Taking the "Convoluted" out of Bernoulli Convolutions

A Combinatorial Approach

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Acknowledgments

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Motivation

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This is very confusing—we do not like it.

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$$F(t) = \frac{1}{2}F\left(\frac{t-1}{q}\right) + \frac{1}{2}F\left(\frac{t+1}{q}\right)$$

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It can be shown that there is a unique continuous solution $F_q(t)$ to the above equation.

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When $0 < q < 1/2$, it is known that $F_q(t)$ is always singular. For these values of q, the solution $F_q(t)$ is an example of a *Cantor function*, a function that is constant almost everywhere.

It is also easy to see that for $q = 1/2$, the solution $F_q(t)$ is absolutely continuous.

Devil's staircase

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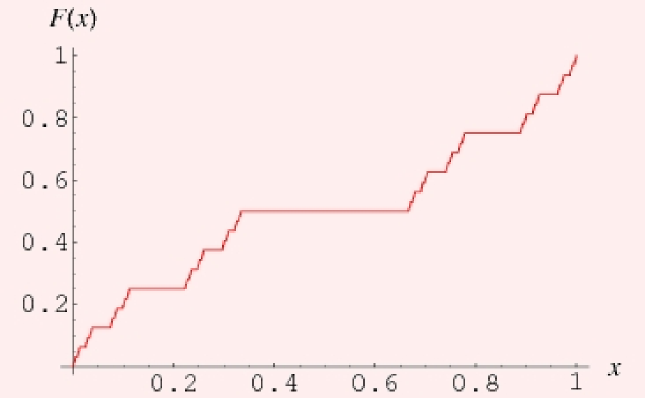

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A *Pisot number* is an algebraic integer greater than 1 in absolute value, whose conjugates are all less than 1 in absolute value.

The classic example is the golden ratio $\tau = (1 + \sqrt{5})/2$. Like all Pisot numbers, τ has the property that large powers of τ approach rational integers.

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Hence it is surprising that no actual example of such a q is known.

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Rather than looking at the function $F_q(t)$, one can also consider its derivative $f_q(t) = F_q'(t)$. Upon differentiating, the functional equation for $F_q(t)$ gives the following equation for $f_q(t)$:

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The question of the existence of an absolutely continuous solution $F_q(t)$ to the previous equation is equivalent to the existence of an $L^1(I_q)$ solution $f_q(t)$ to the above equation.

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to gain a sequence of functions $f_0, f_1, f_2, \ldots$. If this sequence converges, then it converges to the solution of the previous functional equation.

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$T : L^1([0, 1]) \longrightarrow L^1([0, 1])$ where

$$T : f(x) \longmapsto \frac{3}{4} f\left(\frac{3x}{2}\right) + \frac{3}{4} f\left(\frac{3x-1}{2}\right).$$

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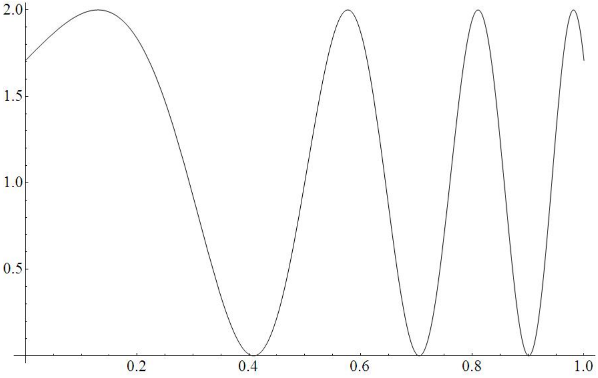
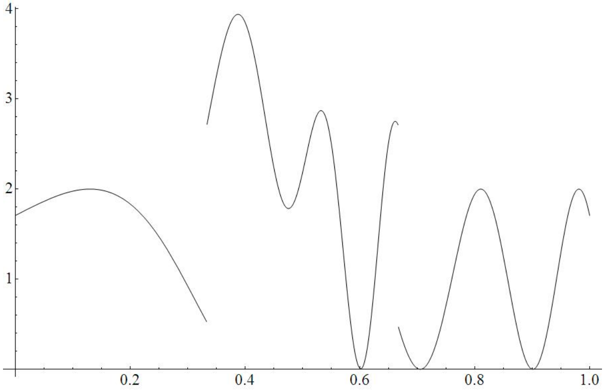
In this setting, the question to be answered is: starting with the function $f_0(x) = 1$, does the iteration determined by this transform converge to a bounded function?

