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Systems Engineering Procedia 4 (2012) 25 – 34

Procedia
Systems Engineering

The First 3-C Risk Forum & 2011 International Conference on Engineering and Risk Management (ERM)

Coherent Distortion Risk Measures in Portfolio Selection

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Abstract

The theme of this paper relates to solving portfolio selection problems using linear programming. We extend the well-known linear optimization framework for Conditional Value-at-Risk (CVaR)-based portfolio selection problems [1, 2] to optimization over a more general class of risk measure known as the class of Coherent Distortion Risk Measure (CDRM). CDRM encompasses many well-known risk measures including CVaR, Wang Transform measure, Proportional Hazard measure, and lookback measure. A case study is conducted to illustrate the flexibility of the linear optimization scheme, explore the efficiency of the 1/n-portfolio strategy, as well as compare and contrast optimal portfolios with respect to different CDRMs.

Keywords: linear programming; coherent risk measure, distortion risk measure; coherent distortion risk measure; portfolio selection

1. Introduction

The problem of optimal portfolio selection is of paramount importance to investors, hedgers, fund managers, among others. Inspired by the seminal work of [3], the research on optimal portfolio selection has been growing rapidly. Researchers and practitioners are constantly seeking better and more sophisticated risk and reward tradeoff in constructing optimal portfolios. The classical Markowitz model used variance as the benchmark for risk measurement and this is perceived to be undesirable since it penalizes equally, regardless of downside risk or upside potential. Consequently, other measures of risk have been proposed in connection to portfolio optimization. These include semi-variance [4], partial moments [5], safety first principle [6], skewness and kurtosis ([7] and [8]), value-at-risk (VaR) [9] and conditional value-at-risk (CVaR) [1].

Let $\rho(\mathbf{x})$ be a function which measures the riskiness of a portfolio $\mathbf{x}$. Then, in general, a portfolio selection problem seeks the solution to

$$\min_{\mathbf{x}\in\mathbf{S}} \rho(\mathbf{x}) \tag{1}$$

where the minimization is taken over all feasible portfolios $\mathbf{S}$. If ρ corresponds to the variance of the return of portfolio **x**, then **Error! Reference source not found.** reduces to the standard Markowitz model. In this paper, we

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2211-3819 © 2011 Published by Elsevier Ltd. Selection and peer-review under responsibility of Desheng Dash Wu.
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are concerned with the portfolio selection problem involving a particular class of risk measure known as the coherent distortion risk measure (CDRM). We demonstrate that the problem can be formulated as a linear programming and hence is computational amenable. CDRM is the intersection of two important families of risk measures: the coherent risk measure (CRM) [10] and the distortion risk measure (DRM) [11, 12]. To the best of our knowledge, [13] was the first to synchronize CRM and DRM and to study the intersection of both classes. CVaR is an example of CDRM while VaR is neither CRM nor DRM, and hence not CDRM.

The rest of the paper is organized as follows. Sections 2 and 3 review, respectively, the CVaR minimization approach developed in [1, 2] and the classes of CRM and DRM. Main contributions of the paper are collected in Section 4. We begin the section by first listing some properties associated with CDRM. We then generalize the finite generation theorem [14] for CDRM. We also show that any CDRM can be defined as a convex combination of ordered portfolio losses and equivalently a convex combination of CVaRs. We solve the CDRM-based portfolio optimization via linear programming and thus generalize the results of [1, 2]. Section 5 complements the paper by providing some numerical examples to illustrate the applicability of our proposed optimization problems. In particular, we implement four different CDRM-based portfolio models and these results are compared to the naive 1/n-portfolio strategy. Section 6 concludes the paper.

2. CVaR-Based Portfolio Optimization Model

Let $\boldsymbol{l} = f(\mathbf{x},\mathbf{y})$ be the portfolio loss associated with the decision vector $\mathbf{x}$, to be chosen from a set $\mathbf{S} \subseteq \mathbb{R}^n$, and the random vector $\mathbf{y} \in \mathbb{R}^m$. The vector $\mathbf{x}$ represents what we may generally call a portfolio, with $\mathbf{S}$ capturing the set of all feasible portfolios subject to certain portfolio constraints. For every $\mathbf{x}$, the loss $f(\mathbf{x},\mathbf{y})$ is a random variable having a distribution in $\mathbb{R}$ induced by the distribution of $\mathbf{y} \in \mathbb{R}^m$. The underlying probability distribution of $\mathbf{y}$ is assumed to be discrete with probability masses $\mathbf{p}$, i.e., $\Pr[\boldsymbol{l} = l(\mathbf{x}, y_i)] = p_i$ for $i = 1,\cdots,m$.

Note that in many cases it is assumed that X, i.e., the portfolio loss has a discrete uniform distribution. This is not a very limiting assumption if we restrict ourselves to discrete portfolio loss distributions, which is typically the case if we are obtaining distributional information via scenario generation or from historical data. In addition, given any arbitrary discrete distribution representable with rational numbers, we may always convert it to discrete uniform distribution for some large enough m. While we impose this assumption in our numerical example in Section 5, we emphasize that we do not rely on such assumption in our proposed risk measure based optimization framework to be discussed in Section 4.

