Rainbow Copies of C_4 in Edge-Colored Hypercubes

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July 10, 2014

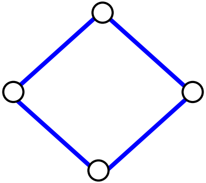
Definitions

Monochromatic Coloring

For a graph G, an edge coloring

$$\varphi : E(G) \to \{1, 2, \dots\}$$

is called **monochromatic** if all edges receive the same color.

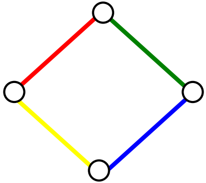


Rainbow Coloring

For a graph G, an edge coloring

$$\varphi : E(G) \to \{1, 2, \ldots\}$$

is called **rainbow** if no two edges receive the same color.



Motivation

Rainbow Variants

Erdős, Simonovits, and Sós introduced the anti-Ramsey number $AR(n, H)$, the maximum number of colors in an edge coloring of K_n such that it contains no rainbow copy of H.

Rainbow Variants

Conjecture (Erdős, Simonovits, and Sós)

It is possible to color the edges of K_n with

$$n\left(\frac{k-2}{2} + \frac{1}{k-1}\right) + O(1)$$

colors such that there is no rainbow C_k.

- True for C_3 (Erdős, Simonovits, and Sós)
- True for C_4 (Alon)
- True in general (Montellano-Ballesteros and Neumann-Lara)

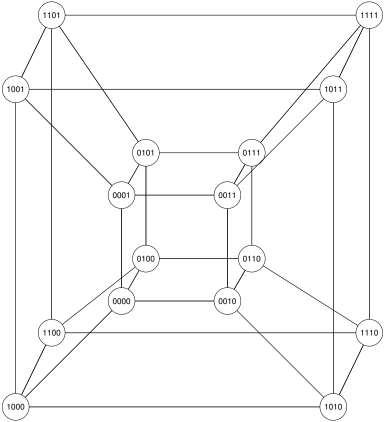
Rainbow Variants

Many people have studied the maximum number of rainbow subgraphs of a certain type in hypercubes.

- C_4 (Faudree, Gyárfás, Lesniak, and Schelp)
- Cycles (Mubayi and Stading)
- $\mathcal{Q}_3$ (Mubayi and Stading)

d-dimensional Hypercube

Let $\mathcal{Q}_d$ have vertices corresponding elements of $\{0, 1\}^d$ and put edges between elements of Hamming distance 1.



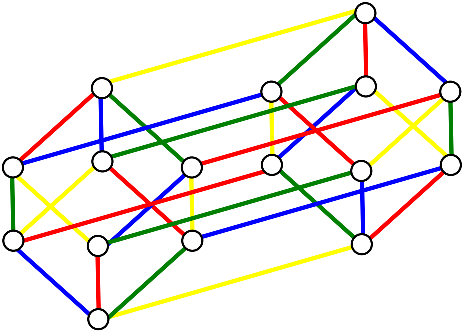
Edge-Colorings of Hypercubes

We were motivated by the work of Faudree, Gyárfás, Lesniak, and Schelp published in 1993.

Theorem (Faudree, Gyárfás, Lesniak, and Schelp)

If $d \in \mathbb{N}$ with $4 \leq d$ and $d \neq 5$, then there is a d-edge-coloring of $\mathcal{Q}_d$ such that every C_4 is rainbow.

$d = 4$



$d = 5$

Faudree, Gyárfás, Lesniak, and Schelp claim that there is no 5-edge-coloring of $\mathcal{Q}_5$ where every copy of C_4 is rainbow.

Using a computer, we find that the maximum number of rainbow copies of C_4 in a 5-edge-coloring of $\mathcal{Q}_5$ is 73 out of the 80 total copies of C_4.

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We would like to understand this case better.

Perhaps the reason for this unusual behavior is the ratio between number of edges and the total copies of C_4.

The number of edges of Q_5 is

$$d2^{d-1} = 80,$$

exactly equal to the total copies of C_4 in Q_5

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Theorem (Balogh, D., Lidický, Palmer, 2013+)

Fix $k, d \in \mathbb{N}$ such that $4 \leq k < d$ and $k \neq 5$. Then the maximum number of rainbow copies of C_4 in a k-edge-coloring of $\mathcal{Q}_d$ is

$$2^{d-2}\left[\binom{d}{2} - k\binom{a}{2} - ba\right]$$

where $d = ka + b$ with $a \in \mathbb{N}$ and $b \in \{0, 1, 2, \ldots, k-1\}$.

Upper bound

Assume that $\mathcal{Q}_d$ is edge-colored with colors

$$[k] = \{1, \ldots, k\}$$

such that the number of rainbow copies of C_4 is maximized.

A vertex in $\mathcal{Q}_d$, say v, has $\binom{d}{2}$ incident copies of C_4.

In the set of t_i edges of color $i \in [k]$ which are incident to v, none of the $\binom{t_i}{2}$ possible pairs can be in a rainbow copy of C_4.

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Upper bound

If the color classes are as equal as possible and

$$t_1 + \ldots + t_k = d = ka + b,$$

then there are at most

$$\begin{aligned} \binom{d}{2} - \sum_{i \in [k]} \binom{t_i}{2} &\leq \binom{d}{2} - (k-b)\binom{a}{2} - b\binom{a+1}{2} \\ &= \binom{d}{2} - k\binom{a}{2} + b\binom{a}{2} - b\binom{a+1}{2} \\ &= \binom{d}{2} - k\binom{a}{2} - ba \end{aligned}$$

rainbow copies of C_4 at v.