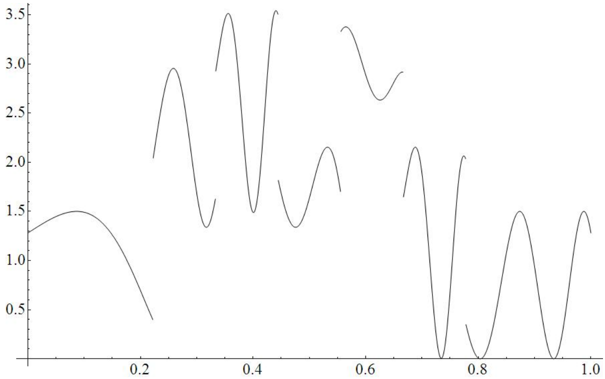
f_0



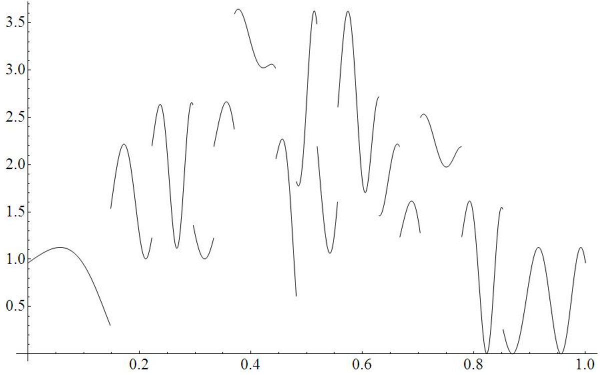
f_1



f_2



f_3



Recursive Algorithms

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Duplicate, Shift, Add

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Looking at this problem through the lens of combinatorics

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$$\mathrm{shf}_n : (a_1, a_2, ..., a_{n-1}, a_n) \longmapsto (\overbrace{0, ..., 0}^{n \text{ times}}, a_1, a_1, a_2, a_2, ..., a_{n-1}, a_{n-1}, a_n, a_n).$$

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$$
\begin{array}{cccccc}
1 & \longrightarrow & 1 & 1 & \\
 & & & 1 & 1 \\
\hline
 & & 1 & 2 & 1
\end{array}
$$

$$
\begin{array}{ccccccccccc}
1 & 2 & 1 & \longrightarrow & 1 & 1 & 2 & 2 & 1 & 1 & & & \\
 & & & & & & & 1 & 1 & 2 & 2 & 1 & 1 \\
\hline
 & & & & 1 & 1 & 2 & 3 & 2 & 3 & 2 & 1 & 1
\end{array}
$$

$$
\begin{array}{cccccccccccccccccccccccccccccccccccccc}
1 & 1 & 2 & 3 & 2 & 3 & 2 & 1 & 1 & \longrightarrow & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & & & & & & & & & \\
 & & & & & & & & & & & & & & & & & & & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 \\
\hline
 & & & & & & & & & & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 2 & 3 & 4 & 4 & 3 & 4 & 3 & 4 & 4 & 3 & 2 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1
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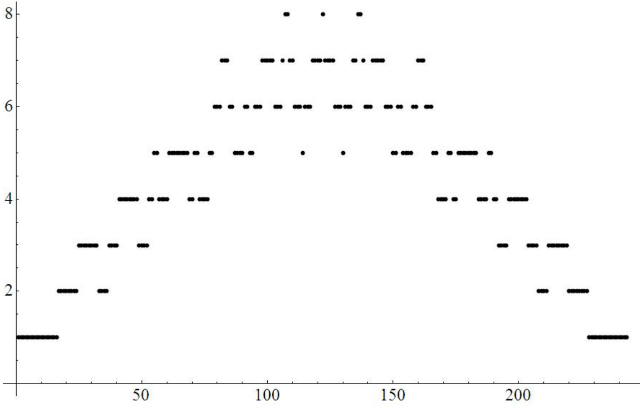
$$\begin{array}{ccccccccccccccccccccccccccccccccccccc} 1 & 1 & 2 & 3 & 2 & 3 & 2 & 1 & 1 & \longrightarrow & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & & & & & & & & & \\ & & & & & & & & & & & & & & & & & & & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 \\ \hline & & & & & & & & & & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 2 & 3 & 4 & 4 & 3 & 4 & 3 & 4 & 4 & 3 & 2 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 \end{array}$$