For every portfolio $\mathbf{x}$, let $\Psi(\mathbf{x},\zeta) = \sum_{i=1}^m p_i \mathbf{1}_{\{l_i \le \zeta\}}$ denote the cumulative distribution function (cdf) of the portfolio loss $\boldsymbol{l} = f(\mathbf{x},\mathbf{y})$, then α-VaR and α-CVaR are defined as follows (see Proposition 8 in [2]):

Definition 2.1 *Suppose for each $\mathbf{x} \in \mathbf{S}$, the distribution of the portfolio loss $\boldsymbol{l} = f(\mathbf{x},\mathbf{y})$ is concentrated in $m < \infty$ points, and $\Psi(\mathbf{x},\cdot)$ is a step function with jumps at these points. Now fixing $\mathbf{x}$ and let $l_{(1)} < \cdots < l_{(m)}$ denote the corresponding ordered portfolio loss points and $p_{(i)} > 0, i = 1,\cdots,m$, represent the probability of realizing loss $l_{(i)}$. If $\min_{\mathbf{x}\in\mathbf{S}} \rho(\mathbf{x})$ denotes the unique index satisfying α then α-VaR and α-CVaR of the portfolio loss are given, respectively, by $\zeta_\alpha(\mathbf{x}) = l_{(i\alpha)}$ and*

$$\phi_\alpha(\mathbf{x}) = \frac{1}{1-\alpha}[(\sum_{i=1}^{i\alpha} p_{(i)} - \alpha) l_{i\alpha} + \sum_{i=i\alpha}^{m} p_{(i)} l_{(i)}] \tag{2}$$

As pointed out in [9], if ρ is set toVaR, then the resulting portfolio problem (1) is numerically challenging due to its lack of convexity. In contrast, the CVaR-based portfolio optimization problem (1) is a convex program and hence is computationally amenable. The CVaR-based portfolio model becomes even more popular and more practical when [1] showed that the convex program can in fact be formulated as a liner program. The key to Rockafellar-Uryasev's linear optimization scheme of CVaR-based portfolio selection problem is expressing $\phi_\alpha(\mathbf{x})$ and $\zeta_\alpha(\mathbf{x})$ in terms of the following special function:

$$F_\alpha(\mathbf{x},\zeta) = \zeta + \frac{1}{1-\alpha} E[(\boldsymbol{l}(\mathbf{x},\mathbf{y}) - \zeta)^+] = \zeta + \frac{1}{1-\alpha} \sum_{i=1}^{m} p_i (l_i - \zeta)^+ \tag{3}$$

As shown in [2], if $f(\mathbf{x},\mathbf{y})$ is convex with respect to $\mathbf{x}$, then $\varphi_\alpha(\mathbf{x})$ is convex with respect to $\mathbf{x}$. In this case, $F_\alpha(\mathbf{x},\zeta)$ is also jointly convex in $(\mathbf{x},\zeta)$. Armed with these findings, they derived the following equivalence formulation (Theorem 14 in [2]):

Theorem 2.1 *Minimizing* $\phi_\alpha(\mathbf{x})$ *with respect to* $\mathbf{x}\in\mathbf{S}$ *is equivalent to minimizing* $F_\alpha(\mathbf{x},\zeta)$ *over all* $(\mathbf{x},\zeta)\in\mathbf{S}\times\mathbb{R}$*, in the sense that*

$$\min_{\mathbf{x}\in\mathbf{S}}\phi_\alpha(\mathbf{x}) = \min_{(\mathbf{x},\zeta)\in\mathbf{S}\times\mathbb{R}} F_\alpha(\mathbf{x},\zeta) \tag{4}$$

where moreover

$$(\mathbf{x}^*,\zeta^*)\in\arg\min_{(\mathbf{x},\zeta)\in\mathbf{S}\times\mathbb{R}} F_\alpha(\mathbf{x},\zeta)\Leftrightarrow \mathbf{x}^*\in\arg\min_{\mathbf{x}\in\mathbf{S}}\phi_\alpha(\mathbf{x}),\ \zeta^*\in\arg\min_{\zeta\in\mathbb{R}} F_\alpha(\mathbf{x}^*,\zeta) \tag{5}$$

The above theorem links the representation (3) explicitly to both VaR and CVaR simultaneously. The theorem asserts that for the purpose of determining an optimal portfolio with respect to CVaR, we can replace $\phi_\alpha(\mathbf{x})$ by $F_\alpha(\mathbf{x},\zeta)$ in portfolio selection problems. More importantly, by exploiting (3) the general convex programming of CVaR portfolio optimization problem can be linearized into a linear objective function with an additional linear auxiliary constraints. With such linear representation we can cast any portfolio selection problem with CVaR objective and linear constraint(s) as a linear program.

3. Coherent Risk Measure (CRM) and Distortion Risk Measure (DRM)

The uncertainty for future value of an investment position is usually described by a function $X:\Omega\mapsto\mathbb{R}$, where Ω is a fixed set of scenarios with a probability space $(\Omega,\mathcal{F},\mathbb{P})$. Let $\mathcal{X}$ be a linear space of random variables on Ω, i.e., a set of functions $X:\Omega\mapsto\mathbb{R}$. Note that X can be thought of as a loss from an uncertain position.

