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Systemic risk components in a network model of contagion

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ABSTRACT
We show how to perform a systemic risk attribution in a network model of contagion with interlocking balance sheets, using the Shapley and Aumann–Shapley values. Along the way, we establish new results on the sensitivity analysis of the Eisenberg–Noe network model of contagion, featuring a Markov chain interpretation. We illustrate the design process for systemic risk attribution methods by developing several examples.

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1. Introduction

Risk attribution is important in systemic risk management, as it is in portfolio risk management. Systemic risk involves risk that arises due to the structure of the financial system and interactions between financial institutions. Systemic risk attribution is decomposing the risk of a system into risk components that are attributed to components of the system. For an introduction and literature review, see Staum (2012, 2013). We build on a basic theory of systemic risk attribution, in which the key tools are the Shapley and Aumann–Shapley values, introduced briefly in Section 2; for further background, see Staum (2012). Our primary contribution is to further develop the theory of systemic risk attribution by showing how to do systemic risk attribution in a network model of interlocking balance sheets.

In the network model presented in Section 3, a graph represents a financial network, with nodes representing firms and directed edges representing loans between firms. Contagion spreads through the network when the default of one firm imposes losses on its creditor, causing the creditor to default. The balance sheets of borrower and lender are interlocking, due to the loan appearing as an asset on the lender's balance sheet and a liability on the borrower's balance sheet. Using the Shapley or Aumann–Shapley values to attribute risk entails imagining a financial network where one of the firms is absent or is smaller. Removing or changing the size of one firm and consequently also the loans this firm has made or received causes a problem of "holes in … banks' balance sheets" (Gauthier *et al.*, 2010, p. 7). We propose a solution to this problem that is similar to that proposed by Drehmann and Tarashev (2013) in simultaneous research. Whereas they use only the Shapley value, we show how to use linear programming sensitivity analysis to compute Aumann–Shapley values for systemic risk components in a network model of contagion. This sensitivity analysis in Section 3.2 is one of our secondary contributions. It extends the results of Liu and Staum (2010) on sensitivity analysis of the Eisenberg and Noe (2001) network model of contagion by including sensitivity to liabilities as well as to income, and it adds a Markov-chain interpretation. We use simple, artificial examples (Section 4.1) to illustrate the workings of our methods, whereas Drehmann and Tarashev (2013) apply their method to real data on major global banks.

Our analysis goes beyond that of Drehmann and Tarashev (2013) in that we explore more than one solution to the problem of holes in banks' balance sheets in Section 4. We contribute to the methodology of the design of risk attribution methods through a discussion of the characteristics of different systemic risk attribution methods. Section 5 offers some guidance about choosing and designing systemic risk attribution methods.

2. Risk attribution

This section presents a risk attribution framework for a financial network with interlocking balance sheets. This framework applies the Shapley or Aumann–Shapley values with the following four steps.

The first step is to select the components of balance sheets to which risk is to be attributed and to create a "balance-sheet scheme" that specifies how the whole time-0 balance sheet depends on the sizes of these components. In Section 4, we explore several balance-sheet schemes that lead to different risk attributions. The remaining three steps are parts of the risk management framework that are required even for purposes other than risk attribution. For concreteness and simplicity, throughout our study we will work with the following choices for these steps.

The second step is to model the time-1 balance sheets as a function of the time-0 balance sheets and a random vector of risk factors that vary across future scenarios. We choose to model the time-1 interlocking balance sheets in a network model of contagion, which will presented in Section 3.

The third step is to associate a cost with the time-1 balance sheet. We take the cost to be the aggregate loss on loans to

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firms in the financial network incurred by external creditors; i.e., lenders that are not firms in the financial network. This is made precise in Section 3. Our approach can be implemented for other choices for the cost, but the results would differ.

The fourth step is to choose a risk measure that maps a random vector whose value in each scenario is cost to a scalar whose interpretation is risk. We take the risk measure to be expected cost. This choice allows us to focus henceforth on cost allocation within a single scenario because, under mild conditions, the risk attributed to a component of the network is the expectation of the cost attributed to that component by the Shapley or Aumann–Shapley value (Staum, 2012). Staum (2013, Section 6) describes how to implement our approach for other risk measures.

Next we briefly summarize a standard framework for cost allocation. Suppose that cost is to be allocated to n components of a system. Let $\boldsymbol{\lambda} \in [0,1]^n$ be a participation vector, where λ_i is interpreted as the participation level of component i in the system. A participation vector $\boldsymbol{\lambda}$ specifies a counterfactual system that is related to the real system. For example, where γ_i is the equity of firm i in the real system, this firm's equity in the counterfactual system specified by $\boldsymbol{\lambda}$ could be $\lambda_i\gamma_i$. When every component fully participates—i.e., $\boldsymbol{\lambda} = \mathbf{1}$—the corresponding system is the real system. The key object in this cost allocation framework is the cost function c, which arises from steps 1 to 3. The cost function maps a participation vector $\boldsymbol{\lambda}$ to the cost $c(\boldsymbol{\lambda})$ of the counterfactual system specified by participation vector $\boldsymbol{\lambda}$. The cost function must make $c(\mathbf{0})$ be zero and $c(\mathbf{1})$ be the cost of the real system. There are two widely used methods that generate a cost allocation given the cost function: the Aumann–Shapley value and the Shapley value.

The Aumann–Shapley value allocates to component i:

$$\phi_i^{\mathrm{AS}} = \int_0^1 \frac{\partial c}{\partial \lambda_i}(t\mathbf{1})\,\mathrm{d}t \text{ for } i = 1, \ldots, n, \tag{1}$$

if the cost function $c(\cdot)$ is sufficiently differentiable. What is required is that $c(\cdot)$ be differentiable at almost every point on the "diagonal," the set $\{\gamma\mathbf{1} : \gamma \in [0,1]\}$, and that the function that maps $\gamma \in [0,1]$ to $c(\gamma\mathbf{1})$ is absolutely continuous. This condition is satisfied if $c(\cdot)$ is differentiable. It is also satisfied in the examples in the present article, even examples where there are some points of non-differentiability on the diagonal. Non-differentiability of the cost function in risk attribution is discussed in more generality by Denault (2001), Buch and Dorfleitner (2008), Tsanakas (2009), Cherny and Orlov (2011), and Boonen *et al.* (2012). Equation (1) equals the partial derivative $(\partial c/\partial \lambda_i)(\mathbf{1})$ if the cost function $c(\cdot)$ is (positively) homogeneous. The partial derivative is the sensitivity of cost to perturbation of the participation of component i.

The Shapley value allocates to component i:

$$\phi_i^{\mathrm{S}} = \frac{1}{n!} \sum_{S \subseteq \{1,\ldots,n\}\setminus\{i\}} (n - |S| - 1)!|S|! \left(c\left(\mathbf{1}_{S\cup\{i\}}\right) - c\left(\mathbf{1}_S\right)\right), \tag{2}$$

where, for a subset $S \subseteq \{1,\ldots,n\}$, $\mathbf{1}_S$ is a n-vector whose ith component equals one if $i \in S$ and zero if $i \notin S$. Equation (2) involves the incremental costs $c\left(\mathbf{1}_{S\cup\{i\}}\right) - c\left(\mathbf{1}_S\right)$ when the participation of component i is added to the participation of a set S of other components. The term $c(\mathbf{1}_{\{i\}}) - c(\mathbf{0})) = c(\mathbf{1}_{\{i\}})$ is the stand-alone cost of component i, and the terms of the form $c\left(\mathbf{1}_{S\cup\{i\}}\right) - c\left(\mathbf{1}_S\right)$ assess the contribution of component i to interactions between the n components of the system.

