GRADUATE STUDENT NUMBER THEORY SEMINAR

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Abstract. This talk is based primarily on the paper *Interlacing Families I: Bipartite Ramanujan Graphs of All Degrees* by Marcus, Spielman, and Srivastava [6]. By proving a variant of a conjecture of Bilu and Linial [1], they show that there exist infinite families of d-regular bipartite Ramanujan graphs for $d \geq 3$. Of particular interest in the paper by Marcus, Spielman, and Srivastava is their "method of interlacing polynomials" [6].

1. Basic Definitions

Think of the adjacency matrix of a graph G as an adjacency operator. Then an eigenvalue of the matrix is an eigenvalue of the operator. If G is d-regular, then the largest eigenvalue is d. If G is connected, then the next eigenvalue is smaller than d.

Definition 1. For a d-regular graph G with eigenvalues $d = \lambda_1 \geq \lambda_2 \geq \dots \lambda_n$, if

$$|\lambda_2|, \dots, |\lambda_n| \leq 2\sqrt{d-1},$$

then G is said to be **Ramanujan**.

If G is bipartite, then the spectrum is symmetric about the origin; therefore if G is a d-regular bipartite graph, then the smallest eigenvalue is $-d$.

Definition 2. For a d-regular bipartite graph G with eigenvalues

$$d = \lambda_1 \geq \lambda_2 \geq \dots \lambda_n = -d,$$

if $|\lambda_2|, \dots, |\lambda_{n-1}| \leq 2\sqrt{d-1}$, then G is said to be **bipartite Ramanujan**.

A simple example of a d-regular Ramanujan graph is K_{d+1}, the complete graph. A simple example of a d-regular bipartite Ramanujan graph is $K_{d,d}$, the complete bipartite graph. The value $2\sqrt{d-1}$ arises naturally.

Theorem 1 (Alon-Boppana)**.** *For all fixed $\varepsilon > 0$, there exists $N > 0$ s.t. for all graphs on $n > N$ vertices, we have $\lambda_2 > 2\sqrt{d-1} - \varepsilon$.*

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Definition 3. If G is a graph on n vertices with maximum degree d and for every $W \subset V$ with $|W| \leq \frac{n}{2}$ the inequality $|N(W)| \geq \lambda|W|$ holds (where $N(W)$ is the neighborhood of W in $V \backslash W$), then G is a (n, d, λ)**-expander**.

Simply speaking, expander graphs are sparse yet highly connected d-regular graphs. Because of these nice properties, expander graphs have many applications in engineering and computer science from network design to cryptography. A large spectral gap indicates high connectivity. Ramanujan graphs are "good quality" expanders.

Theorem 2. *For the following values of d, there exist infinite families of d-regular Ramanujan graphs:*

(1) $d = p + 1, p$ *an odd prime (1988 Lubotzky-Phillips-Sarnak* [5] *and 1988 Margulis* [8]*)*
(2) $d = 2 + 1 = 3$ *(1992 Chiu* [2]*),*
(3) $d = q + 1, q$ *a prime power(1994 Morgenstern* [9]*).*

Theorem 3 (Marcus, Spielman, and Srivastava[6])**.** *For all $d \geq 3$, there are infinitely many d-regular, bipartite Ramanujan graphs.*

While the construction of Ramanujan graphs is fairly simple, proving they have the desired properties is not and relies heavily on group theory, modular forms, and even the Riemann Hypothesis for curves over finite fields. The name "Ramanujan" comes from the constructions' dependence on Ramanujan's conjecture (solved by Eichler in 1954) concerning coefficients of modular forms with weight 2. Eichler related the eigenvalues of Hecke operators T_m acting on spaces of cusp forms to the zeros of zeta functions of modular curves over the fields $\mathbb{F}_p$. For m prime, varying the space on which the Hecke operators T_m act, we obtain a large family of Ramanujan graphs. For T_m, m not prime, we are able to associate an "almost Ramanujan" graph.[10].

Curiously some applications actually require bipartite or non-bipartite Ramanujan graphs. Explicit constructions of the error correcting codes of Sipser and Spielman actually require non-bipartite expanders [11] whereas improvements of this construction require bipartite Ramanujan expanders [12].