The first few maximums m_n are $1, 2, 3, 4, 6, 8, 11, \ldots$.

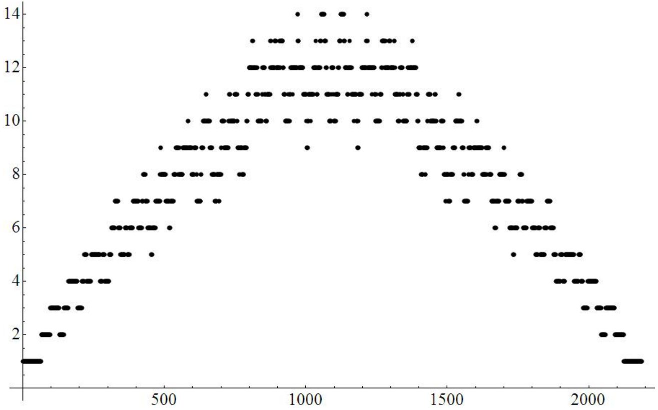
Level $n = 5$

The plot shows the index on the horizontal axis and the B_5 entry on the vertical axis.



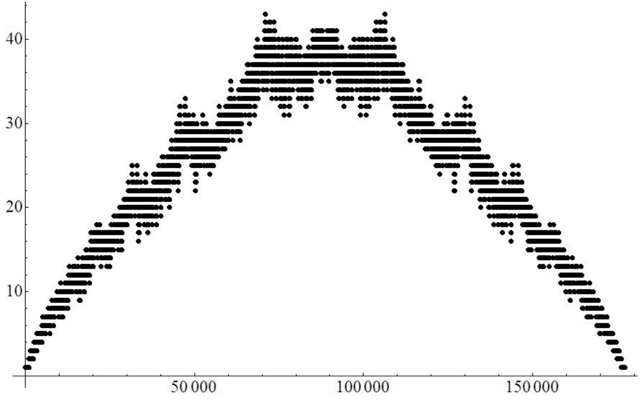
Level $n = 7$

The plot shows the index on the horizontal axis and the B_7 entry on the vertical axis.



Level $n = 11$

The plot shows the index on the horizontal axis and the B_{11} entry on the vertical axis.



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The reason for this is because under the DSA process, the length of a Bernoulli sequence grows by a factor of three while the sum of the terms increases by a factor of four.

Does m_n also grow like $(4/3)^n$?

Double, Enlarge, Merge

Recursive Algorithms Continued

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The process we call *double, enlarge, merge*, abbreviated DEM, is a way of encoding the Bernoulli sequence B_n as a sequence of length $2(2^n - 1)$.

The advantage with DEM is that the sequence grows in size like 2^n as opposed to the 3^n size increase required for the DSA process.

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1, 1, 1, 1, 2, 2, 3, 3, 2, 3, 4, 4, 3, 4, 3, 4, 4, 3, 2, 3, 3, 2, 2, 1, 1, 1, 1

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Given the Bernoulli sequence B_n, the DEM representation is $(d_1, ..., d_r)$ where d_i is the index of the i^{th} jump in B_n up to a sign. Suppose the i^{th} jump occurs at index j in B_n, that is b_j and b_{j+1} are different. Explicitly

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For example, the DEM representation of $B_2 = (1, 1, 2, 3, 2, 3, 2, 1, 1)$ is $(2, 3, -4, 5, -6, -7)$.

