All work and no play makes Jack enumerate maps

Michael La Croix PhD

University of Waterloo

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Outline

My Perspective

- I'm a mathematician.
- I study combinatorial enumeration.
- I think a lot of problems are best understood via pictures.

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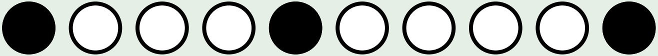
Example

How many ways are there to arrange 7 black and 3 white marbles in a row?



Generating Series (or Partition Functions)

A **generating series** is an algebraic tool for recording a sequences of numbers. Using such tools, counting problems become algebra problems.

Example

If $a_{n,k}$ is the number of ways to arrange n black and k white marbles, then

$$\sum_{n,k\geq 0} a_{n,k} x^n y^k = \sum_{n,k\geq 0} \binom{n+k}{k} x^n y^k = \frac{1}{1-x-y}$$

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Why Should We Count Things?

- It could be the key to computing probabilities.
- It is a first step in listing all the possible cases for a case analysis.
  - By counting we may implicitly describe how to list the objects.
  - A count lets us check that our list is complete.
- The behaviour of a sequence of numbers can describe physical properties.
  - This is the idea behind **Statistical mechanics**, a branch of physics that qualitatively models phase transitions.

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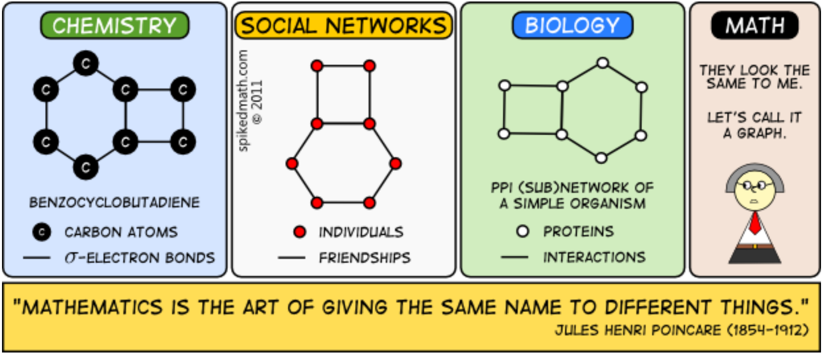
What is mathematics ?

Mathematics is a way of abstractly studying relationships between objects.

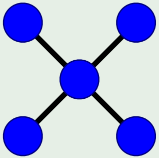


From the webcomic **Spiked Math** at `spikedmath.com`

Extra Information

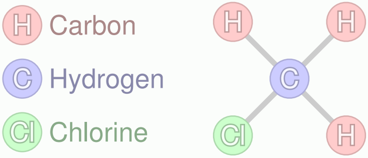
By studying the abstraction, we can more easily recognize what extra information is essential to the structure of a problem.

Example (A Graph)



The graph $K_{4,1}$.

Example (Chemistry)



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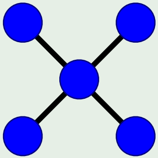

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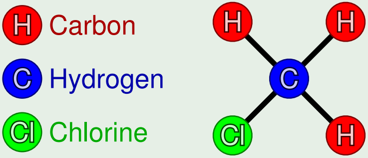

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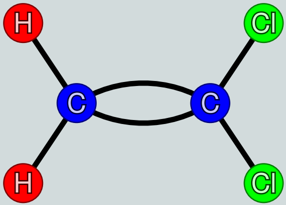
Labelled Vertices

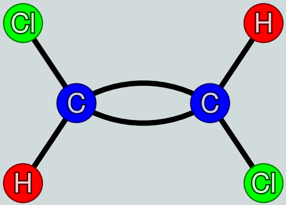
With labelled graphs, we can distinguish between some kinds of isomers.

Example (Isomers of dichloroethene)

1,1-dichloroethene



1,2-dichloroethene



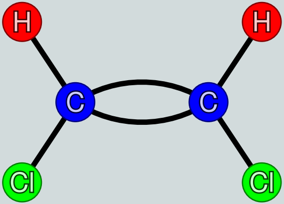
The 1, 1 and 1, 2 isomers of $C_2H_2Cl_2$ are represented by different graphs.

Embeddings

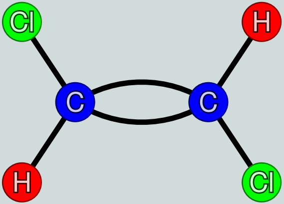
For other isomers, we actually need to draw the graphs.