It is well-known that a CRM satisfies properties of *monotonicity, translation invariance, positive homogeneity,* and *subaddivity* (see [10]) while a DRM satisfies properties of conditional *state independence, monotonicity, comonotonic additivity* and *continuity* (see [15]). The notion of comonotonicity is central in risk measures (see [16] and [17]). [15] imposed axiom comonotonic additivity based on the argument that the comonotonic random variables do not hedge against each other, leading to additivity of risks. They also proved (Theorem 3 in [15]) that if $\mathcal{X}$ contains all the *Bernoulli*(p) random variables, $0\le p\le 1$, then a DRM ρ satisfies $\rho(1)=1$ if and only if ρ has a Choquet integral representation with respect to a distorted probability; i.e.,

$$\rho_g(X)=\int Xd(g\circ\mathbb{P})=\int_{-\infty}^0[g(\mathbb{P}(X>x))-1]dx+\int_0^\infty g(\mathbb{P}(X>x))dx \tag{6}$$

where $g(\cdot)$ is known as the distortion function which is nondecreasing with $g(0)=0$ and $g(1)=1$ and $g\circ\mathbb{P}(A):=g(\mathbb{P}(A))$ is called the distorted probability. The Choquet integral representation of DRM can be used to explore its mathematical properties. Furthermore, calculations of DRMs can easily be done by taking the expected value of X under probability measure $\mathbb{P}^*:=g\circ\mathbb{P}$ (see Theorem 1 in [11] and Definition 4.2 in [12]). Here we list some commonly used distortion functions:

- CVaR distortion: $g_{CVaR}(x,\alpha)=\min\{x/(1-\alpha),1\}$ *with* $\alpha\in[0,1)$ (7)
- Wang Transform (WT) distortion: $g_{WT}(x,\beta)=\Phi[\Phi^{-1}(x)-\Phi^{-1}(\beta)]$ *with* $\beta\in[0,1)$ (8)
- Proportional hazard (PH) distortion: $g_{CVaR}(x,\alpha)=\min\{x/(1-\alpha),1\}$ *with* $\alpha\in[0,1)$ (9)
- Lookback (LB) distortion: $g_{CVaR}(x,\alpha)=\min\{x/(1-\alpha),1\}$ *with* $\alpha\in[0,1)$ (10)

The connection between CVaR and the distortion function (10) was observed in [18] while the remaining three distortion functions were proposed in [11], [19], and [20], respectively.

For discretely distributed portfolio losses random variable $\boldsymbol{l}=(l_1,\cdots,l_m)$ with probability masses $\Pr[\boldsymbol{l}=l_i]=p_i$ for $i=1,\cdots,m$, cdf $F_{\boldsymbol{l}}(l)=\sum_{i=1}^m p_i\mathbf{1}_{\{l_i\le l\}}$, and the survival function $S_{\boldsymbol{l}}(l)=1-F_{\boldsymbol{l}}(l)$ (6) becomes

$$\rho_g(\boldsymbol{l})=\int_{-\infty}^0[g(S_{\boldsymbol{l}}(l))-1]dl+\int_0^\infty g(S_{\boldsymbol{l}}(l))dl=E^*[\boldsymbol{l}]=\sum_{i=1}^m p_{(i)}^* l_{(i)} \tag{11}$$

where

$$p_{(i)}^*=\begin{cases}1-g(S_{\boldsymbol{l}}(l_{(1)})), & \textit{for } i=1\\ p_{(i)}^*=g(S_{\boldsymbol{l}}(l_{(i-1)}))-g(S_{\boldsymbol{l}}(l_{(i)})), & \textit{for } i=2,\cdots,m.\end{cases} \tag{12}$$

Since g is non-decreasing with $g(0)=0$ and $g(1)=1$, then $p_i^*\ge 0$ for $i=1,\cdots,m,$ and $\sum_{i=1}^m p_i^*=1-g(S_{\boldsymbol{l}}(l_{(m)}))=1$.

4. Coherent Distortion Risk Measure (CDRM)-based Portfolio Selection

Recall that CDRM is the intersection of CRM and DRM. There are two ways to derive and define CDRM:

Definition 4.1 *We say* ρ *is a coherent distortion risk measure (CDRM) if:*

- ρ_g is a distortion risk measure (DRM) with a concave distortion function g, see [13]; or equivalently,
- ρ is a coherent risk measure (CRM) that is also comonotonic and law-invariant[2], see [14].

The following representation theorem for CDRM is the key result that enables us to develope a convex optimization framework for any CDRM portfolio selection problem.

Theorem 4.1 *For any random variable X and a given concave distortion function g, risk measure ρ_g is a CDRM if and only if there exists a function $w:[0,1]\mapsto[0,1]$, satisfying $\int_{\alpha=0}^{1} w(\alpha)d\alpha = 1$, such that*

$$\rho_g(X) = \int_{\alpha=0}^{1} w(\alpha)\phi_\alpha(X)d\alpha \tag{13}$$

where $\phi_\alpha(X)$ is the α-CVaR of X.

This representation theorem says that any CDRM can be represented as a convex combination of $CVaR_\alpha(X)$, $\alpha\in[0,1]$ and we can construct any CDRM based on a convex combination of $CVaR_\alpha(X)$. Such result was proved by [21] for continuous portfolio loss distributions. [14] proved and strengthened the representation theorem that any CDRM can be represented as a convex combination of finite number of $CVaR_\alpha(X)$ under the assumption that the portfolio loss has a discrete uniform distribution. In addition to developing our convex programming formulation CDRM portfolio selection problem, we also generalize the finite generation theorem for CDRM to general discrete loss distributions. Before stating the main results of the paper, it is useful to state the following definition:

Definition 4.2 *For a given loss observation $\boldsymbol{l} = (l_1,\cdots,l_m)$ and its ordered losses $l_{(1)} < l_{(2)} < \cdots < l_{(m)}$. Let $p_{(i)}$ be the probability of realizing $l_{(i)}$, $i = 1,\cdots,m$ and let $S_l(l_{(i)}) = 1-\sum_{j=1}^{i} p_{(i)}$. Define a CVaR-matrix $\mathbf{Q}\in\mathbb{R}^m\times\mathbb{R}^m$ with columns $\mathbf{Q}_i\in\mathbb{R}^m$, $i=1,\cdots,m$ as*

$$\mathbf{Q} = [\mathbf{Q}_1,\mathbf{Q}_2,\cdots,\mathbf{Q}_m] = \begin{bmatrix} p_{(1)} & 0 & 0 & \cdots & 0 \\ p_{(2)} & \dfrac{p_{(2)}}{1-S_l(l_{(1)})} & 0 & \cdots & 0 \\ p_{(3)} & \dfrac{p_{(3)}}{1-S_l(l_{(1)})} & \dfrac{p_{(3)}}{1-S_l(l_{(2)})} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{(m)} & \dfrac{p_{(m)}}{1-S_l(l_{(1)})} & \dfrac{p_{(m)}}{1-S_l(l_{(2)})} & \cdots & \dfrac{p_{(m)}}{1-S_l(l_{(m-1)})} = 1 \end{bmatrix} \tag{14}$$