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The process of add translates to *merge*: we discard the original elements and concatenate the two new lists attained in the *double* and *enlarge* processes. We then add two additional elements $-2(3^n)$ and 3^n to the list. Finally, we merge sort the elements according to their absolute value.

Data

Using Palmetto...

Data

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Level n	Maximum Value	$m_n(3/4)^n$
0	1	1
1	2	1.5
2	3	1.6875
3	4	1.6875
4	6	1.8984375
5	8	1.8984375
6	11	1.957763672
7	14	1.868774414
8	18	1.802032471
9	25	1.877117157

Data

Level n	Maximum Value	$m_n(3/4)^n$
10	33	1.858345985
11	43	1.816110849
12	56	1.773875713
13	75	1.781794801
14	99	1.763976853
15	131	1.750613395
16	176	1.763976853
17	232	1.743931662
18	309	1.742052425

Data

Level n	Maximum Value	$m_n(3/4)^n$
19	410	1.733595860
20	545	1.728310507
21	728	1.731481719
22	962	1.716022061
23	1283	1.716468012
24	1705	1.710782128
25	2266	1.705263476
26	3024	1.706768563

Translating DSA as a Polynomial Recursion

Polynomial Approach

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Consider the polynomial $p_n(x) := b_0 + b_1 x + ... + b_t x^t$.

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By adding the duplicate and the shift of the sequence, we get the sequence on the next level.

This yields the recurrence relation

$$p_{n+1}(x) = (1 + x)p_n(x^2)\left(1 + x^{3^n}\right).$$

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Theorem

(Calkin-Lennard) The polynomials $p_n(x)$ satisfy

$$p_n(x) = \prod_{i=0}^{n-1} \left(1 + x^{2^i}\right) \prod_{j=0}^{n-1} \left(1 + x^{2^{n-1}(3/2)^j}\right).$$

Proof

We proceed by induction on n. When $n = 1$, we have that

$$1 + 2x + x^2 = (1 + x)(1 + x) = (1 + x^{2^0})(1 + x^{2^0(3/2)^0}).$$

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$$\begin{aligned} p_{n+1}(x) &= (1+x)p_n(x^2)\left(1+x^{3^n}\right) \\ &= (1+x)\prod_{i=0}^{n-1}\left(1+(x^2)^{2^i}\right)\prod_{j=0}^{n-1}\left(1+(x^2)^{2^{n-1}(3/2)^j}\right)\left(1+x^{3^n}\right) \end{aligned}$$

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A Bound on the Coefficients

By factoring p_n in a clever way, we can put a bound on how fast the coefficients grow with the level n.

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Theorem

(Calkin-Lennard) The maximum values satisfy $m_n = O((\sqrt{2})^n)$.

Proof

To start, define polynomials q_n, r_n, s_n by

$$q_n(x) = \prod_{i=0}^{n-1}\left(1 + x^{2^i}\right) \qquad s_n(x) = \prod_{\substack{1 \le j \le n-1 \\ j \text{ odd}}}\left(1 + x^{2^{n-1}(3/2)^j}\right)$$

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$$r_n(x) = \prod_{\substack{1\le j\le n-1 \\ j \text{ even}}}\left(1 + x^{2^{n-1}(3/2)^j}\right) = \prod_{j=1}^{\lfloor (n-1)/2 \rfloor}\left(1 + x^{2^{n-1}(9/4)^j}\right).$$

Proof

We see that

$$p_n(x) = q_n(x)\left(1 + x^{2^{n-1}}\right) r_n(x) s_n(x).$$

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In other words, the coefficients are all either 0 or 1. Hence the coefficients of $q_n(x)r_n(x)(1 + x^{2^{n-1}})$ are all either 0, 1, or 2.

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Hence there are at most $2^{n/2}$ nonzero terms in the polynomial $s_n(x)$ since we have 2 choices from each term in the product.

Therefore the coefficients of $p_n(x)$ are all $O(2^{n/2}) = O((\sqrt{2})^n)$.

PIP

Polynomial Isolated Point

An Algorithmic Implementation: PIP

The fact that our sequence can be realized as the coefficients of an explicitly defined polynomial provides us with an algorithm for computing isolated points on high levels.