Upper bound

Summing up this for each of the 2^d vertices of $\mathcal{Q}_d$, we over count by a factor of four.

Thus, the maximum number of rainbow copies of C_4 in a k-edge-coloring of $\mathcal{Q}_d$ is at most

$$2^{d-2}\left[\binom{d}{2} - k\binom{a}{2} - ba\right],$$

as desired.

Lower bound

We would like to use edge-coloring of $\mathcal{Q}_k$ to color edges of $\mathcal{Q}_d$.

Now we give a construction using a “blow-up technique”.

Thinking of vertices of $\mathcal{Q}_d$ as elements of $\{0, 1\}^d$, we want to partition each string into k “blocks” of consecutive binary digits of length either a or $a + 1$.

We partition the first $(k - b)a$ binary digits into $(k - b)$ blocks of length a and the last $b(a + 1)$ digits into b blocks of length $a + 1$.

Lower bound

We associate an element of $\{0,1\}^k$ with each vertex of $\mathcal{Q}_d$ by computing the sum of the terms in each block modulo 2.

This process gives a map

$$h : V(\mathcal{Q}_d) \to V(\mathcal{Q}_k).$$

For example, consider $d = 10$ and $k = 3$:

$$\begin{array}{ccccc} 111 & | & 011 & | & 1011 \\ 1 & & 0 & & 1 \end{array}$$

and

$$h(1110111011) = 101.$$

Lower bound

Furthermore, h preserves edges.

If $u, v \in V(\mathcal{Q}_d)$ have Hamming distance 1, then $h(u)$ and $h(v)$ differ exactly in one block and have Hamming distance 1.

Again consider $d = 10$ and $k = 3$:

$$\begin{array}{ccccc} 111 \mid 011 \mid 1{\color{red}0}11 & & & & 111 \mid 011 \mid 1{\color{red}1}11 \\ 1 \quad 0 \quad {\color{red}1} & & & & 1 \quad 0 \quad {\color{red}0} \end{array}$$

Lower bound

Faudree, Gyárfás, Lesniak, and Schelp showed there is a k-edge-coloring of $\mathcal{Q}_k$, say φ, such that every C_4 is rainbow.

Color edges of $\mathcal{Q}_d$ with the color of their image under h in $\mathcal{Q}_k$, i.e. the color of an edge e in $\mathcal{Q}_d$ is $\varphi(h(e))$.

Using this coloring, each vertex in $\mathcal{Q}_d$ is incident to d edges, a edges of each of $k-b$ colors and $a+1$ edges of each of the remaining b colors.

Lower bound

We must check that for each vertex v in $\mathcal{Q}_d$, each pair of edges with different colors incident to v is actually in a rainbow C_4.

Note that among the four vertices in any C_4 the maximum Hamming distance is 2.

Thus, all differences among elements of $\{0, 1\}^d$ of the four vertices of the C_4 occur in at most 2 blocks.

Lower bound

If all the differences occur in the same block,
then the four edges of the C_4 are mapped to the same edge in $\mathcal{Q}_k$, and thus, the C_4 is monochromatic in $\mathcal{Q}_d$.

If the differences occur in 2 **distinct blocks,**
then the four edges of the C_4 are mapped to a C_4 in $\mathcal{Q}_k$, and thus, receive different colors in the coloring of $\mathcal{Q}_d$.

Further Directions

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$k = 5$

Upper Bound

For $k = 5$, flag algebra methods did not improve the upper bound obtained from our main result.

We actually suspect that the upper bound might be the correct order of magnitude for large d.

Lower Bound

For a lower bound, our blow-up method can be applied to a 5-edge-coloring of $\mathcal{Q}_5$ with 73 rainbow copies of C_4.

This, however, is very far from our upper bound.

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Edge-Coloring Subgraphs

Let G and H be graphs and $|E(H)| \geq q \in \mathbb{N}$.

Denote the minimum number of colors required to edge-color G such that the edges of every copy of H in G receive at least q colors by

$$f(G, H, q).$$

In this context, Faudree, Gyárfás, Lesniak, and Schelp show

$$f(\mathcal{Q}_d, C_4, |E(C_4)|) = f(\mathcal{Q}_d, C_4, 4) = d,$$

for integer $4 \leq d$ with $d \neq 5$.

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Mubayi and Stading generalized this result.

They proved that there are positive constants, say c_1 and c_2, depending only on k such that

$$c_1 d^{k/4} < f(\mathcal{Q}_d, C_k, k) < c_2 d^{k/4}$$

for $k \equiv 0 \pmod 4$.

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Mubayi and Stading showed that

$$f(\mathcal{Q}_d, C_6, 6) = f(\mathcal{Q}_d, \mathcal{Q}_3, |E(\mathcal{Q}_3)|) = f(\mathcal{Q}_d, \mathcal{Q}_3, 12).$$

They were able to show that for every $\varepsilon > 0$, there exists d_0 such that for $d > d_0$

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Edge-Coloring Subgraphs

Problem

Determine the value of

$$f(\mathcal{Q}_d, \mathcal{Q}_\ell, |E(\mathcal{Q}_\ell)|) = f(\mathcal{Q}_d, \mathcal{Q}_\ell, \ell 2^{\ell-1})$$

for $\ell \geq 3$.

Perhaps a generalization of our blow-up technique could be used to determine the maximum number of rainbow copies of $\mathcal{Q}_\ell$ in a k-edge-coloring of $\mathcal{Q}_d$ in general.

Edge-Coloring Subgraphs

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Thank you for listening!