Since portfolio losses are discretely distributed at m points, there are m jumps in the cumulative function of $\boldsymbol{l}$. By defining

$$\alpha_i = \begin{cases} 0 & for\ i = 1 \\ \sum_{j=1}^{i-1} p_{(j)} & for\ i = 2,\cdots,m \end{cases} \tag{15}$$

at these m jumps, then the m CVaRs at these probability levels are given by

$$\varphi_{\alpha_i}(\boldsymbol{l}) = \frac{1}{1-\alpha_i}\sum_{j=i}^{m} p_{(j)}l_{(j)} = \sum_{j=i}^{m}\frac{p_{(j)}}{1-S_l(l_{(m-1)})}l_{(j)} = \sum_{j=i}^{m} Q_{ij}l_{(i)}, \tag{16}$$

for $i=1,\cdots,m$ and Q_{ij} is the (i,j)-th entry of $\mathbf{Q}$. Note that column $\mathbf{Q}_i$ is essential to the calculation of $CVaR_{(i-1)/m}(\boldsymbol{l})$ and hence explains the name of the matrix.

We now give a finite generation result for the CDRM, which generalizes Theorem 4.2 of [14] to general discrete loss distributions.

Theorem 4.2 *For a give portfolio loss sample $\boldsymbol{l} = (l_1,\cdots,l_m)$, the corresponding ordered losses $l_{(1)},\cdots,l_{(m)}$ and a given concave distortion function g, the resulting CDRM ρ_g is given by*

$$\rho_g(l) = \sum_{i=1}^{m} q_i l_{(i)} \tag{17}$$

where q_i, $i=1,\cdots,m$ are defined in (15). Moreover, every such $\mathbf{q}$ can be written in the form

$$\mathbf{q} = \mathbf{Q}\mathbf{w} \tag{18}$$

[2]Definition 4.5 in [14] should be defining CDRM as oppose to defining DRM.

where $\mathbf{w}^T = (w_1, \cdots, w_m)$ *denotes the convex weights satisfying* $w_i \ge 0$ *,* $i = 1, \cdots, m$ *, and* $\sum_{i=1}^m w_i = 1$ *, and* $\mathbf{Q}$ *corresponds to the CVaR-matrix (17). The convex weights* $\mathbf{w}$ *are given by*

$$w_i = \begin{cases} \dfrac{q_1}{p_{(1)}} & \text{if} \quad i = 1 \\ \left(q_i - \dfrac{p_{(i)}}{p_{(i-1)}} q_{i-1} \right) \dfrac{S_l(l_{(i-1)})}{p_{(i)}} & \text{if} \quad i = 2, \cdots, m. \end{cases} \tag{19}$$

We now make the following observations. First, it is easy to verify that the convex weights defined in (19) satisfy $w_i \ge 0$ for $i = 1, \cdots, m$ and $\sum_{i=1}^m w_i = 1$. Theorem 4.2 implies that every CDRM can be defined as a convex combination of the ordered losses $l_{(1)}, \cdots, l_{(m)}$ via (17) or equivalently as a convex combination of CVaRs via (18). The latter formulation is what we adopt in our CDRM portfolio optimization model.

Motivated by Theorem 2.1 and Theorem 4.2, we consider the following special function for some $w(\alpha) \ge 0$ and $\int_{\alpha=0}^1 w(\alpha) d\alpha = 1$.

$$M_g(\mathbf{x}, \zeta) = \int_{\alpha=0}^1 w(\alpha) F_\alpha(\mathbf{x}, \zeta_\alpha) d\alpha \tag{20}$$

Theorem 4.1 of CDRM ensures the existence of $w(\alpha)$, $\alpha \in [0,1]$ and defines CDRM for a given set of weights. For each α there is a corresponding auxiliary variable ζ_α. Taking partial derivatives w.r.t. all ζ_α and setting them equal to zeros give the extremal properties of $M_g(\mathbf{x}, \boldsymbol{\zeta})$. This provides more insights about the connection between a particular CDRM, $\rho_g(\mathbf{x})$, and its convex representation $M_g(\mathbf{x}, \boldsymbol{\zeta})$. Yet $\boldsymbol{\zeta}$ may have infinite many entries ζ_α. Taking partial derivative w.r.t. all ζ_α for $\alpha \in [0,1]$ requires calculus of variations, which is outside the scope of this thesis. We alleviate such difficulty by applying properties of Choquet integrals because CDRM is a subclass of DRM.