Example (cis-trans isomers)

cis-1,2-dichloroethene



trans-1,2-dichloroethene



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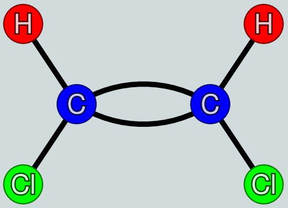

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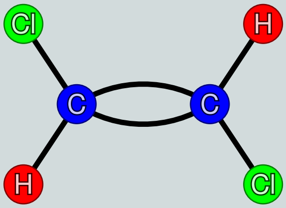
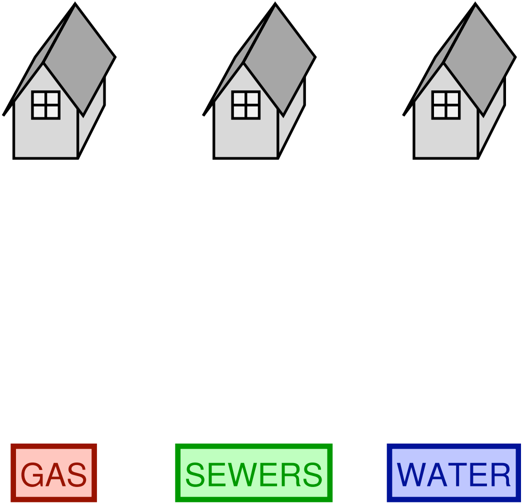


cis-trans isomers have the same graph, but different embeddings.

Conclusion: Sometimes the way a graph is drawn is as important as the graph itself.

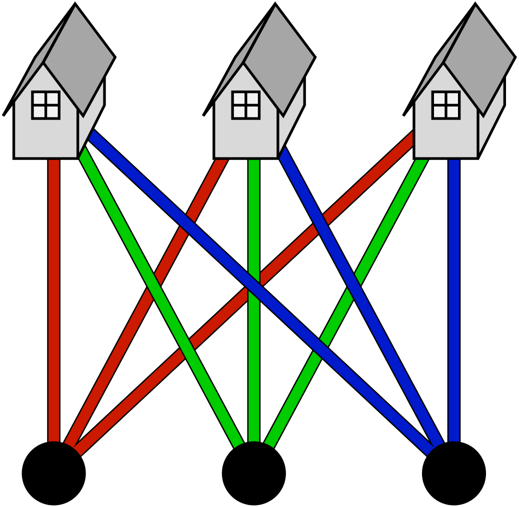
Three Utilities



A classical challenge is how to place conduits so that three utilities can be connected to three houses.

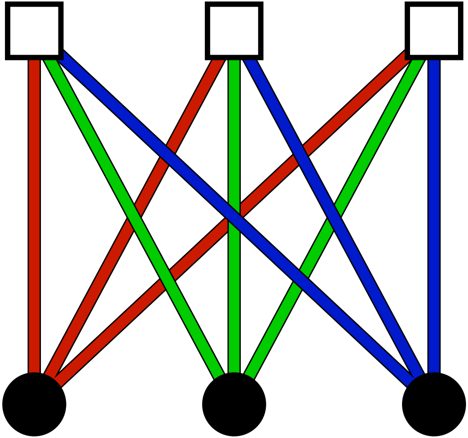
This should be done so that no conduits cross.

Three Utilities



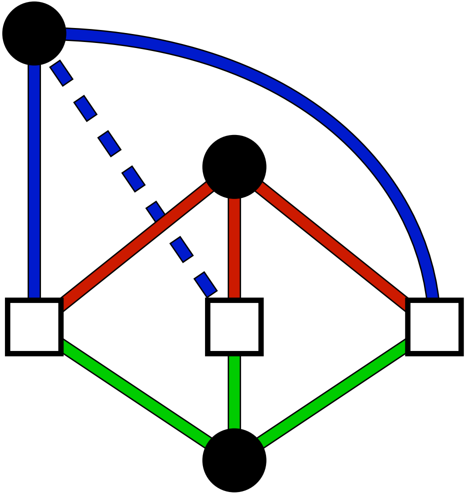
Together, the houses, utilities, and conduits define a graph.

Three Utilities

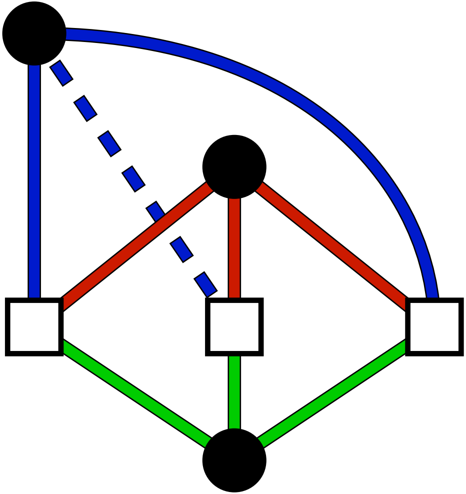


We want to embed the graph.

Three Utilities



After a lot of effort, we conclude that the problem cannot be solved as it appears to be stated.



After a lot of effort, we conclude that the problem cannot be solved as it appears to be stated.

This is actually a classic result (Kuratowski's Theorem). The given graph is one of two obstacles to being able to draw a graph on the plane.