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The algorithms DSA and DEM are useful for computing entire levels, but this becomes impossible for large n due to the recursive nature of the algorithm.

The following algorithm, which we call PIP, allows us to compute with much larger n values. The algorithm is nonrecursive—we need no previous levels to compute entries on B_n. The name PIP stands for *polynomial isolated point* because this algorithm computes the entry corresponding to a given index and given level number.

An Algorithmic Implementation: PIP

Our algorithm is based on the following idea. Suppose $S = \{a_1, ..., a_n\}$ is a sequence of positive integers. Consider the polynomial

$$f(x) = \prod_{i=1}^{n}(1 + x^{a_i}) = \sum_{j=1}^{m} \alpha_j x^j.$$

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This idea is applicable to the coefficients of our polynomial because our polynomial is a product of terms of the form $(1 + x^a)$.

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Hence we get that the coefficient b_j of x^j in $p_n(x)$ (which is the j^{th} entry on the n^{th} Bernoulli sequence) is precisely the number of ways that j can be written as a sum of distinct terms in the sequence

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$$S = \{1, 2, 4, ..., 2^{n-2}, 2^{n-1}, 2^{n-1}, 2^{n-2}3, 2^{n-3}3^2, ..., 2^2 3^{n-3}, 2^1 3^{n-2}, 3^{n-1}\}.$$

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- Let $S = \{a_1, ..., a_n\}$ where each $a_i > 0$. Let $\mathrm{N}_S(k)$ denote the number of ways to write k as a sum of elements from S. Then for any $i \in \{1, ..., n\}$, the following holds

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$$\mathrm{N}_S(k) = \mathrm{N}_{S\setminus\{a_i\}}(k) + \mathrm{N}_{S\setminus\{a_i\}}(k - a_i).$$

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$$\mathrm{N}_S(k) = \mathrm{N}_{S\setminus\{a_i\}}(k) + \mathrm{N}_{S\setminus\{a_i\}}(k - a_i).$$

- If $k > \sum_{s \in S} s$, then $\mathrm{N}_S(k) = 0$.

An Algorithmic Implementation: PIP

- If $k < 0$, then $\mathrm{N}_S(k) = 0$.

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- If $k < 0$, then $\mathrm{N}_S(k) = 0$.
- We see that if $0 < k < 2^{n-1}$, then $\mathrm{N}_S(k) = \mathrm{N}_{S'}(k)$ where $S' = \{1, 2, 4, ..., 2^{n-2}\}$ since all other elements of S are too large. However, every k with $0 < k < 2^{n-1}$ can be written uniquely as a sum from elements of S'; this is simply the binary expansion of k.

Flowchart

The following flowchart shows an explicit example of our implementation of the PIP algorithm in use.

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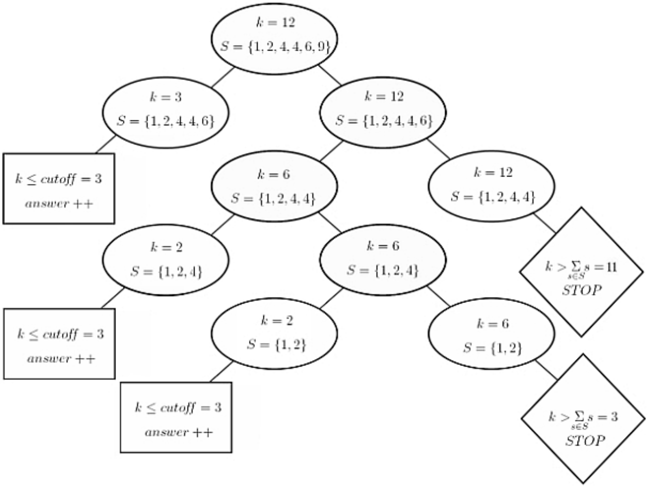
The following flowchart shows an explicit example of our implementation of the PIP algorithm in use.