We conclude this section by presenting the key result of the paper. This generalizes the CVaR-based portfolio model of [1] to the more general class of CDRM-based portfolio model:

Theorem 4.3 *Let* $\rho_g(\mathbf{x})$ *be a CDRM with a corresponding distortion function* g *. Minimizing* $\rho_g(\mathbf{x})$ *with respect to* $\mathbf{x} \in \mathbf{S}$ *is equivalent to minimizing* $M_g(\mathbf{x}, \boldsymbol{\zeta})$ *over all* $(\mathbf{x}, \boldsymbol{\zeta}) \in \mathbf{S} \times \mathbb{R}^{|\boldsymbol{\zeta}|}$*, in the sense that*

$$\min_{\mathbf{x} \in \mathbf{S}} \rho_g(\mathbf{x}) = \min_{(\mathbf{x}, \boldsymbol{\zeta}) \in \mathbf{S} \times \mathbb{R}^{|\boldsymbol{\zeta}|}} M_g(\mathbf{x}, \boldsymbol{\zeta}) \tag{21}$$

where moreover

$$(\mathbf{x}^*, \zeta^*) \in \arg\min_{(\mathbf{x}, \boldsymbol{\zeta}) \in \mathbf{S} \times \mathbb{R}^{|\boldsymbol{\zeta}|}} M_g(\mathbf{x}, \boldsymbol{\zeta}) \Leftrightarrow \mathbf{x}^* \in \arg\min_{\mathbf{x} \in \mathbf{S}} \rho_g(\mathbf{x}), \zeta^* \in \arg\min_{\boldsymbol{\zeta} \in \mathbb{R}^{|\boldsymbol{\zeta}|}} M_g(\mathbf{x}^*, \boldsymbol{\zeta}) \tag{22}$$

Proof. Since CDRM is a subclass of DRM, all results of DRM and of Choquet integrals can be applied. In particular, one of the properties of Choquet integral states that if a random variable X_n has a finite number of values and converges to X, i.e., $X_n \xrightarrow{W} X$, then $\rho_g(X_n) \xrightarrow{W} \rho(X)$ provided that $\rho_g(X)$ exists,. This property implies that it is sufficient to prove the statement for the discrete random variables, and then carry over the result to the general continuous case. Consider a discrete portfolio loss random variable $\boldsymbol{l} = (l_1, \cdots, l_m)$ induced by the choice of portfolio $\mathbf{x} \in \mathbb{R}^n$ and the random vector $\mathbf{y} \in \mathbb{R}^m$; i.e. $l_i = l(\mathbf{x}, y_i)$. It follows from Theorem 4.2 that $\rho_g(\mathbf{x}) = \sum_{i=1}^m q_i l_{(i)} = \sum_{i=1}^m w_i \phi_{\alpha i}(\mathbf{x})$ Consider now the discrete analog of (23), i.e. $M_g(\mathbf{x}, \boldsymbol{\zeta}) = \sum_{i=1}^m w_i F_{\alpha i}(\mathbf{x}, \zeta_{\alpha i})$, where $F_{\alpha i}(\mathbf{x}, \zeta_{\alpha i})$ and α_i, $i = 1, \cdots, m$ are defined by (3) and (15) respectively. Since $F_{\alpha i}(\mathbf{x}, \zeta_{\alpha i})$, $i = 1, \cdots, m$ are all joint convex functions of $\mathbf{x}$ and $\zeta_{\alpha i}$ and $M_g(\mathbf{x}, \boldsymbol{\zeta})$ is a convex combination of $F_{\alpha i}(\mathbf{x}, \zeta_{\alpha i})$ for $i = 1, \cdots, m$, then $M_g(\mathbf{x}, \boldsymbol{\zeta})$ is a joint convex function of $\mathbf{x}$ and $\boldsymbol{\zeta}$.

For a given portfolio $\mathbf{x}$, we want to find $\boldsymbol{\zeta}^*$ that minimizes $M_g(\mathbf{x}, \boldsymbol{\zeta})$. Since $M_g(x, \zeta)$ is a convex function of $\boldsymbol{\zeta}$, we can simply set the gradient of $M_g(\mathbf{x}, \boldsymbol{\zeta})$ with respect to ζ equal to zero. This leads to

$$
\begin{aligned}
0 &= \frac{\partial M_g(\mathbf{x},\boldsymbol{\zeta})}{\partial \zeta} \\
0 &= \frac{\partial}{\partial \zeta_{\alpha i}} w_j[\zeta_{\alpha i} + \frac{1}{1-\alpha_i}\sum_{i=1}^{m} p_i (l_i - \zeta_{\alpha i})^+], \quad i = 1,\cdots,m \\
0 &= w_j[1 - \frac{1}{1-\alpha_i}\sum_{i=1}^{m} p_i \mathbf{1}_{(l_i - \zeta_{\alpha i})}], \qquad i = 1,\cdots,m \\
&\Leftrightarrow \begin{cases} \zeta_{\alpha i}^* \in [l_i, l_{i+1}) & \text{if} \quad w_i \neq 0 \\ \zeta_{\alpha i}^* \text{unconstrainted} & \text{if} \quad w_i = 0. \end{cases}
\end{aligned}
$$

Substituting these extremal conditions into $M_g(\mathbf{x},\boldsymbol{\zeta})$, we have

$$
\begin{aligned}
\min_{\boldsymbol{\zeta}\in\mathbb{R}^m} M_g(\mathbf{x},\boldsymbol{\zeta}) &= \sum_{i=1}^{m} w_i[\zeta_{\alpha i}^* + \frac{1}{1-\alpha_i}\sum_{j=1}^{m} p_j (l_j - \zeta_{\alpha i}^*)^+] = \sum_{i=1}^{m} w_i[\zeta_{\alpha i}^* + \frac{1}{1-\alpha_i}\sum_{j=i}^{m} p_{(j)} l_{(j)} - \frac{\sum_{j=i}^{m} p_{(j)}}{\alpha_i}\zeta_{\alpha i}^*] \\
&= \sum_{i=1}^{m} w_i[\frac{1}{1-\alpha_i}\sum_{j=i}^{m} p_{(j)} l_{(j)}] = \sum_{i=1}^{m} w_i \phi_{\alpha i}(\mathbf{x}) = \rho(\mathbf{x})
\end{aligned}
$$

The minimum value of $M_g(\mathbf{x},\boldsymbol{\zeta})$ is precisely $\rho_g(\mathbf{x})$ and such result holds for any portfolio $\mathbf{x}$. Therefore the equivalences in Theorem 4.3 hold. The proof is complete.

