2k}{k}$ edges

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4. Other Settings

- Let S_n denote the **symmetric group** on $[n]$
- $\mathcal{F} \subseteq S_n$ is **intersecting** if $\forall\ \sigma, \pi \in \mathcal{F}$

$$|\sigma \cap \pi| := |\{i \in [n] : \sigma(i) = \pi(i)\}| \geq 1$$

- For example,
  1 2 3 4
  1 3 4 2
  2 3 1 4

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- $\mathcal{F} \subseteq S_n$ is **trivial** if $\exists\, i, j \in [n]$ such that $\pi(i) = j \quad \forall\, \pi \in \mathcal{F}$
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  1 2 4 3
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  1 3 2 4
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  1 4 2 3
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How large can an intersecting family be?

1	2	3	4
1	2	4	3
1	3	2	4
1	3	4	2
1	4	2	3
1	4	3	2

for fixed $i, j \in [n]$,

$$|\{\sigma \in S_n : \sigma(i) = j\}| = (n-1)!$$

$$(4-1)! = 3! = 6$$

Theorem (Frankl–Deza 1977)

If $\mathcal{F} \subseteq S_n$ is intersecting, then $|\mathcal{F}| \leq (n-1)!$.

Theorem (Cameron–Ku / Larose–Malvenuto 2003)

We have equality above only if $\mathcal{F} \subseteq S_n$ is trivial.

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How large can a non-trivial, intersecting family be?

Theorem (Ellis 2008)

For n sufficiently large, the largest non-trivial intersecting $\mathcal{F} \subseteq S_n$ have size

$$\left(1 - \frac{1}{e} + o(1)\right)(n-1)!.$$

$\{\sigma \in S_n : \sigma(1) = 1, \sigma(j) = j \text{ for some } j > 2\} \cup \{(1\ 2)\}$

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$\{\sigma \in S_n : \sigma(1) = 1, \sigma(j) = j \text{ for some } j > 2\} \cup \{(1\ 2)\}$

How large can a non-trivial, intersecting family be?

Theorem (Ellis 2008)

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Theorem (BDDLS 2015)

1. *The number of intersecting families of permutations is*

$$(n^2 + o(1))2^{(n-1)!}.$$

2. *Almost every intersecting family of permutations is trivial.*

Maximal intersecting families

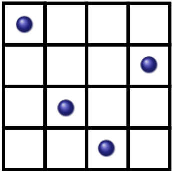
Proposition (BDDLS 2015)

The number of maximal intersecting $\mathcal{F} \subseteq S_n$ is at most

$$\sum_{i=0}^{\frac{1}{2}\binom{2n}{n}} \binom{n!}{i} < n^{n2^{2n-1}}.$$

1 3 4 2

$\{(1,1),(2,3),(3,4),(4,2)\}$



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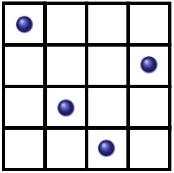

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t-intersecting families

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$\{1,2,5\}$
$\{1,2,6\}$
$\{1,2,7\}$

Definition

$\mathcal{F} \subseteq \binom{[n]}{k}$ *is t-**intersecting** if* $\forall F, G \in E(\mathcal{F}),\ |F \cap G| \geq t$

Definition

t-intersecting $\mathcal{F} \subseteq \binom{[n]}{k}$ *is **trivial** if all edges share some t vertices*

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t-intersecting set-pairs inequality

Theorem (Füredi 1984)

Let $A_1, \dots, A_m$ be sets of size a and $B_1, \dots, B_m$ be sets of size b such that

$$|A_i \cap B_i| < t \text{ and } |A_i \cap B_j| \geq t$$

for every $1 \leq i < j \leq m$. Then $m \leq \binom{a+b-2t+2}{a-t+1}$.

Results

Theorem (BDDLS 2015)

1. *The number of t-intersecting families of $\binom{[n]}{k}$ is*
$$\left(\binom{n}{t} + o(1)\right) 2^{\binom{n-t}{k-t}}.$$
2. *Almost every t-intersecting family is trivial.*

Results

Theorem (BDDLS 2015)

1. *The number of t-intersecting families of S_n is*

$$\left(\binom{n}{t}^2 t! + o(1)\right) 2^{(n-t)!}.$$

2. *Almost every t-intersecting family of permutations is trivial.*

Method of graph containers

Theorem (Kohayakawa–Lee–Rödl–Samotij 2013)

Let G be a graph on N vertices, let R and ℓ be integers, and let $\beta > 0$ be a positive real. Then, provided

$$e^{-\beta\ell} N \leq R, \tag{1}$$

and, for every subset $S \subset V(G)$ of at least R vertices, we have

$$e(S) \geq \beta \binom{|S|}{2}, \tag{2}$$

there is a collection of sets $C_i \subset V(G)$, $1 \leq i \leq \binom{N}{\ell}$, such that $|C_i| \leq R + \ell$ for every i and, for every independent set $I \subset V(G)$, there is some i satisfying $I \subset C_i$.

Method of graph containers

Proposition (BDDLS 2015)

The number of intersecting families of permutations is

$$2^{(1+o(1))(n-1)!}.$$

Theorem (Alon–Chung Expander Mixing Lemma (form in Alon–Balogh–Morris–Samotij 2014))

Let G be a D-regular graph on N vertices with second largest eigenvalue (in absolute value) λ. Then for all $S \subseteq V$,

$$e(G[S]) \geq \frac{D}{2N}|S|^2 + \frac{\lambda}{2N}|S|(N - |S|).$$

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Consider the graph Γ with $V = S_n$ and edges non-intersecting pairs.

Theorem (Ellis 2008)

$\lambda = -(\frac{1}{e} + o(1))(n-1)!$

$N = n!$ and $D = (1 - \frac{1}{e} + o(1))N$.
Pick S with $|S| = (1 + o(1))(n-1)!$.
Because $G[S]$ spans 'many' edges then G does not have 'many' independent sets.

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Thank you for listening!