It shows the computation of $\mathrm{N}_S(k)$ for $k = 12$ and $S = \{1, 2, 4, 4, 6, 9\}$. As the diagram suggests, $\mathrm{N}_S(k) = 3$, corresponding to the fact that there are three boxes containing the word *answer* $++$

Flowchart


$k = 12$
$S = \{1, 2, 4, 4, 6, 9\}$
$k = 3$
$S = \{1, 2, 4, 4, 6\}$
$k = 12$
$S = \{1, 2, 4, 4, 6\}$
$k \leq cutoff = 3$
$answer++$
$k = 6$
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$k = 12$
$S = \{1, 2, 4, 4\}$
$k = 2$
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$k = 6$
$S = \{1, 2, 4\}$
$k > \sum_{s \in S} s = 11$
$STOP$
$k \leq cutoff = 3$
$answer++$
$k = 2$
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$k = 6$
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$answer++$
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$STOP$

Data

In this table, we consider various values for α appearing in the top row. Then we compute the k^{th} entry b_k of the Bernoulli sequence B_n for $k = \lceil \alpha(3^n - 1) \rceil$. Finally, we compute $b_k(3/4)^n$ for $n = 1, 2, ..., 40$. The data suggests this quantity converges as n grows large.

Level	0.38	0.4	0.42	0.44	0.46	0.48	0.5
1	1.500000	1.500000	1.500000	1.500000	1.500000	1.500000	1.500000
2	1.687500	1.687500	1.687500	1.687500	1.125000	1.125000	1.125000
3	1.687500	1.687500	1.687500	1.687500	1.265625	1.265625	1.687500
4	1.582031	1.582031	1.265625	1.582031	1.265625	1.582031	1.898438
5	1.186523	1.661133	1.423828	1.898438	1.423828	1.423828	1.898438
6	1.423828	1.779785	1.423828	1.779785	1.423828	1.423828	1.779785
7	1.334839	1.735291	1.601807	1.601807	1.334839	1.601807	1.601807
8	1.301468	1.802032	1.701920	1.501694	1.401581	1.501694	1.802032
9	1.351524	1.877117	1.802032	1.501694	1.351524	1.576778	1.802032
10	1.407838	1.858346	1.689405	1.464151	1.520465	1.520465	1.576778
11	1.435995	1.816111	1.562700	1.393759	1.520465	1.647170	1.689405
12	1.393759	1.742199	1.615494	1.488789	1.520465	1.647170	1.710523
13	1.401679	1.710523	1.591737	1.472950	1.520465	1.591737	1.520465
14	1.425436	1.674887	1.567979	1.496708	1.478890	1.621433	1.639251
15	1.469981	1.643706	1.576888	1.496708	1.496708	1.630342	1.630342
16	1.433231	1.603615	1.573548	1.533457	1.493367	1.583570	1.583570
17	1.450771	1.578559	1.563525	1.525940	1.510906	1.563525	1.623661

Data

Level	0.38	0.4	0.42	0.44	0.46	0.48	0.5
18	1.460167	1.578559	1.544733	1.516544	1.482718	1.572921	1.668762
19	1.429160	1.594063	1.530638	1.530638	1.496812	1.577149	1.640574
20	1.436559	1.585606	1.525353	1.534867	1.474614	1.569750	1.617318
21	1.446073	1.591156	1.534074	1.541209	1.479370	1.560236	1.584020
22	1.455586	1.589372	1.539425	1.539425	1.478776	1.566182	1.641102
23	1.446221	1.581345	1.547898	1.547898	1.485019	1.577331	1.650913
24	1.453914	1.585358	1.551243	1.535189	1.483012	1.578334	1.609440
25	1.455419	1.592382	1.563785	1.537446	1.478748	1.576579	1.610443
26	1.448270	1.603482	1.563409	1.533495	1.479877	1.581470	1.609690
27	1.448129	1.609832	1.559881	1.526864	1.478607	1.577237	1.601789
28	1.444848	1.615017	1.556601	1.532790	1.485486	1.579142	1.612160
29	1.448896	1.616049	1.554379	1.529853	1.476517	1.576046	1.618192
30	1.451872	1.616346	1.554914	1.527055	1.471516	1.577594	1.610810
31	1.449729	1.617150	1.550718	1.531565	1.470757	1.577639	1.608846
32	1.447017	1.616983	1.552593	1.536219	1.473034	1.575094	1.610252
33	1.447419	1.616631	1.551990	1.531498	1.474767	1.572030	1.613467
34	1.447589	1.615463	1.549579	1.532515	1.474315	1.575458	1.615463
35	1.449637	1.616141	1.549989	1.529139	1.473750	1.574780	1.617582
36	1.449245	1.617667	1.550127	1.530548	1.474640	1.574123	1.616396
37	1.450556	1.617897	1.552057	1.530150	1.474251	1.574417	1.617445
38	1.450008	1.619012	1.551682	1.529727	1.473464	1.574763	1.617743
39	1.450535	1.619714	1.550310	1.529070	1.473370	1.575921	1.617233
40	1.450019	1.620015	1.550705	1.529959	1.475150	1.575666	1.616817