According to Theorem 4.3, we can replace $\rho_g(\mathbf{x})$ with $M_g(\mathbf{x},\boldsymbol{\zeta})$ in portfolio selection problems. Since $M_g(\mathbf{x},\boldsymbol{\zeta})$ is a joint convex function w.r.t $(\mathbf{x},\boldsymbol{\zeta})$, a portfolio selection problem therefore becomes a convex programming problem if the feasible set $\mathbf{S}$ is convex. Since Theorem 4.3 relates closely to the Rockafellar-Uryasev CVaR optimization approach, we can cast portfolio selection problems with CDRM objective/constraint(s) similar to those with CVaR objective/constraint(s).

5. Case Study: Optimal Investments with CDRMs

The "universe" of stocks in our studies consists of 20 stocks from S&P 500 with 2 stocks randomly selected from each of the 10 sectors defined in Global Industry Classification Standard (GICS). This is to ensure that the our universe of stocks mimics the market index. Weekly closing prices (adjusted for dividends and splits) from 02/01/2001 to 31/05/2011 (a total of 543 weeks; i.e. 542 weekly returns) for these stocks were obtained from finance.yahoo.com. Figure 1 shows the time series for the sum of these 20 stock prices. The market has been sufficiently volatile due to various events including the 9/11terrorist attack in 2001 and the sub-prime mortgage financial crisis (2007 to 2009). Instead of using scenario generations, we use historical data of stock returns as our samples and ues the corresponding empirical distributions.

Subsection 5.1 addresses the efficiency of 1/n-portfolio strategy, a portfolio strategy that is commonly used in practice. Subsection 5.2 then compares and contrasts the optimal portfolios arising from various specifications of CDRMs. All programming problems are solved with AMPL using the Gurobi 4.5.1 solver.

5.1. Efficiency of 1/n-Portfolio Strategy

The initial wealth of a 1/n-portfolio is invested equally, in monetary amount, in all available stocks. [22] observed that many participants in defined contribution pension plans used this simple strategy. For our selected universe of assets, we derive the efficient frontiers (Figure 2) at beginning of years 2003, 2005, 2007, and 2009 (defined as return v.s. 95%-CVaR) using the linear programming discussed in Section 2. Note that while [23] preformed an empirical study across 14 portfolio selection models and concluded that none of them consistently outperforms the 1/n-portfolio in terms of the Sharpe Ratio, yet Figure 2 suggests that the 1/n-portfolio (shown as solid diamond) is far from being efficient. In fact, three out of four times the 1/n-portfolio lie below the minimum-risk portfolio. Moreover, the 1/n-portfolio lies significantly farther from the efficient frontier in periods of market declines (beginning of years 2003 and 2009) than in periods of market increases (beginning of years 2005 and 2007). One possible explanation is that, assets are more correlated when the market performs poorly hence the benefit of risk diversification for 1/n-portfolio becomes the disadvantage of risk aggregation.

Figure 1: Time series plot of the selected portfolio

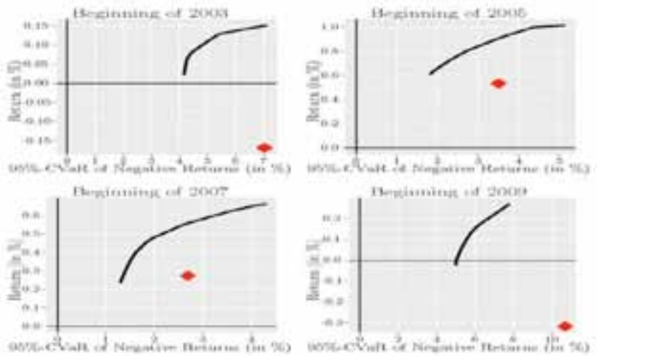

Figure 2: Efficiency of 1/n-Portfolio relative to Efficiency Frontier at different times.

5.2. Comparisons among Different CDRMs

We assume the initial portfolio consists of \$100 cash and is rebalanced weekly according to the respective optimal CDRM criterion. For each chosen CDRM (and subject to various constraints), we determine the minimum risk portfolios on 442 overlapping 100-week periods. We impose a budget constraint $\sum_{i=1}^{20} x_i = 1$, no-short selling constraints $\mathbf{x} \geq \mathbf{0}$, upper-limit constraints $\mathbf{x} \leq \mathbf{0.2}$ so that no more than 20% of the total portfolio value should be invested in one single stock, and a return constraint $R(\mathbf{x}) \geq \mu$ where μ is the expected return of the 1/n-portfolio. We compare the four members of CDRMs: CVaR measure (2) with $\alpha \in \{0.9, 0.95, 0.99\}$, Wang Transform (WT) (8) with $\beta \in \{0.75, 0.85, 0.95\}$, Proportional Hazard (PH) transform (9) with $\gamma \in \{0.1, 0.5, 0.9\}$, and Lookback (LB) measure (10) with $\delta \in \{0.1, 0.5, 0.9\}$. We implement the linear programming of CVaR portfolio model using the approach of [1]. For the latter three risk measures, we use Theorem 4.3 to determine the optimal portfolios. In these cases, we need to determine weight vectors $\mathbf{q}$ and $\mathbf{w}$ (see Figure 3). The summary statistics for the expected returns and realized returns of the optimal portfolios under each CDRM risk measure are listed in Table 1.