3. A network model of contagion

Consider a system containing n firms whose balance sheets are interlocking due to loans between firms. Part of the model of the system is a network or graph, in which firms are nodes and directed edges represent the flow of money at time 0 when one firm lends to another; the flow of money at time 1 when loans mature is in the opposite direction. It is important to distinguish between internal assets, which are claims on other nodes, and external assets, which are claims on entities outside the system, and between internal liabilities, which are obligations to other nodes, and external liabilities, which are obligations to entities outside the system. The vector of equity issued by each node is $\boldsymbol{\gamma}$. The vector of debt at each node and held externally is $\boldsymbol{\delta}$. Internal liabilities are described by an $n \times n$ matrix $\boldsymbol{L}$, where L_{ij} is the principal borrowed by node i from node j. The nodes' external assets include the vector $\boldsymbol{\eta}$ of cash and the matrix $\boldsymbol{\Theta}$ of risky asset holdings, where Θ_{ij} is the amount of asset j held by node i. The accounting equation is

$$\boldsymbol{\varsigma} = \boldsymbol{\gamma} + \boldsymbol{\delta} + \boldsymbol{L}\mathbf{1} = \boldsymbol{\eta} + \boldsymbol{\Theta}\mathbf{1} + \boldsymbol{L}^{\top}\mathbf{1}, \tag{3}$$

where $\mathbf{1}$ is a vector of ones of appropriate dimension. At time 1, the external assets of each node are worth

$$\boldsymbol{e} = \boldsymbol{\eta} + \boldsymbol{\Theta}\boldsymbol{a}, \tag{4}$$

the external liabilities are $\boldsymbol{\delta}$, and the internal liabilities are $\boldsymbol{L}\mathbf{1}$. Henceforth, we make the following assumption.

Assumption 1: The balance-sheet primitives $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, $\boldsymbol{L}$, $\boldsymbol{\eta}$, and $\boldsymbol{\Theta}$ are non-negative. Furthermore, every time-1 external asset value in the vector $\boldsymbol{e}$ is strictly positive.

For some of our methods, the existence of the gradient in Equation (1) may require that other quantities, such as cash holdings $\boldsymbol{\eta}$ or external debt $\boldsymbol{\delta}$, be strictly positive. These additional assumptions are stated explicitly when introducing those methods.

For the sake of simplicity in the model and analysis, we make several strong assumptions in our modeling to justify Equations (3) and (4) above and Equations (5) to (7) below. These are not limitations on the scope of the applicability of our approach to systemic risk components, but carrying through the approach in a more complicated model takes more space. We assume that internal assets and liabilities include only debt, not equity. Elsinger (2007) shows how to extend the analysis of Section 3.1 when equity is held inside the system. Liu and Staum (2011) handle time-1 liabilities that result from internal financial relationships other than loans, such as swaps. We assume that all loans have equal seniority. This assumption too can be lifted using the results of Elsinger (2007). Liu and Staum (2011) and Drehmann and Tarashev (2013) study systemic risk components in models where external liabilities are senior to internal liabilities. We assume that interest rates are zero, so that time-1 liabilities equal time-0 liabilities. This assumption is sufficiently realistic when interest rates are low enough and the time horizon is short enough. Liu and Staum (2012) allow for positive interest rates but do not allow them to depend on the borrower's asset risk or leverage. This is a weakness for their systemic risk attribution methods, because they neglect the effects of changes to time-0 balance sheets on interest rates and, through these, on systemic risk. We have omitted all of these interesting features of real financial systems to study systemic risk attribution in a simple model that coincides with the much-used Eisenberg–Noe model (Eisenberg and Noe, 2001) and captures the most-discussed feature of bilateral contagion in financial systems: loss given default on interbank loans.

The external creditors' loss in a financial system, in which the nodes have time-1 external assets $\boldsymbol{e}$, external liabilities $\boldsymbol{\delta}$, and internal liabilities $\boldsymbol{L}$, is $\ell(\boldsymbol{e}, \boldsymbol{\delta}, \boldsymbol{L}) = \boldsymbol{\delta}^\top(\mathbf{1} - \boldsymbol{f}^*)$, where the vector $\boldsymbol{f}^*$ contains the fractions $\boldsymbol{f}^*$ of their liabilities that the nodes pay. The nodes' capacities to pay their liabilities depend on the values of their assets, internal as well as external. As each internal loan appears as an asset on one balance sheet and as a liability on an interlocking balance sheet, contagion is a factor and it is not simple to determine $\boldsymbol{f}^*$. The next subsection describes how to compute these fractions. The cost of the counterfactual system specified by participation vector $\boldsymbol{\lambda}$ is $c(\boldsymbol{\lambda}) = \ell(\boldsymbol{e}(\boldsymbol{\lambda}), \boldsymbol{\delta}(\boldsymbol{\lambda}), \boldsymbol{L}(\boldsymbol{\lambda}))$. The cost function c is a function of the participation vector, whereas the loss function ℓ is a function of the interlocking balance sheets.

3.1. *Clearing payments*

We use the analysis of Eisenberg and Noe (2001) of the flows of money in the network at time 1. The inflows from external assets make up the vector $\boldsymbol{e}$, and $\boldsymbol{L}$ is a matrix of maximum flows from one node to another. Let the external creditors be represented by a sink node. Then $\boldsymbol{\delta}$ contains maximum flows to the sink node. Define $\bar{\boldsymbol{p}} = \boldsymbol{\delta} + \boldsymbol{L}\mathbf{1}$ as the vector of total time-1 liabilities; i.e., maximum outflows from each node. Let $\boldsymbol{p}$ be the vector of outflows from each node. Define the matrix $\boldsymbol{\Pi} = [\Pi_{ij}]_{i,j=1}^n$ whose element $\Pi_{ij} = L_{ij}/\bar{p}_i$ is the fraction of the outflow from node i that goes to node j. Similarly, $\boldsymbol{\Pi}^0 = [\delta_i/\bar{p}_i]_{i=1}^n$ contains the fractions of the nodes' outflows that go to external creditors. For any i such that $\bar{p}_i = 0$, we define $\Pi_i^0 = 0$, $\boldsymbol{\Pi}_{i\cdot} = \mathbf{0}$, and $f_i = 1$. The flows from node i to node j and to external creditor are $p_i\Pi_{ij}$ and $p_i\Pi_i^0$, respectively.

The vector of internal inflows to nodes is $\boldsymbol{\Pi}^\top\boldsymbol{p}$. The flows must satisfy the capacity constraints

$$\mathbf{0} \le \boldsymbol{p} \le \bar{\boldsymbol{p}} \quad \text{and} \quad (\mathbf{I} - \boldsymbol{\Pi}^\top)\boldsymbol{p} \le \boldsymbol{e}. \tag{5}$$

The first capacity constraint says that the outflow from each node can neither be negative nor exceed its liabilities. The second says that the outflow of each node can not exceed its inflow from internal and external sources: $\boldsymbol{p} \le \boldsymbol{\Pi}^\top\boldsymbol{p} + \boldsymbol{e}$. The time-1 equity value of the nodes is $\boldsymbol{v} = \boldsymbol{e} + \boldsymbol{\Pi}^\top\boldsymbol{p} - \boldsymbol{p}$. The flows must also satisfy the priority constraint

$$\operatorname{diag}(\boldsymbol{v})(\bar{\boldsymbol{p}} - \boldsymbol{p})^+ = \mathbf{0}, \tag{6}$$

which says that any node whose outflow is less than its liabilities is left with zero equity value. Eisenberg and Noe (2001) define a clearing payment vector as a value of $\boldsymbol{p}$ that satisfies the capacity and priority constraints.