Better Bound

Normalize

Seeing that our sequence on level n has length 3^n, we naturally index it by the first 3^n nonnegative integers.

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In certain circumstances, it is advantageous to normalize the indexing in such a way that each index is on the interval $[0, 1]$.

To this end, we can simply take the image of $k \in \{0, 1, 2, ..., 3^n - 1\}$ under the map $k \mapsto k/3^n$.

Notation

To emphasize our new indexing scheme, let $g_n(x)$ denotes the n^{th} level Bernoulli sequence where now $x \in [0,1]$. In other words,

$$g_n\left(\frac{k}{3^n}\right) = b_k \qquad \text{for } k = 0, 1, \ldots 3^n - 1.$$

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$$g_n\left(\frac{k}{3^n}\right) = b_k \qquad \text{for } k = 0, 1, ...3^n - 1.$$

For a subset $S \subset [0,1]$, we define

$$\Gamma_n(S) = \max_{x \in S} g_n(x)$$

An in depth example

We now walk through an in depth example to demonstrate our algorithm.

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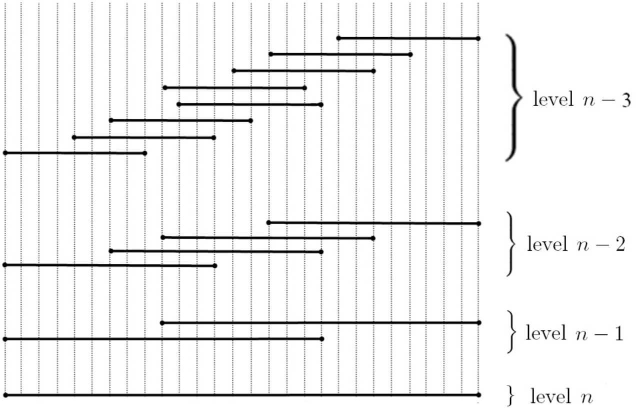
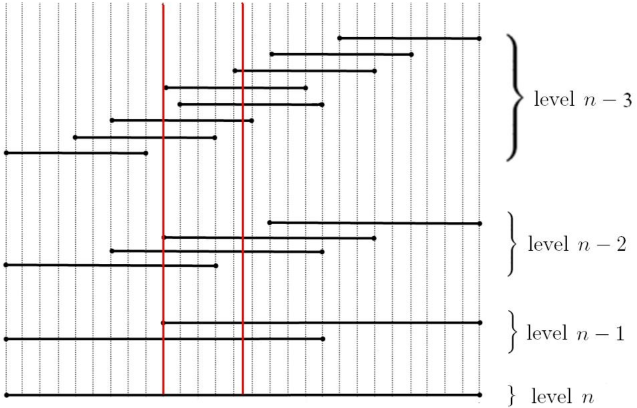
We break up the interval $[0, 1]$ into subintervals of length $1/81$. Let's see what we get.