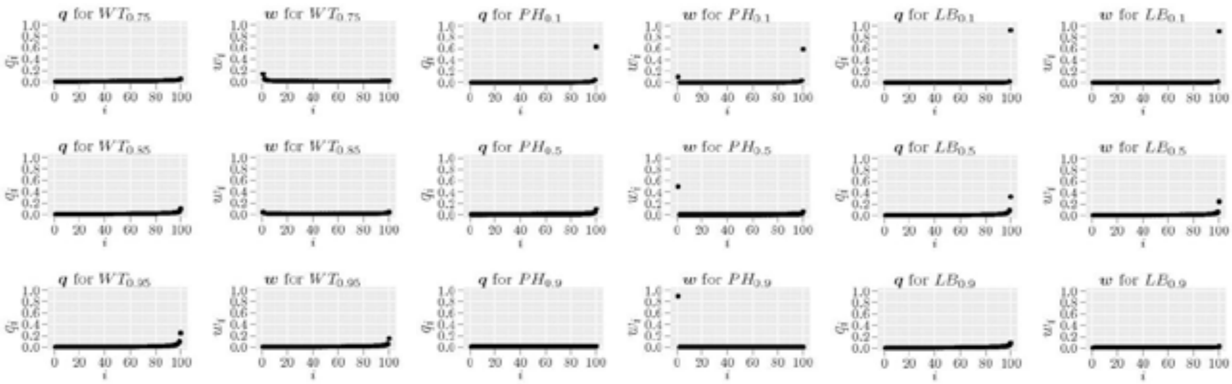

Figure 3: The respective weights q and w for WT, PH and LH

- It is of interest to note that the PH portfolio optimization is almost equivalent to optimizing over two extreme CVaR-based portfolios: one with $\alpha = 0.99$ and the other with $\alpha = 0$. Recall that minimizing CVaR with high value of α implies that you are someone who is very risk averse and hence is interested in risk minimization. In contrast, minimizing CVaR with α close to 0 implies an investor is risk seeker and is only interested in maximizing expected return.
- Consistent with the classical tradeoff theory on risk and reward, a more risk averse investor seeks an optimal portfolio with lower risk (as measured by the respective CDRM) but at the expense of lower expected return. Hence the expected return of the optimal portfolio decreases with α for CVaR, decreases with β for WT,

increases with γ for PH, and increases with δ for LB. The reported Sharpe ratios (assuming zero risk-free interest rate) in Table 1 are also consistent with these observations.

- Figure 4 produces the out-of-sample realized returns (on the constructed optimal portfolios) over 442 overlapping 100-week periods. We observe that while an optimal portfolio with a higher risk is compensated with a higher expected return, the portfolio does not necessary lead to higher realized out-of-sample return, as confirmed in some of these graphs.

Table 1Summary statistics for the returns of optimal portfolios w.r.t various CDRMs

	Expected	*Realized*	*Expected*	*Realized*	*Expected*	*Realized*
CVaR	$\alpha = 0.90$		$\alpha = 0.95$		$\alpha = 0.99$	
Mean	0.0025	0.0015	0.0023	0.0012	0.0021	0.0014
Standard Deviation	0.0021	0.0189	0.0023	0.0205	0.0026	0.0224
Skewness Coefficient	0.3975	-0.9370	0.1961	-0.5674	-0.1758	-0.2081
Kurtosis Coefficient	0.5856	6.0820	0.1382	4.9651	0.2113	4.4711
Sharpe Ratio	1.1710	0.0783	0.9738	0.0572	0.7853	0.0622
WT	$\beta = 0.75$		$\beta = 0.85$		$\beta = 0.95$	
Mean	0.0031	0.0016	0.0026	0.0014	0.0023	0.0014
Standard Deviation	0.0022	0.0192	0.0021	0.0191	0.0023	0.0211
Skewness Coefficient	-0.0019	-1.0024	0.3737	-0.7753	0.1095	-0.3052
Kurtosis Coefficient	-0.2304	7.0607	0.3865	5.8864	0.3137	5.4681
Sharpe Ratio	1.4267	0.0856	1.2493	0.0748	0.9964	0.0663
PH	$\gamma = 0.1$		$\gamma = 0.5$		$\gamma = 0.9$	
Mean	0.0021	0.0013	0.0030	0.0015	0.0061	0.0028
Standard Deviation	0.0026	0.0222	0.0023	0.0209	0.0035	0.0262
Skewness Coefficient	-0.1909	-0.2616	-0.1962	-0.8342	1.5106	-0.9574
Kurtosis Coefficient	0.3613	5.1429	-0.1318	8.5093	3.9670	6.7800
Sharpe Ratio	0.8243	0.0584	1.2824	0.0709	1.7294	0.1057
LB	$\delta = 0.1$		$\delta = 0.5$		$\delta = 0.9$	
Mean	0.0021	0.0013	0.0023	0.0014	0.0026	0.0014
Standard Deviation	0.0026	0.0223	0.0024	0.0213	0.0021	0.0189
Skewness Coefficient	-0.1683	-0.2288	0.0358	-0.3401	0.4153	-0.8040
Kurtosis Coefficient	0.2944	4.6000	0.2819	5.1539	0.4736	6.0423
Sharpe Ratio	0.7970	0.0600	0.9559	0.0644	1.2609	0.0764