Lemma 1. *There exists a unique clearing payment vector.*

Proof. By Theorem 1 of Eisenberg and Noe (2001), a clearing vector exists. Under Assumption 1, the external inflows in $\boldsymbol{e}$ are all strictly positive. Therefore, the financial system is regular (Eisenberg and Noe, 2001, Definition 2); i.e., when you trace where payments flow from any node, you encounter some node with positive external inflow. By Theorem 2 of Eisenberg and Noe (2001), the clearing vector is unique. □

Call this unique clearing payment vector $\boldsymbol{p}^*$. Let $\boldsymbol{f}^*$ and $\boldsymbol{v}^*$ be the corresponding vectors of payment fractions and equity, respectively. From them we can compute the external creditors' loss:

$$\ell(\boldsymbol{e}, \boldsymbol{\delta}, \boldsymbol{L}) = \boldsymbol{\delta}^\top(\mathbf{1} - \boldsymbol{f}^*) = \boldsymbol{\delta}^\top\mathbf{1} - {\boldsymbol{\Pi}^0}^\top\boldsymbol{p}^*, \tag{7}$$

which is the basis for the cost function c used in cost attribution.

3.2. *Sensitivity analysis*

Computing the Aumann–Shapley value requires a sensitivity analysis. The following classification of nodes is useful in sensitivity analysis.

Definition 1. Node i is <u>green</u> if $v_i^* > 0$, <u>red</u> if $f_i^* < 1$, and <u>borderline</u> if $v_i^* = 0$ and $f_i^* = 1$.

Let $\mathcal{R}$ and $\mathcal{G}$ be the sets of red and green nodes, respectively. Define the corresponding indicator vectors by $\mathbf{1}_{\mathcal{R}}$ and $\mathbf{1}_{\mathcal{G}}$. Due to the priority constraint (6), a green node i does not default ($f_i^* = 1$) and a red node i has zero time-1 equity ($v_i^* = 0$).

The sensitivity analysis involves the matrix $\nabla_{\boldsymbol{e}}\boldsymbol{p}^*$ whose (i, j)th element is the partial derivative of the clearing payment made by node i to the income of node j. The formula for $\nabla_{\boldsymbol{e}}\boldsymbol{p}^*$ in Corollary 1 agrees with that of Liu and Staum (2010, Proposition 2). It is derived here via Proposition 1 as the Markov-chain computation yields insight and a good way to implement the "fictitious default algorithm" of Eisenberg and Noe (2001), which they gave in an inexplicit form. Our proof of Proposition 1 is similar to the proof of Lemma 3 of Eisenberg and Noe (2001) and is provided for the sake of completeness. In the algorithm, k is an iteration counter, $\mathcal{D}_k$ represents a set of defaulting nodes in this iteration, and $\mathcal{N}_k$ represents a set of nodes that have not yet been identified as defaulting. During the algorithm, $\mathcal{D}_k$ grows from iteration to iteration as more nodes are identified as defaulting. When the algorithm terminates, $\mathcal{D}_k = \mathcal{R}$; i.e., all red nodes are identified as defaulting nodes.

Proposition 1. *The following algorithm terminates with iteration counter $k \le n$ and then the clearing payment vector $\boldsymbol{p}^* = \boldsymbol{p}^k$.*

1. *Initialize $k \leftarrow 0$ and set $\mathcal{D}_0 = \emptyset$ and $\mathcal{N}_0 = \{1, \ldots, n\}$.*
2. *Set $\boldsymbol{p}^k_{\mathcal{N}_k} = \bar{\boldsymbol{p}}_{\mathcal{N}_k}$ and*

$$\boldsymbol{p}^k_{\mathcal{D}_k} = (\mathbf{I} - (\boldsymbol{\Pi}_{\mathcal{D}_k\mathcal{D}_k})^\top)^{-1}(\boldsymbol{e}_{\mathcal{D}_k} + (\boldsymbol{\Pi}_{\mathcal{N}_k\mathcal{D}_k})^\top\bar{\boldsymbol{p}}_{\mathcal{N}_k}). \tag{8}$$

3. *Set $\mathcal{D}'_k = \{i \in \mathcal{N}_k : \bar{p}_i > e_i + (\boldsymbol{\Pi}_{\cdot i})^\top\boldsymbol{p}^k\}$.*
4. *If $\mathcal{D}'_k = \emptyset$, terminate the algorithm. Otherwise, set $\mathcal{D}_{k+1} = \mathcal{D}_k \cup \mathcal{D}'_k$ and $\mathcal{N}_{k+1} = \mathcal{N}_k \setminus \mathcal{D}'_k$, then update $k \leftarrow k + 1$ and return to Step 2.*

Proof. See Online Supplement section A. □

Equation (8) has an interpretation in terms of a Markov chain. Its states correspond to nodes, including the sink node, which represents external creditors. The transitions of the Markov chain represent the movements of a dollar from one node to another at time 1. The sink node and other non-defaulting nodes correspond to absorbing states; as they are already making payments equal to their liabilities, increasing the inflow to such a node does not increase its outflow. The defaulting nodes correspond to transient states; an extra dollar received by a defaulting node will be paid out to its creditors and eventually be absorbed elsewhere. Let the matrix of transition probabilities from transient states be $[\mathbf{\Pi}^{\mathbf{0}}_{\mathcal{D}}\ \mathbf{\Pi}_{\mathcal{D}\cdot}]$, assigning index 0 to the sink node. If nodes i and j default, the (i, j)th element of the matrix $(\mathbf{I} - (\mathbf{\Pi}_{\mathcal{DD}})^\top)^{-1}$ is the expected number of visits by the Markov chain to state i given that the chain's initial state is j. This can also be interpreted as the expected number of visits to node i of a dollar that starts at node j before absorption by the sink node or another non-defaulting node. The vector $\boldsymbol{e}_{\mathcal{D}} + (\mathbf{\Pi}_{\mathcal{ND}})^\top \bar{\boldsymbol{p}}_{\mathcal{N}}$ contains the inflow to each defaulting node from outside the system and from non-defaulting nodes. Therefore, the right-hand side of Equation (8) contains the outflows from each defaulting node; if node i defaults, its outflow equals its inflow, and the ith element of the right-hand side of Equation (8) is the inflow, the number of times that a dollar enters node i from any source. This Markov-chain analysis provides expressions for the sensitivity of payments to income.

Corollary 1. *The right derivatives $\nabla^+_{\boldsymbol{e}}\boldsymbol{p}^*$ are given by $(\nabla^+_{\boldsymbol{e}}\boldsymbol{p}^*)_{\mathcal{RR}} = (\mathbf{I} - \mathbf{\Pi}^\top_{\mathcal{RR}})^{-1}$, whereas the rest of the elements of $\nabla^+_{\boldsymbol{e}}\boldsymbol{p}^*$ are zero. The left derivatives $\nabla^-_{\boldsymbol{e}}\boldsymbol{p}^*$ are given by the same formula, but with $\mathcal{R}$ replaced by $\mathcal{G}^{\complement} = \{i : v_i = 0\}$, the set of red and borderline nodes. If there are no borderline nodes, then $\nabla_{\boldsymbol{e}}\boldsymbol{p}^*$ equals the left and right derivatives.*

Proof. A sufficiently small positive perturbation in external asset value $\boldsymbol{e}$ does not change the set $\mathcal{R}$ of red nodes. Therefore, the statement about right derivatives follows from Proposition 1, as Equation (8) holds for $\boldsymbol{p}^*$ and $\mathcal{D} = \mathcal{R}$ when the algorithm terminates. If node i is borderline, then the time-1 equity value $v_i = 0$ and $\bar{p}_i = p^*_i = e_i + (\mathbf{\Pi}_{\cdot i})^\top \boldsymbol{p}^*$. It follows that Equation (8) also holds for $\boldsymbol{p}^*$ but with $\mathcal{D}$ replaced by $\mathcal{G}^{\complement}$, the set of nodes that are not green. A sufficiently small negative perturbation in external asset value does not change the set $\mathcal{G}$ of green nodes. This establishes the statement about left derivatives. □

Next, we accumulate some results that are useful in computing Shapley and Aumann–Shapley values by considering how changes to the external income $\boldsymbol{e}$, external liability $\boldsymbol{\delta}$, and internal liability $\boldsymbol{L}$ affect the cost $\ell(\boldsymbol{e}, \boldsymbol{\delta}, \boldsymbol{L})$ defined in Equation (7).