2. 2-Lifting

In 2006 Bilu and Linial suggested constructing Ramanujan graphs by iteratively applying 2-lifts to a base graph[1]. Given a graph $G = (V, E)$, we construct a 2-lift of G, say $\widetilde{G} = (\widetilde{V}, \widetilde{E})$, in the following manner. We can think of $\widetilde{G}$ as being a covering space of G. Let vertex set of $\widetilde{G}$ be a multiset containing two copies of V, say $\widetilde{V} = V_0 \cup V_1$. Each pair of vertices in $\widetilde{V}$ is the **fiber** over the original vertex in V, and each edge in E corresponds to two edges in $\widetilde{E}$. For each edge $uv \in E$, if $\{u_0, u_1\}$ is the fiber of u and $\{v_0, v_1\}$ is the fiber of v, then $\widetilde{G}$ contains either the pair of edges u_0v_0 and u_1v_1 or the pair of edges u_0v_1 and u_1v_0. Note that if only edges of the first type appear, then $\widetilde{G}$ consists of two copies of G. If only edges of the second type appear, then $\widetilde{G}$ is the **double cover** of G. Also, a 2-lift of a bipartite graph by definition is also bipartite.

Bilu and Linial also introduce the notion of signings of E to calculate the eigenvalues of $\widetilde{G}$. Let $s : E \to \{\pm 1\}$ with $s(uv) = 1$ if the fiber of uv is a pair of the first type and $s(uv) = -1$ if the fiber of uv is a pair of the second type.

Definition 4. Let G be a graph with adjacency matrix A. The entries of the **signed adjacency matrix** A_s associated with a 2-lift $\widetilde{G}$ are

$$(A_s)_{uv} = \begin{cases} s(uv), & \text{if } uv \in E \\ 0, & \text{otherwise.} \end{cases}$$

The eigenvalues of $\widetilde{G}$ are related to A and A_s. We think of the eigenvalues of A as "old" eigenvalues and the eigenvalues of A_s as "new" eigenvalues of $\widetilde{G}$.

Lemma 1. *Let A be the adjacency matrix of a graph G on n vertices, and A_s the signed adjacency matrix associated with a 2-lift $\widetilde{G}$. Then every eigenvalue of A and every eigenvalue of A_s are eigenvalues of $\widetilde{G}$. Furthermore, the multiplicity of each eigenvalue of $\widetilde{G}$ is the sum of the multiplicities in A and A_s.*

Proof. The adjacency matrix of $\widetilde{G}$ is

$$\widetilde{A} = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}$$

where A_1 is the adjacency matrix of $(V, s^{-1}(1))$ and A_2 is the adjacency matrix of $(V, s^{-1}(-1))$. Note that $A = A_1 + A_2$ and $A_s = A_1 - A_2$. If v is an eigenvector of A with eigenvalue μ, then $\widehat{v} = (v\ v)$ is an eigenvector of $\widetilde{A}$ with eigenvalue μ. If u is an eigenvector of A_s with eigenvalue λ, then $\widehat{u} = (u - u)$ is an eigenvector of $\widetilde{A}$ with eigenvalue λ. Because the $\widehat{u}$'s and $\widehat{v}$'s are perpendicular and there are $2n$ of them, they span all eigenvectors of $\widetilde{A}$. □

Example 1. Consider the weighted graph and the corresponding 2-lift in Figure 1 below. Then

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad A_s = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \text{ and}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Also

$$\widetilde{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

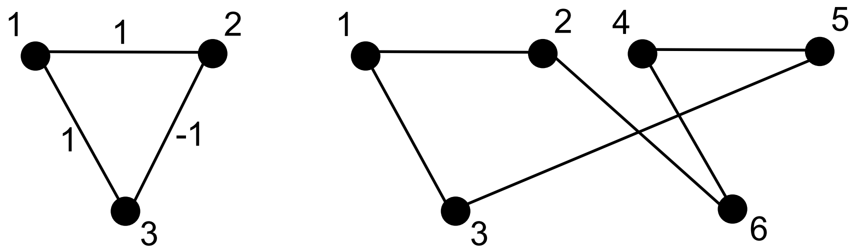

FIGURE 1.

Bilu and Linial suggest the following method to attempt to construct (n, d, λ)-expanders. Let $G_0 = K_{d+1}$, the d-regular complete graph. Note that the eigenvalues of G_0 are d with multiplicity 1 and -1 with multiplicity d. Let G_i be a 2-lift of G_{i-1} with all new eigenvalues in the range $[-\lambda, \lambda]$. If such 2-lifts always exist, then $\{G_i\}_{i=0}^{\infty}$ is an infinite family of (n, d, λ)-expanders.

We care about the smallest $\lambda = \lambda(d)$ such that every graph with degree at most d has a 2-lift with new eigenvalues in the desired range, $[-\lambda, \lambda]$. In other words, we want a signing with spectral radius $\leq \lambda$. Note that $\lambda(d) \geq 2\sqrt{d-1}$, otherwise we would be able to construct graphs which violate Alon-Boppana.