Pullback diagram



Pullback diagram



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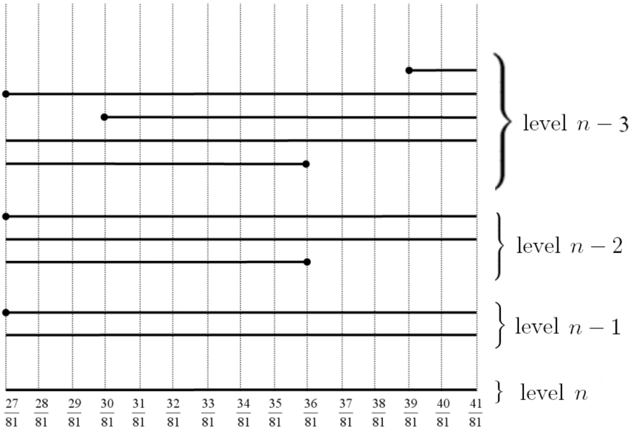

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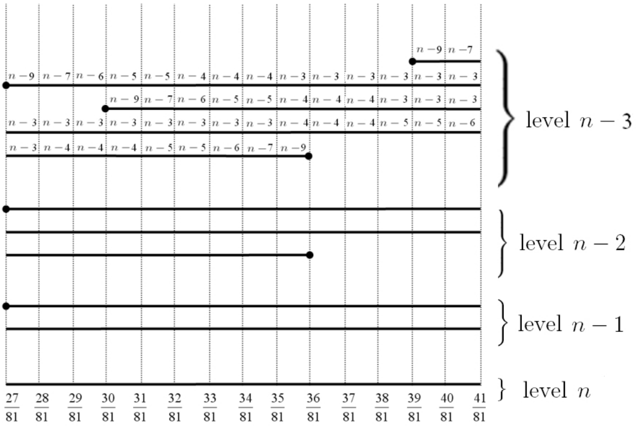

Largest real root

Interval	Polynomial	Largest real root
1	$x^n - 2x^{n-3} - x^{n-9}$	1.301688030
2	$x^n - x^{n-3} - x^{n-4} - x^{n-7}$	1.288452726
3	$x^n - x^{n-3} - x^{n-4} - x^{n-6}$	1.304077155
4	$x^n - x^{n-3} - x^{n-4} - x^{n-5} - x^{n-9}$	1.349240712
5	$x^n - x^{n-3} - x^{n-7} - 2x^{n-5}$	1.342242489
6	$x^n - x^{n-3} - x^{n-4} - x^{n-5} - x^{n-6}$	1.380277569
7	$x^n - x^{n-3} - x^{n-4} - x^{n-5} - x^{n-6}$	1.380277569
8	$x^n - x^{n-3} - x^{n-4} - x^{n-5} - x^{n-7}$	1.366811194
9	$x^n - x^{n-3} - 2x^{n-4} - x^{n-9}$	1.375394454
10	$x^n - x^{n-3} - 2x^{n-4}$	1.353209964
11	$x^n - x^{n-3} - 2x^{n-4}$	1.353209964
12	$x^n - 2x^{n-3} - x^{n-5}$	1.363964602
13	$x^n - 2x^{n-3} - x^{n-5} - x^{n-9}$	1.385877646
14	$x^n - 2x^{n-3} - x^{n-6} - x^{n-7}$	1.383834352

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The actual proof uses induction on n. The details are rather involved......if anyone is interested we would be glad to go through the proof after the talk.

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$$X^{33} - 752X^8 - 520X^7 - 319X^6 - 231x^5 - 141X^4 - 101X^3 - 54X^2 - 50X - 83$$

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Studying Bernoulli convolutions through the lens of combinatorics sheds much new insight on the subject.

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Many of our algorithms would not have been discovered without combinatorial thinking (for example the PIP algorithm). The combinatorial point of view is a very simple way to think about Bernoulli convolutions (the *duplicate, shift, add* method could be explained to a small child), but a computer has trouble computing more than a handful of Bernoulli sequences.

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In particular, studying Bernoulli convolutions via combinatorics has led to the discovery and development of three elegant algorithms (DEM, PIP, and the bound improvement algorithm).

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We conjecture $m_n = O((4/3)^n)$. Using our three algorithms, we have sufficient data to support this claim.

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However, we have provided partial answers for these questions through our algorithms and data.

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1.41421356237

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1.33997599527

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1.41421356237

1.33997599527

1.33333333333

Thank you for listening!