For simplicity, Corollary 2 and Proposition 2 are presented for cases where there is no borderline node. When borderline nodes are present, left and right derivatives can be found using Corollary 1.

Corollary 2. *The clearing payment made by node i in a network system is $\bar{p}_i$ if $i \in \mathcal{G}$; i.e., node i is green in the system. If $i \in \mathcal{R}$—i.e.; node i is red in this system—then:*

$$p^*_i = \sum_{j\in\mathcal{R}} \frac{\partial p^*_i}{\partial e_j}\left(e_j + \sum_{k\in\mathcal{G}} L_{kj}\right). \tag{9}$$

Proof. This follows from Proposition 1, Equation (8), and Corollary 1. In deriving Equation (9) from Equation (8), observe that $\bar{p}_k \Pi_{kj} = L_{kj}$. □

Proposition 2. *If there are no borderline nodes, then*

$$-\boldsymbol{\zeta} = \nabla_{\boldsymbol{e}}\ell(\boldsymbol{e}, \boldsymbol{\delta}, \boldsymbol{L}) = -{\mathbf{\Pi}^0}^\top \nabla_{\boldsymbol{e}}\boldsymbol{p}^*, \tag{10}$$

$$\nabla_{\boldsymbol{\delta}}\ell(\boldsymbol{e}, \boldsymbol{\delta}, \boldsymbol{L}) = (\mathbf{1} - \boldsymbol{f}^*) + \mathrm{diag}(\boldsymbol{f}^*)\boldsymbol{\zeta}, \tag{11}$$

and

$$\nabla_{\boldsymbol{L}}\ell(\boldsymbol{e}, \boldsymbol{\delta}, \boldsymbol{L}) = \mathrm{diag}(\boldsymbol{f}^*)(\boldsymbol{\zeta}\mathbf{1}^\top - \mathbf{1}\boldsymbol{\zeta}^\top). \tag{12}$$

Proof. See Online Supplement section B. □

A fundamental quantity in the sensitivity analysis is ζ_i, the marginal price of wealth at node i. It is the rate of decrease of cost as the inflow to node i increases. Its Markov chain interpretation is the probability of reaching state 0 (the sink node) starting from state i (node i). If node i is red, then $\zeta_i = \sum_{j=1}^n \Pi_{ij}\zeta_j$. Therefore, the formula for $\boldsymbol{\zeta}$ given by Equation (10) is the solution to the system of equations

$$\boldsymbol{\zeta}_{\mathcal{G}} = \mathbf{0} \quad \text{and} \quad \boldsymbol{\zeta}_{\mathcal{R}} = \mathbf{\Pi}^{\mathbf{0}}_{\mathcal{R}} + \mathbf{\Pi}_{\mathcal{R}\cdot}\boldsymbol{\zeta}. \tag{13}$$

The sensitivity $\partial\ell/\partial\delta_i = (1 - f^*_i) + f^*_i\zeta_i$ in Equation (11) contains a first term for the direct sensitivity of node i's external creditors' loss to increasing its external debt and a second term for the marginal cost of moving wealth away from node i to repay external creditors, thereby indirectly affecting external creditors through effects on other nodes. In Equation (12), the (i, j)th element of the matrix $\nabla_{\boldsymbol{L}}\ell$ is $\partial\ell/\partial L_{ij} = f^*_i(\zeta_i - \zeta_j)$, which can be interpreted in terms of the marginal cost of moving wealth from node i to node j due to repayment of the liability that node i owes to node j.

4. Balance-sheet schemes and systemic risk components

The first and the most crucial step in designing a systemic risk attribution method is creating a balance-sheet scheme. For the model of interlocking balance sheets, a balance-sheet scheme specifies how every entry on the balance sheet of every node in the network depends on the vector $\boldsymbol{\lambda}$ of participation levels of some components of the system: equity $\boldsymbol{\gamma}(\boldsymbol{\lambda})$, external liability $\boldsymbol{\delta}(\boldsymbol{\lambda})$, internal liability $\boldsymbol{L}(\boldsymbol{\lambda})$, cash $\boldsymbol{\eta}(\boldsymbol{\lambda})$, and risky external assets $\boldsymbol{\Theta}(\boldsymbol{\lambda})$. In general, let $(\boldsymbol{\lambda})$ denote a quantity in the counterfactual system specified by participation vector $\boldsymbol{\lambda}$. The notation $(\mathbf{1})$ describing the real system may be omitted for brevity. Such balance-sheet schemes must satisfy the accounting equation (3). Satisfying the accounting equation when balance sheets are interlocking is more difficult, due to the internal liabilities affecting two balance sheets, those of the lender and of the borrower. In this section, we design a few balance-sheet schemes for the model of interlocking balance sheets, derive systemic risk attribution methods from them, and illustrate these methods in some numerical examples. In designing a balance-sheet scheme, two main questions arise.

First, are balance-sheet sizes variable or fixed, and how is the accounting equation enforced? One way is that when the size of one entry in the balance sheet changes, another entry on the

Table 1. Overview of methods

Balance sheet scheme	*Value*	*Responsibility for internal liabilities*	*Balance sheet size*	*Accounting equation enforcement*
External assets	Shapley A–S	None	Fixed	Cash substitutes for risky external assets
Transmission Leverage	Shapley A–S	Borrower	Fixed	Equity substitutes for liability Cash substitutes for internal loans
Intermediation Solvency Absorption	Shapley A–S A–S A–S	Shared Borrower Lender	Variable	External liability substitutes for internal liability Cash substitutes for internal loans
Funding	A–S	Lender	Variable	Borrowing provides cash

same side of the balance sheet compensates, so that the other side of the balance sheet need not change. For example, in Section 4.2 we present a fixed-size balance-sheet scheme where cash serves as a substitute for risky assets. Another way is that when the size of one or more entries in the balance sheet changes, the size of the balance sheet changes, leading to a change in one, some, or all entries on the other side. For example, in Section 4.5, a decrease in the liabilities of a node leads to decrease in its balance sheet's size and changes in all its assets.

Second, when risk is attributed to nodes, how does the size of an internal liability depend on the participation levels of the borrower and lender? It can depend on neither, on the borrower, on the lender, or on both. Consider a liability that node i owes to node j. The participation level of the borrower is λ_i and that of the lender is λ_j. To formalize the attribution of responsibility for this internal liability, we let the size of the internal liability $L_{ij}(\boldsymbol{\lambda})$ be $\lambda_i L_{ij}$ for borrower responsibility, $\lambda_j L_{ij}$ for lender responsibility, and $\sqrt{\lambda_i \lambda_j} L_{ij}$ for shared responsibility with the Shapley value. Shared responsibility for internal liabilities with the Aumann–Shapley value yields an allocation that is the average of the allocations from borrower responsibility and lender responsibility (Liu and Staum, 2012).

Table 1 provides an overview of the methods based on these criteria. Within each group of methods marked off by horizontal lines, the approach is the same, except for differences of responsibility for internal liabilities and Shapley value or Aumann–Shapley value. Therefore, a single explanation of how the accounting equation is enforced is given for the entire group.

By combining a balance-sheet scheme with the sensitivity analysis of cost to income, external liability, and internal liability in Section 3.2, using Equations (7), (9), and (10), we derive the cost function c as

$$c(\boldsymbol{\lambda}) = \sum_{i \in \mathcal{R}(\boldsymbol{\lambda})} \left(\delta_i(\boldsymbol{\lambda}) - \Pi_i^0(\boldsymbol{\lambda}) p_i(\boldsymbol{\lambda}) \right)$$

$$= \sum_{i \in \mathcal{R}(\boldsymbol{\lambda})} \left(\delta_i(\boldsymbol{\lambda}) - \zeta_i(\boldsymbol{\lambda}) \left(e_i(\boldsymbol{\lambda}) + \sum_{j \notin \mathcal{R}} L_{ji}(\boldsymbol{\lambda}) \right) \right). \quad (14)$$

Given a balance-sheet scheme, the Shapley value can be computed from the cost function and Equation (2). However, further work is required to derive the Aumann–Shapley value in Equation (1) as the gradient of the cost function must be taken. This is done in Online Supplement section C.