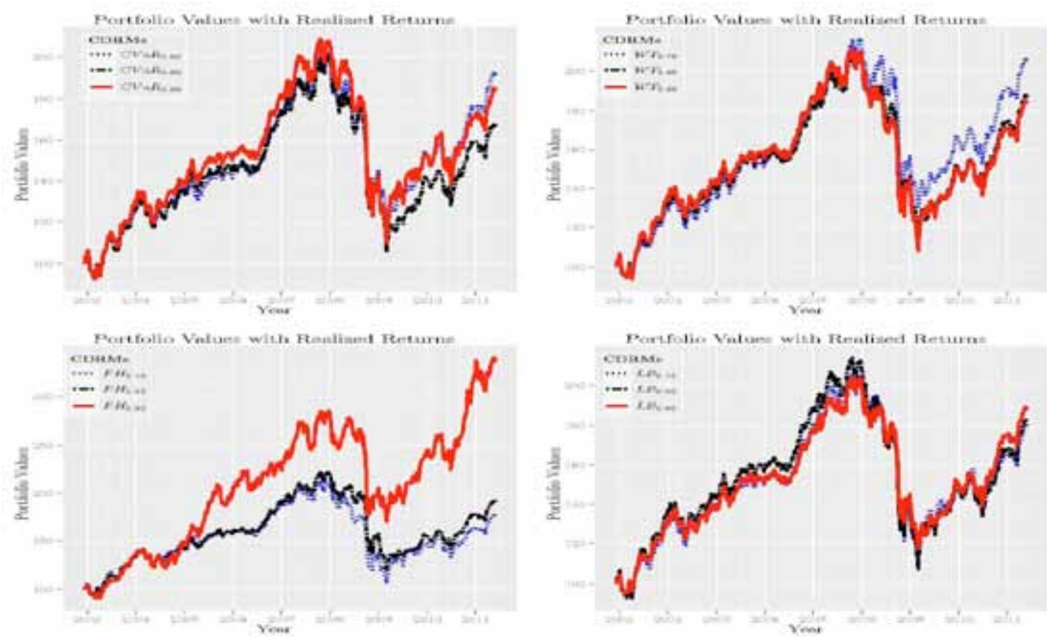

Figure 4 Time Series Plots for Various CDRM Optimal Portfolios

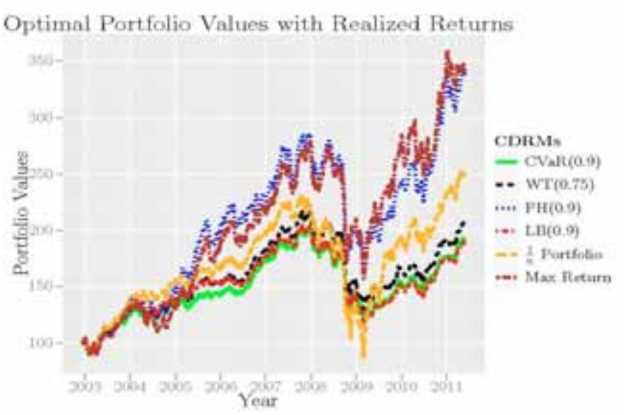

Figure 5: Time Series Plots of Optimal Portfolio Values.

Table 2 Summary Statistics for 1/n-Portfolio and Return Maximization Portfolio

	1/n-Portfolio		*Max Return Portfolio*	
	Expected	Realized	Expected	Realized
Mean	0.00168	0.00208	0.00639	0.00279
Standard Deviation	0.00279	0.03038	0.00352	0.02947
Skewness Coefficient	-0.33372	0.25175	1.52513	-0.73268
Kurtosis Coefficient	0.64192	13.73943	4.02045	4.95543
Sharpe Ratio	0.60129	0.06854	1.81415	0.09480

To conclude our analysis, we perform a final comparison among the best performing portfolios in the aforementioned four members of CDRMs to the 1/n-portfolio, and the return maximization portfolio. Note that we can also view the return maximization problem as a CDRM minimization problem by minimizing $CVaR_0$ of the negative returns. The resulting time series are plotted in Figure 5. The summary statistics for 1/n-portfolio and profit maximization portfolio is given in Table 2.

We see that for our selection of stocks, the return maximization produces the best portfolio in terms of its terminal wealth and the mean portfolio returns. However, this should not be a practical recommendation for portfolio manager because it might bear unacceptably high risks. For instance, the standard deviation of the expected returns for return maximization portfolios is the highest among all portfolios we have discussed in this section. Investors should consider their own risk appetites and choose an appropriate risk measure in every investment decision. It is also of interest to note that the performance of PH with $\gamma = 0.9$ is very similar to the return maximization strategy. This should not be surprising since with such a high value of γ, the PH-based portfolio model is similar to return maximization.

6. Concluding Remarks

This paper extended the well-known linear optimization framework for CVaR (see [1]) to a general class of risk measure known as the CDRM. We first generalized the finite generation theorem for CDRM in [14] and showed that any CDRM can be defined as a convex combination of ordered portfolio losses and equivalently a convex combination of CVaRs. We make use of the latter to develop a CDRM-based portfolio optimization framework. We solved CDRM-based portfolio optimization via linear programming, which could handle problems with large number of variables and/or constraints. A case study was conducted on constructing a portfolio consisting of 20 S&P 500 stocks. Our empirical analysis suggested the importance of active risk management since naive portfolio construction, such as the 1/n-portfolio strategy, can be very inefficient. We also compared and contrasted optimizations over four different members of CDRM. Our numerical shows that different CDRMs reflecte different risk appetites and hence different optimization focuses. Choosing risk measures wisely enables risk managers to achieve higher Sharpe ratios and/or other risk adjusted returns on their investments.

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