Bilu and Linial conjecture that every d-regular graph has a signing in which all new eigenvalues have absolute value at most $2\sqrt{d-1}$. By iteratively applying 2-lifts to K_{d+1} we would obtain an infinite sequence of d-regular Ramanujan graphs. Marcus, Spielman, and Srivastava show the weaker statement that every d-regular graph has a signing in which all new eigenvalues are at most $2\sqrt{d-1}$. In other words, they do not control the smallest new eigenvalue. Because a 2-lift of a bipartite graph is bipartite, by iteratively applying 2-lifts to $K_{d,d}$, the d-regular complete bipartite graph, we obtain an infinite sequence of d-regular bipartite Ramanujan graphs.

Interestingly the double cover of a d-regular non-bipartite Ramanujan graph is a d-regular bipartite Ramanujan graph; we see that constructing bipartite Ramanujan graphs is at least as easy as constructing non-bipartite Ramanujan graphs. In applications of expanders requiring only upper bounds on the second eigenvalue, we may use bipartite Ramanujan graphs.

3. Expectation Matching Polynomial

In 1972 Heilmann and Lieb [4] defined the **matching polynomial**

$$\mu_G(x) = \sum_{i\geq 0} x^{n-2i}(-1)^i m_i$$

where m_i is the number of matchings with i edges and n is the number of vertices of G. For simplicity define $m_0 = 1$. Heilman and Lieb were studying the monomer-dimer partition function and ferromagnetism; in particular, they give an alternative proof of the Yang-Lee Circle Theorem. They also proved the following two theorems which allow Marcus, Spielman, and Srivastava to show the existence of infinite families of d-regular bipartite Ramanujan graphs.

Theorem 4. *For every graph G, all roots of $\mu_G(x)$ are real.*

Theorem 5. *For every graph G with maximum degree Δ, all roots of $\mu_G(x)$ have absolute value at most $2\sqrt{\Delta - 1}$.*

There is actually an important relationship between the matching polynomial $\mu_G(x)$ and the 2-lifts of G. Order the m edges of G arbitrarily, say $e_1, \ldots, e_m$. Given a signing $s \in \{\pm 1\}^m$, let A_s denote the signed adjacency matrix corresponding to s and $f_s(x)$ be the characteristic polynomial of A_s.

Theorem 6.

$$\mathbb{E}_s[f_s(x)] = \mu_G(x).$$

Proof. For $\pi \in \text{sym}(S)$, let $(-1)^\pi$ denote the **sign** of π (the number of inversions). Note that for $S \subseteq [n]$ with $|S| = k$ and π the part of σ such that $\sigma(i) \neq i$

$$\begin{aligned}
\mathbb{E}_s[f_s(x)] = \mathbb{E}_s\left[\det(xI - A_s)\right] &= \mathbb{E}_s\left[\sum_{\sigma\in\text{sym}([n])} (-1)^\sigma \prod_{i=1}^n (xI - A_s)_{i,\sigma(i)}\right] \\
&= \sum_{k=0}^n x^{n-k} \sum_S \sum_{\pi\in\text{sym}(S)} \mathbb{E}_s\left[(-1)^\pi \prod_{i\in S}(A_s)_{i,\pi(i)}\right] \\
&= \sum_{k=0}^n x^{n-k} \sum_S \sum_{\pi\in\text{sym}(S)} \mathbb{E}_s\left[(-1)^\pi \prod_{i\in S} s_{i,\pi(i)}\right].
\end{aligned}$$

Note that $s_{i,j}$ are independent, $\mathbb{E}[s_{i,j}] = 0$, $\mathbb{E}_s[s_{i,j}^2] = 1$. Only products with even powers (0 or 2) of $s_{i,j}$ survive. So consider π with orbits of size two. In other words we care about the perfect matchings of S. The number of perfect matchings is 0 if $|S|$ is odd and each matching has $\frac{|S|}{2}$ inversions if $|S|$ is even. For a matching M in S,

$$\mathbb{E}_s[f_s(x)] = \mathbb{E}_s\left[\det(xI - A_s)\right] = \sum_{k=0}^{n} x^{n-k} \sum_S \sum_M (-1)^{\frac{|S|}{2}} \cdot 1 = \mu_G(x).$$

□

In order to be able to construct good 2-lifts, we need to find a signing s so that the largest root of $f_s(x)$ is at most the largest root of $\mu_G(x)$. In other words, we need to show that $\{f_s(x)\}_{s\in\{\pm 1\}^m}$ are an interlacing family.

4. Interlacing Property

Roughly speaking, Bilu and Linial's conjecture [1] requires that every graph has a signed adjacency matrix with all eigenvalues within a small range. Marcus, Spielman, and Srivastava [6] show that the roots of the expected characteristic polynomial of a randomly signed adjacency matrix lie in the desired range. At first glance, this appears to be useless. The roots of a sum of polynomials are not necessarily related to the roots of the polynomials in the sum. However, using the "method of interlacing polynomials," Marcus, Spielman, and Srivastava show that an interlacing family always contains a polynomial whose largest root is at most the largest root of the sum. They bound the largest root of the sum of the characteristic polynomials of the signed adjacency matrices of a graph by observing that this is the matching polynomial of the graph, a well studied object.