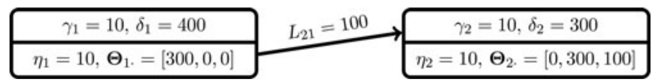

Figure 1. Two-node network in Example 1, in thousands. Arrows in the direction of lending.

4.1. *Numerical examples*

Our examples of financial systems are artificial, not realistic. They are designed to clearly illustrate the different behaviors of the systemic risk attribution methods we introduce.

Example 1: Consider the two node network in Fig. 1 and the four scenarios for external assets' returns in Table 2. The amounts shown in Fig. 1 are in thousands. The two nodes have the same amount of cash, equity, and total debt. The "downstream" node 1 has lent 100 000 to the "upstream" node 2. Both nodes have default probability 8% and their loss given default is 25%. There is no loss to any external creditor in the first scenario and there is no contagion effect in the second scenario. The contagion effects are shown in the third and fourth scenarios.

Table 2. Scenarios in Example 1

	External asset returns (%)			*Nodes' loss (%)*	
Probability (%)	$a_1 - 1$	$a_2 - 1$	$a_3 - 1$	$1 - f_1^*$	$1 - f_2^*$
88	5	5	5	0	0
4	−59.17	10	−10	41.875	0
4	10	−2.5	−62.5	0	15
4	−2.5	−59.17	27.5	8.125	35

Table 3. Systemic risk components in Example 1

Method		*Downstream node 1*	*Upstream node 2*
External creditors' loss	*Both*	*8000*	*6000*
External assets	Shapley Aumann–Shapley	6750 6917	7250 7083
Transmission Leverage	Shapley Aumann–Shapley	7350 7740	6650 6260
Intermediation	Shapley Aumann–Shapley	6350 6300	7650 7700
Solvency	Aumann–Shapley	6600	7400
Absorption	Aumann–Shapley	6000	8000
Funding	Aumann–Shapley	6150	7850

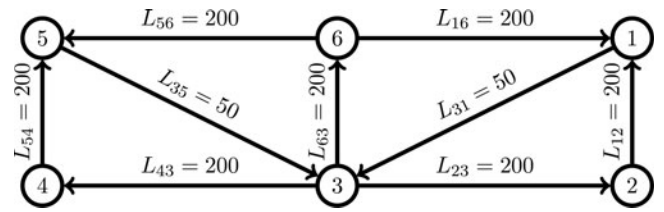

Figure 2. Interbank loans in Example 2, in thousands. Arrows in the direction of lending.

Table 4. Example 2: balance sheet components, in thousands, and outcomes

Node	Size	Equity	External debt	Cash	Risky assets	Portfolio return (%)	Payment fraction f^*	Sensitivity ζ
1	600	50	150	50	500	−40	0.723	0.635
2	1000	50	750	50	750	10	1	0
3	1000	25	875	50	350	−20	0.951	0.981
4	1000	50	750	800	0	0	0.994	0.996
5	600	50	150	50	500	−40	0.723	0.997
6	1000	50	750	50	550	10	0.994	0.996

Table 3 presents the systemic risk components for Example 1. The expected external creditors' loss is only 6000 for the upstream node, whereas it is 8000 for the downstream node, a result of the debt of the upstream node being only 75% of the external debt. The systemic risk components provided by other methods are discussed below, as those methods are introduced. For the present, we merely remark on the diversity of the attributions; just the opposite of the expected external creditors' loss, another method attributes 6000 to the downstream node and 8000 to the upstream node.

Example 2: This example is of a single scenario involving six nodes. The network structure and the interbank loans are shown in Fig. 2, and the other components of the balance sheets as well as the returns on the risky portfolios are summarized in Table 4. Nodes 1 and 3 have the smallest balance sheets and node 5 has a relatively small risky asset portfolio. In this scenario, only nodes 1, 3, and 5 incur losses in their risky asset portfolios. However, in addition to these three nodes, nodes 4 and 6 default, even though node 4 does not have any risky assets and node 6 makes a profit in its risky asset portfolio. A systemic event occurs due to losses incurred by relatively small nodes.

Table 5 presents the cost allocations for Example 2. Again, for the present, we merely comment on the diversity of the allocations. The system's cost is 134 571, but some methods attribute more than this to a single node, whereas other methods give a single node a negative allocation whose magnitude is close to the system's cost. Being a red node, node 6 has positive allocations under some methods and negative allocations under others. The allocations to a node under different methods can vary by more than an order of magnitude. The methods disagree about whether node 4 contributes to, mitigates, or does not affect cost. Its allocation is positive for the leverage method because it has a high leverage. Its allocation is zero under methods that focus on external assets, due to it having no risky assets. Its allocation is negative for other methods, due to it mitigating systemic risk by absorbing losses with its equity.

4.2. *External assets*

This balance-sheet scheme shows how systemic risk arises from the risks that nodes take in investing in external assets. The participation level λ_i scales the size of risky asset portfolio of node i; i.e., $\boldsymbol{\Theta}_{i\cdot}(\boldsymbol{\lambda}) = \lambda_i \boldsymbol{\Theta}_{i\cdot}$. Balance-sheet sizes and liabilities are fixed: $\varsigma_i(\boldsymbol{\lambda}) = \varsigma_i$, $\gamma_i(\boldsymbol{\lambda}) = \gamma_i$, $\delta_i(\boldsymbol{\lambda}) = \delta_i$, and $L_{ij}(\boldsymbol{\lambda}) = L_{ij}$. Cash substitutes for risky external assets: $\eta_i(\boldsymbol{\lambda}) = \varsigma_i - \sum_{j=1}^{n} L_{ji} - \sum_{j=1}^{m} \lambda_i \Theta_{ij}$. The external asset value $e_i(\boldsymbol{\lambda}) = \varsigma_i - \sum_{j=1}^{n} L_{ji} - \sum_{j=1}^{m} \lambda_i \Theta_{ij}(a_j - 1)$. As $\boldsymbol{e}(\mathbf{0}) \geq \boldsymbol{\delta}$, zero participation implies no defaults, so the cost function satisfies $c(\mathbf{0}) = 0$. This scheme can be applied with the Shapley or Aumann–Shapley value.

The cost function $c(\cdot)$ is not homogeneous. It is somewhat involved to compute the Aumann–Shapley value in Equation (1) by integrating ∇c along the "diagonal" $\{t\mathbf{1} : 0 \leq t \leq 1\}$, due to $c(\cdot)$ not needing to be differentiable everywhere on the diagonal, due to borderline nodes in counterfactual systems. As shown in Online Supplement section C.1, there are borderline nodes in the counterfactual system specified by participation vector $\boldsymbol{\lambda} = t\mathbf{1}$ only for finitely many values of $t \in [0, 1]$. Let these values be arranged in the decreasing sequence $1 = \tilde{t}_1 \geq \tilde{t}_2 \geq \ldots \geq \tilde{t}_m \geq \tilde{t}_{m+1} = 0$. For any other value of t, in the counterfactual system specified by participation vector $t\mathbf{1}$, there are sensitivities $\boldsymbol{\zeta}(t\mathbf{1}) = -\nabla_{\boldsymbol{e}} \ell(\boldsymbol{e}(t\mathbf{1}), \boldsymbol{\delta}(t\mathbf{1}), \boldsymbol{L}(t\mathbf{1}))$; cf. Equation (10). The sensitivities are piecewise constant in t, with points of discontinuity contained in the set $\{\tilde{t}_2, \ldots, \tilde{t}_m\}$, which can be found by a method explained in Online Supplement section C.1. Define

Table 5. Cost allocations in Example 2

Method		Node 1	Node 2	Node 3	Node 4	Node 5	Node 6
External creditors' loss	Both	41 572	0	42 517	4286	41 572	4625
External assets	Shapley	54 575	−2410	44 273	0	56 945	−18 812
	A–S	45 404	0	41 899	0	50 669	−3400
Transmission and leverage	Shapley	42 941	0	42 227	2383	45 307	1713
	A–S	41 739	0	41 878	4358	42 656	3940
Intermediation	Shapley	96 674	−27 561	44 571	−25 005	99 221	−53 327
	A–S	122 619	−27 714	44 575	−49 901	149 783	−104 791
Solvency	A–S	95 238	0	44 151	−49 801	149 567	−104 583
Absorption	A–S	150 000	−55 429	45 000	−50 000	150 000	−105 000
Funding	A–S	109 574	−35 193	44 918	−49 844	149 726	−84 609

$\omega_i = \tilde{t}_i - \tilde{t}_{i+1}$ and $\mu_i = (\tilde{t}_i + \tilde{t}_{i+1})/2$, the width and midpoint of the ith interval.

In finding the gradient of the cost function in Equation (14), because participation does not affect liability, the sensitivities of external asset value to participation are $\partial e_i(\boldsymbol{\lambda})/\partial\lambda_j = \sum_{j=1}^m \Theta_{ik}(a_k - 1)$ if $i = j$ or zero if $i \neq j$. Therefore, $\partial c(t\mathbf{1})/\partial\lambda_i = -(\sum_{j=1}^m \Theta_{ik}(a_k - 1))\zeta_i(t\mathbf{1})$. The Aumann–Shapley allocation to node i is given by

$$\phi_i^{\text{AS.ExA}} = \left(\sum_{j=1}^m \Theta_{ij}(1 - a_j)\right) \int_0^1 \zeta_i(t\mathbf{1})\,\mathrm{d}t = \left(\sum_{j=1}^m \Theta_{ij}(1 - a_j)\right)\left(\sum_{j=1}^m \omega_j \zeta_i(\mu_j \mathbf{1})\right).$$

The allocation weights each node's profit or loss on external assets by an average sensitivity. The sensitivity is averaged over counterfactual systems where profit or loss is scaled down to zero.

In Example 1, the external assets of the downstream node pose a greater systemic risk than do those of the upstream node. The reason for this behavior is that a loss in the upstream node may be partially absorbed by the equity of the downstream node, whereas any loss incurred by the downstream node is felt entirely by the external creditors. In this scheme, the stand-alone risk (see Section 2) of a node is the aggregate loss to external creditors when the other node's external assets are replaced by cash. The stand-alone risk of the downstream node is $6700 = 4\% \times 167\,500$ from the second scenario and, for the upstream node, the stand-alone risk $7200 = 4\% \times 50\,000 + 4\% \times 130\,000$ from the third and fourth scenarios. The systemic risk of the real system is 14 000, implying that the interaction of the two nodes via interbank loans has increased the overall systemic risk by $100 = 14\,000 - (6700 + 7200)$. The Shapley value splits the additional risk equally between the two nodes. However, the Aumann–Shapley value attributes less systemic risk to the downstream node than its stand-alone risk, due to its external assets making a profit in the third scenario. In this scenario, the downstream node's external assets protect its external creditors from losses.

In Example 2, the Aumann–Shapley value allocates zero cost to node 2, due to the system's cost being marginally insensitive to node 2's external assets: if the investment in these assets were perturbed, node 2 would still be green. However, node 2 is allocated a negative cost by the Shapley value, as if its external assets were replaced by cash, it would default in the absence of the profits. Although node 6 defaults, it receives a negative allocation in this scheme due to its external assets' profit. Either a large or small reduction in its investment in external assets would decrease the fraction of node 6's liabilities that it could pay, leading to larger losses for external creditors. This scheme allocates zero cost to any node that does not have any external asset, such as node 4.

4.3. *Transmission and leverage*

This balance-sheet scheme leads to attribution of systemic risk to the substitution of each node's liability for equity. There are two associated effects: the general risk-increasing effect of a node's leverage on its creditors and the potential to transmit losses to internal creditors. Balance-sheet sizes are fixed, $\varsigma_i(\boldsymbol{\lambda}) = \varsigma_i$. Each node is responsible for its liability: $\delta_i(\boldsymbol{\lambda}) = \lambda_i\delta_i$ and $L_{ij}(\boldsymbol{\lambda}) = \lambda_i L_{ij}$. Therefore, equity $\gamma_i(\boldsymbol{\lambda}) = \varsigma_i - \lambda_i(\delta_i + \sum_{j=1}^n L_{ij})$. Risky external assets are fixed, $\boldsymbol{\Theta}_{i\cdot}(\boldsymbol{\lambda}) = \boldsymbol{\Theta}_{i\cdot}$, and cash substitutes for internal assets: $\eta_i(\boldsymbol{\lambda}) = \varsigma_i - \sum_{j=1}^n \Theta_{ij} - \sum_{j=1}^n \lambda_j L_{ji}$. It follows that the external asset value $e_i(\boldsymbol{\lambda}) = e_i + \sum_{j=1}^n (1 - \lambda_j) L_{ji}$. This scheme can be used with the Shapley or Aumann–Shapley values, which are termed the transmission and the leverage methods, respectively, due to their different emphases in measuring systemic risk. The transmission method describes the impact of eliminating leverage, equivalently, eliminating the transmission of losses through loans. The leverage method describes the impact of perturbing each node's leverage.

The cost function used in the transmission method is the same as that of Drehmann and Tarashev (2013), although our risk measure is different. The external creditors of a non-participating node suffer no loss. A non-participating node transmits no losses internally: its internal creditors have cash in place of the internal loan asset. Therefore, the losses transmitted by a borrower to its non-participating internal creditor count for nothing. Thus, non-participation of a node is equivalent to elimination of the losses it transmits or receives, externally or internally.

The cost function $c(\cdot)$ is not homogeneous, so as in Section 4.2, the Aumann–Shapley value for node i takes the form

$$\phi_i^{\text{AS.Lvg}} = \int_0^1 \frac{\partial c}{\partial\lambda_i}(t\mathbf{1})\,\mathrm{d}t = \sum_{j=1}^m \omega_j \frac{\partial c}{\partial\lambda_i}(\mu_j\mathbf{1}).$$

Online Supplement section C.2 shows that there is a finite number m of intervals, and it shows how to compute the interval widths $\omega_1, \ldots, \omega_m$ and midpoints $\mu_1, \ldots, \mu_m$. It also derives the gradient formula $\partial c(\boldsymbol{\lambda})/\partial\lambda_i = \mathbb{1}\{i \in \mathcal{R}(\boldsymbol{\lambda})\}(\delta_i + \sum_{j=1}^n \zeta_j(\boldsymbol{\lambda}) L_{ij})$. This formula states that the cost is sensitive to the leverage of red nodes, in proportion to the external liability plus their internal liability, weighted by the marginal prices of wealth at their internal creditors.

In Example 1, the stand-alone risk of node 1 is 6700, due to the loss in the second scenario, and the stand-alone risk of node 2 is 6000, due to the losses in the third and fourth scenarios. Based on this scheme, the internal loan increases the systemic risk by $1300 = 14\,000 - (6700 + 6000)$. The transmission method—i.e., the Shapley value—splits the additional risk caused by this interaction equally between the two nodes. The leverage method—i.e., the Aumann–Shapley value—attributes more systemic risk to node 1 than does the Shapley value. Although the two nodes have the same leverage, node 1 has more external debt than does node 2, so perturbing the leverage of node 1 has a larger impact on the external creditors' loss.

In Example 2, the transmission and leverage methods are the only methods that allocate non-negative costs to all nodes in the network. Non-negativity may be a desired property of systemic risk attribution (Staum, 2015). This scheme yields a non-negative allocation, due to the leverage always increasing the systemic risk in this model.