Definition 5. A polynomial $g(x) = \prod_{i=1}^{n-1}(x - \alpha_i)$ **interlaces** a polynomial $f(x) = \prod_{i=1}^{n}(x - \beta_i)$ if

$$\beta_1 \leq \alpha_1 \leq \ldots \leq \alpha_{n-1} \leq \beta_n.$$

If all of these inequalities are strict, then g **strictly interlaces** f. Furthermore, $\{f_i\}_{i=1}^k$ have a **common interlacing** if there is a polynomial g such that g interlaces f_i for each $i \in [k]$.

Lemma 2. *Let* $\{f_i\}_{i=1}^k$ *be real-rooted polynomials of the same degree with positive leading coefficients. Let*

$$f_\emptyset = \sum_{i=1}^k f_i.$$

Then, if $\{f_i\}_{i=1}^k$ *have a common interlacing, then there exists* $i \in [k]$ *such that the largest root of* f_i *is at most the largest root of* $f_\emptyset$*.*

Proof. Let the degree of f_i be n and g be the polynomial interlacing them. Let α_{n-1} be the largest root of g. Because f_i has a positive leading coefficient, $f_i(x)$ is positive for x sufficiently large. Because f_i has exactly one root at least as large as α_{n-1}, $f_i(x)$ must be non-positive at α_{n-1}. Thus, $f_\emptyset(x)$ is non-positive at α_{n-1} but eventually becomes positive. Therefore $f_\emptyset$ has a root that is at least α_{n-1}. Thus, the largest root of $f_\emptyset$, say β_n is at least α_{n-1}. By definition of $f_\emptyset$, there must be some $i \in [k]$ such that $f_i(\beta_n) \geq 0$. Because f_i has exactly one root at least as large as α_{n-1} and $f_i(\alpha_{n-1}) \leq 0$, the largest root of f_i is at least α_{n-1} and at most β_n, as desired. □

Example 2. We must require a common interlacing. Consider

$$f_1(x) = x^3 - 14x^2 - 5x + 450 = ((x+5)(x-9)(x-10)),$$

$$f_2(x) = x^3 - 3x^2 - 46x + 48 = ((x+6)(x-1)(x-8)), \text{ and}$$

$$\begin{aligned} f_1(x) + f_2(x) &= ((x+5)(x-9)(x-10)) + ((x+6)(x-1)(x-8)) \\ &= (x^3 - 14x^2 - 5x + 450) + (x^3 - 3x^2 - 46x + 48) \\ &= 2x^3 - 17x^2 - 51x + 498. \end{aligned}$$

Note that in this example the polynomial $f_1(x) + f_2(x)$ has roots at approximately $-5.3, 6.4$, and 7.4, but $7.4 < 8$ and $7.4 < 10$.

If we pick each sign independently with any probabilities, then the resulting polynomial is also real rooted.

Theorem 7. *Let $p_1, \ldots, p_m \in [0,1]$. Then for $s \in \{\pm 1\}^m$*

$$\sum_s \left(\prod_{i:s_i=1} p_i \right) \left(\prod_{i:s_i=-1} (1-p_i) \right) f_s(x)$$

is real rooted.

Corollary 1. *The polynomials $\{f_s\}_{s\in\{\pm1\}^m}$ are an interlacing family.*

Thus, every graph has a signed adjacency matrix with eigenvalues as desired. Note that this shows the existence of a desired signed adjacency matrix not that almost every signed adjacency matrix is as desired.

Also using the method of interlacing polynomials, Marcus, Spielman, and Srivastava [6] establish the existence of infinite families of so-called "irregular Ramanujan" graphs, graphs with eigenvalues bounded by the spectral radius of the universal cover. In particular Marcus, Spielman, and Srivastava prove the existence of infinite families of (c,d)-biregular bipartite graphs with all non-trivial eigenvalues bounded by $\sqrt{c-1} + \sqrt{d-1}$ for $c, d \geq 3$.

5. Open problems

- Find computationally efficient analogues of the method of Marcus, Spielman, and Srivastava [6].
- Construct an infinite family of d-regular Ramanujan graphs which are not bipartite for all $d \geq 3$ [3], [5].
- Use the method of interlacing polynomials to solve other problems. For instance, using this method Marcus, Spielman, and Srivastava [7] are also able to show Weavers conjecture KS_2 which implies a positive solution to the Kadison-Singer problem concerning extensions of pure states in C^*-algebras.

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