4.4. *Intermediation*

This scheme plugs the holes left in balance sheets by substituting cash for internal loan assets and substituting external liability for internal liability. This substitution of liabilities keeps leverage constant, and it means that internal loans decrease systemic risk. An internal loan reduces external creditors' exposure to the borrower. Intermediation via internal loans can divert losses away from external creditors to other nodes and, at least in part, absorbed by equity instead of eventually reaching external creditors. This scheme differs from the one in Section 4.3 by attributing to internal loans the effect of protecting the borrowers' external creditors. The non-participation of a node corresponds to its removal from the network; it emits no loss to internal and external creditors and diverts the losses that it receives on internal loans to its borrowers' external creditors.

Each node's participation controls its size, and the proportions of risky assets and of equity are fixed; i.e., for all $i = 1, \dots, n$:

$$\varsigma_i(\boldsymbol{\lambda}) = \lambda_i \varsigma_i, \quad \boldsymbol{\Theta}_{i\cdot}(\boldsymbol{\lambda}) = \lambda_i \boldsymbol{\Theta}_{i\cdot}, \quad \text{and} \quad \gamma_i(\boldsymbol{\lambda}) = \lambda_i \gamma_i. \tag{15}$$

Once the internal liabilities $L_{ij}(\boldsymbol{\lambda})$ are specified, cash and external liability are determined by the accounting equation; that is,

$$\eta_i(\boldsymbol{\lambda}) = \lambda_i \left(\varsigma_i - \sum_{j=1}^{n} \Theta_{ij} \right) - \sum_{j=1}^{n} L_{ji}(\boldsymbol{\lambda}), \quad \text{and}$$

$$\delta_i(\boldsymbol{\lambda}) = \lambda_i \left(\delta_i + \sum_{j=1}^{n} L_{ij} \right) - \sum_{j=1}^{n} L_{ij}(\boldsymbol{\lambda}). \tag{16}$$

The intermediation scheme considers shared responsibility for internal liabilities: $L_{ij}(\boldsymbol{\lambda}) = \sqrt{\lambda_i \lambda_j} L_{ij}$. This scheme can be used with either the Shapley or the Aumann–Shapley value. The latter is the average of the schemes with borrower and lender responsibility (Sections 4.5 and 4.6).

With the balance sheet components specified in Equations (15) and (16), the Shapley value can be applied only with shared responsibility. Under shared responsibility, in a counterfactual system any non-participating node is absent, and so are any loans in which it is involved. Attributing responsibility to the borrower or lender only results in infeasible counterfactual systems. For example, consider borrower responsibility and a system where node i has lent $L_{ji} > 0$ to node j. In a counterfactual system where node i does not participate ($\lambda_i = 0$) but node j does ($\lambda_j = 1$), the balance sheet of node i has size $\varsigma_i(\boldsymbol{\lambda}) = 0$; however, it contains a loan to node j of size $L_{ji}(\boldsymbol{\lambda}) = L_{ji}$, which is impossible.

In Example 1, the stand-alone risk for the downstream node is 6700, due to the loss in the second scenario. The stand-alone risk is 8000 for the upstream node, due to losses in the third and fourth scenarios. Because the systemic risk for the real system is 14 000, the interaction of the two nodes via interbank lending reduces systemic risk by 700. The Shapley value splits this reduction equally between the two nodes. Although the absorption and solvency methods, presented in the following sections, use the Aumann–Shapley value, it is still helpful to compare their allocations to the stand-alone risks and the reduction in risk of 700 due to interaction. The absorption method, which assigns responsibility for internal loans to the lender, allocates to the upstream node its stand-alone risk, thus attributing the entire interaction to the downstream node, which is the lender. On the other hand, the solvency method, which assigns responsibility for internal loans to the borrower, attributes most (600) of the interaction to the upstream node, the borrower.

Example 2 illustrates the difference between the transmission and leverage methods, on the one hand, and the intermediation and related methods, on the other, in a network where each node is both a borrower and a lender. The transmission and leverage methods allocate costs that are small or zero to nodes 2, 4, and 6; the intermediation methods allocate large negative costs to those nodes, about twice as much to node 6 as to nodes 2 and 4. This is because nodes 2 and 4 absorb large losses on their lending to nodes 1 and 5, respectively, and node 6 absorbs losses from lending to both nodes 1 and 5. Based on the intermediation scheme, this protects external creditors who would have lent to nodes 1 and 5 if nodes 2, 4, and 6 had not lent to them. Accordingly, nodes 1 and 5 receive much larger allocations under the intermediation methods than they do under the transmission and leverage methods.

4.5. *Solvency*

In Section 4.4, although the Shapley value cannot be applied with borrower responsibility, the Aumann–Shapley value can, if the cash holdings in $\boldsymbol{\eta}$ are all strictly positive. This guarantees feasibility of the counterfactual systems in a neighborhood of the diagonal $\{t\mathbf{1} : 0 \le t \le 1\}$ (Staum, 2012). With borrower responsibility, internal liabilities are $L_{ij}(\boldsymbol{\lambda}) = \lambda_i L_{ij}$ and other components of the balance sheets are as specified in Equations (15) and (16). In Online Supplement section C.3, the allocation of Aumann–Shapley values all to node i is shown to be

$$\phi_i^{\text{AS.Sol}} = -\zeta_i \left(\gamma_i + \sum_{j=1}^{m} \Theta_{ij}(1 - a_j) \right), \quad i = 1, \dots, n.$$

This is proportional to the marginal price of wealth and net contribution of the node to the solvency of the system. The net contributions $\gamma_i + \sum_{j=1}^{m} \Theta_{ij}(1 - a_j)$ are equity plus profit or loss on external assets. If node i is green, it will be allocated zero cost, as its marginal price of wealth $\zeta_i = 0$.

In Example 2, the solvency method allocates very different costs of 95 238 and 149 567 to nodes 1 and 5, even though the balance sheets of these nodes are identical except for the identities of their creditors. The other methods discussed so far gave very similar allocations to these nodes. The difference between nodes 1 and 5 is simply their position in the network. One of the internal creditors of node 1 does not default, whereas all of the internal creditors of node 5 default. The extent to which losses transmitted by node 5 are more damaging is quantified by $\zeta_1 = 0.635$ and $\zeta_5 = 0.997$.

4.6. *Absorption*

In Section 4.4, although the Shapley value cannot be applied with lender responsibility, the Aumann–Shapley value can, if the external liabilities in $\boldsymbol{\delta}$ are all strictly positive. With lender responsibility, internal liabilities are $L_{ij}(\boldsymbol{\lambda}) = \lambda_j L_{ij}$ and other components of the balance sheets are as specified in Equations

(15) and (16). In Online Supplement section C.4, the Aumann–Shapley value is derived to be

$$\phi_i^{\text{AS.Abs}} = \left(1 - f_i^*\right)\left(\delta_i + \sum_{j=1}^{n} L_{ij}\right) - \sum_{j=1}^{n}(1 - f_j^*)L_{ji}, \quad i = 1, \ldots, n.$$

The first term contains the losses transmitted to external and internal creditors of each node. The second term contains the losses absorbed by each node in internal lending. For green nodes, the first term is zero, so the allocation is non-positive. It can be shown that if node i is red, the absorption method allocates to it $\phi_i^{\text{AS.Abs}} = -\gamma_i + \sum_{j=1}^{m} \Theta_{ij}(1 - a_j)$. The sum of equity and profit on external assets, $-\phi_i^{\text{AS.Abs}}$, is the contribution to the solvency of the system by node i.

The absorption method is unlike the solvency method in two regards. The absorption method gives credit to green nodes to the extent that they absorb losses on internal lending, preventing these losses from reaching external creditors. In the absorption method, any red nodes with identical balance sheets will be allocated the same cost, regardless of their positions in the network.

In Example 2, the absorption method gives the most extreme allocations. The only green node, node 2, receives a larger negative allocation under this scheme than under any other.

4.7. *Funding*

In Sections 4.4 to 4.6, internal liability and external liability are substitutes. That is, the allocation rests on the assumption about counterfactual systems that if an internal lender reduced its funding to a borrower, the borrower would replace the missing funding with external liability. As an alternative, one might assume that missing funding cannot be replaced. In the scheme developed in this section, there is no substitute for internal liability as a liability. Instead, in a counterfactual system where an internal loan is smaller, the borrower is smaller. The corresponding adjustment on the asset side of the borrower's balance sheet is made in cash.

Participation determines the balance sheets as follows. Each node's participation controls the amount of its equity, external liabilities, and external risky assets: $\gamma_i(\boldsymbol{\lambda}) = \lambda_i\gamma_i$, $\delta_i(\boldsymbol{\lambda}) = \lambda_i\delta_i$, and $\boldsymbol{\Theta}_{i\cdot}(\boldsymbol{\lambda}) = \lambda_i\boldsymbol{\Theta}_{i\cdot}$. Lender responsibility for internal loans, $L_{ij}(\boldsymbol{\lambda}) = \lambda_j L_{ij}$, makes size $\varsigma_i(\boldsymbol{\lambda}) = \lambda_i(\gamma_i + \delta_i) + \sum_{j=1}^{n} \lambda_j L_{ij}$. Cash is determined by the accounting equation.

It is not generally feasible to apply the Shapley value with this scheme. Suppose that in the actual system, node 1 lends 10 to node 2, which holds 5 in cash. Then eliminating node 1 would leave node 2 with negative cash, which is infeasible.

However, the Aumann–Shapley value can be applied if the cash holdings in $\boldsymbol{\eta}$ are all strictly positive. Online Supplement section C.5 shows that the Aumann–Shapley value is

$$\phi_i^{\text{AS.Fund}} = \left(1 - f_i^*\right)\delta_i + \left(1 - f_i^*\right)\varsigma_i \sum_{j=1}^{n} L_{ij} - \sum_{j=1}^{n}\left(1 - f_j^*\right)\varsigma_j L_{ji}, \quad i = 1, \ldots, n.$$

The first term contains the external creditors' loss, the second term relates to the losses transmitted to internal creditors, and the third term relates to the losses received on internal lending. The allocation to a green node consists only of the third term, which is non-positive. The result is similar to Section 4.6 but with the difference that in this scheme, the losses are weighted by the marginal prices of wealth at the borrower.

5. Choosing a method

We have explored several different methods for systemic risk attribution. There are several different methods, resulting from there being multiple causes of default, namely, leverage and risky investment, and there are multiple perspectives on responsibility for contagion, including borrower responsibility, lender responsibility, and shared responsibility. Which method should be used by a systemic risk manager?

Section 4 showed how different methods serve different purposes. It is possible to design a systemic risk attribution method that targets external assets, leverage, interbank borrowing, or other phenomena as the source of systemic risk. The appropriate design depends on what the risk manager wants to know or intends to change. For example, the external-assets scheme (Section 4.2) is designed to attribute systemic risk to assets outside the financial system that are held by financial firms. It reveals who is making the investments that endanger the system but does not reveal other aspects of contagion; it may attribute little systemic risk to a node that takes on little risk in its external assets but plays a large role in channeling losses through the financial network to external creditors. One may also desire a general-purpose method to use when there is no specific purpose in mind. One good general-purpose method is the transmission method (Section 4.3), introduced by Drehmann and Tarashev (2013). It focuses on the effects of transmission of losses, with the borrower and lender sharing responsibility for internal loans. Another good general-purpose method is the Shapley value in the intermediation scheme (Section 4.4). It is similar to the transmission method, but it also considers how internal loans can reduce systemic risk by channeling losses to firms in the system whose equity can help to absorb the losses, thus protecting the system's external creditors.

One consideration in choosing a method is whether to use the Shapley or Aumann–Shapley value. The Shapley value may seem to be expensive to compute in a network with a large number n of nodes. However, this is not a reason to avoid the Shapley value, which can be approximated efficiently by using a Monte Carlo approach (Castro *et al.*, 2009). As the Shapley value considers counterfactual systems that are very different from the real system, it is difficult to be confident that those counterfactual systems are well-specified. For example, in a richer model of the system than the one presented here, how would the interest rate paid by a firm change if it replaced all of its external assets with cash? The Aumann–Shapley value considers counterfactual systems that are more similar to the real system. If the cost function is homogeneous, it considers only perturbations of the real system.

One possible regulatory application of systemic risk attribution is to provide firms with incentives to lower systemic risk.

Difficult questions surround the attempt to do this for contagion (Staum, 2012, Section 2). However, if this is the goal, then the Shapley and Aumann–Shapley values each have some but not all of the desirable properties with respect to incentives (Staum, 2012, Section 6). One form of incentive is systemic risk charges based on systemic risk components. In such a regulatory framework, it might be desirable to have a method that ensures that the systemic risk components are non-negative, avoiding political problems that could arise from payments (negative systemic risk charges) made by the regulator to financial firms. Liu and Staum (2011) developed methods that yield non-negative systemic risk components based on Staum (2015), and the same approach could be applied to the schemes in Section 4.

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References

Boonen, T., de Waegenaere, A. and Norde, H. (2012) A generalization of the Aumann-Shapley value for risk allocation problems. Center discussion paper 2012-091, Tilburg University, Tilburg, The Netherlands.

Buch, A. and Dorfleitner, G. (2008) Coherent risk measures, coherent capital allocations and the gradient allocation principle. *Insurance: Mathematics and Economics*, **42**, 235–242.

Castro, J., Gómez, D. and Tejada, J. (2009) Polynomial calculation of the Shapley value based on sampling. *Computers and Operations Research*, **36**, 1726–1730.

Cherny, A. and Orlov, D. (2011) On two approaches to coherent risk contribution. *Mathematical Finance*, **21**, 557–571.

Denault, M. (2001) Coherent allocation of risk capital. *Journal of Risk*, **4**(1), 1–34.

Drehmann, M. and Tarashev, N. (2013) Measuring the systemic importance of interconnected banks. *Journal of Financial Intermediation*, **22**(4), 586–607.

Eisenberg, L. and Noe, T.H. (2001) Systemic risk in financial systems. *Management Science*, **47**(2), 236–249.

Elsinger, H. (2007) Financial networks, cross holdings, and limited liability. Available at ssrn.com/abstract=916763 (accessed October 6, 2010).

Gauthier, C., Lehar, A. and Souissi, M. (2010) Macroprudential regulation and systemic capital requirements. Working paper 2010-4, Bank of Canada, Ottawa, ON, Canada. Available at http://ssrn.com/abstract=1663993.

Liu, M. and Staum, J. (2010) Sensitivity analysis of the Eisenberg–Noe model of contagion. *Operations Research Letters*, **38**(5), 489–491.

Liu, M. and Staum, J. (2011) Systemic risk components and deposit insurance premia. Working paper 2011-03, Center for Financial Research, Federal Deposit Insurance Corporation. Available at http://ssrn.com/abstract=1982131. (Accessed December 10, 2010).

Liu, M. and Staum, J. (2012) Systemic risk components in a network model of contagion. Available at http://ssrn.com/abstract=1726107 (Accessed January 29, 2012).

Staum, J. (2012) Systemic risk components and deposit insurance premia. *Quantitative Finance*, **12**(4), 651–662.

Staum, J. (2013) Counterparty contagion in context: contributions to systemic risk, in *Handbook on Systemic Risk*, Fouque, J.-P. and Langsam, J. (eds), Cambridge University Press, Cambridge, UK.

Staum, J. (2015) Non-negative risk components, *Journal of Risk*, **18**(2), 1–29.

Tsanakas, A. (2009) To split or not to split: capital allocation with convex risk measures. *Insurance: Mathematics and Economics*, **44**, 268–277.