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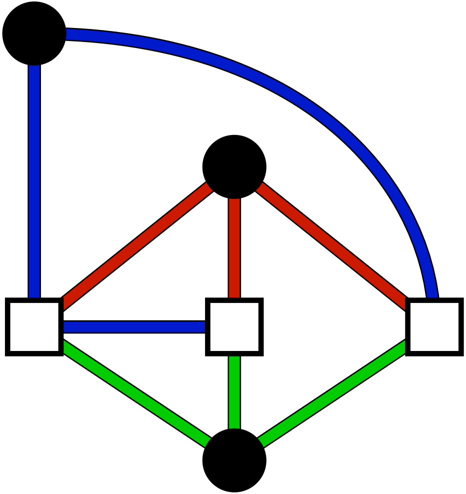

Solving the problems relies on finding a loop-hole in its statement.

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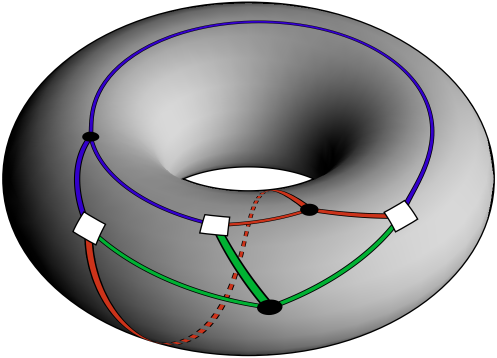
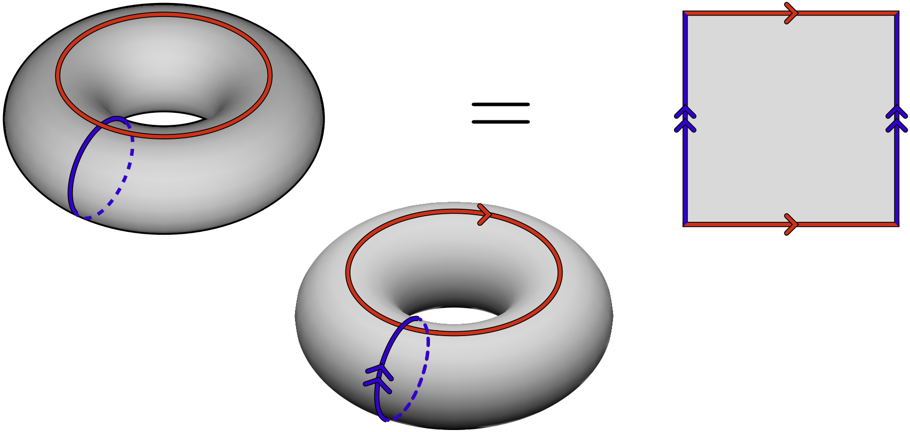


Solving the problems relies on finding a loop-hole in its statement.

One solution is that, as stated, the problem does not say that the houses are on a plane. We can draw them on a torus.

The torus (or any surface) can be represented schematically in terms of the surgery required to stitch it together from a rubber sheet.



Representing Surfaces

The torus (or any surface) can be represented schematically in terms of the surgery required to stitch it together from a rubber sheet.

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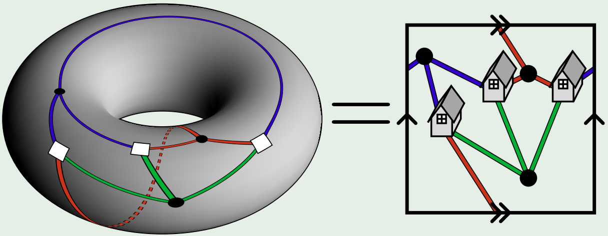
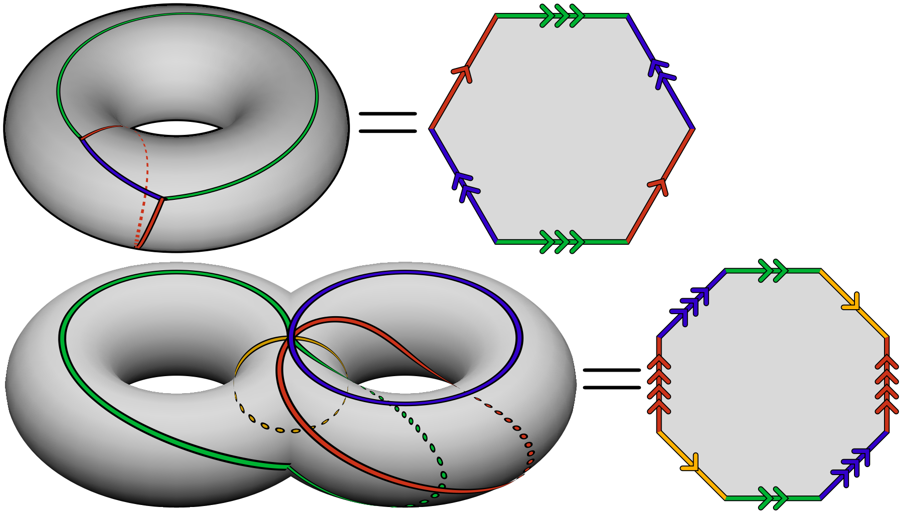


Other Surfaces are Also Obtained by Surgery



Graphs, Surfaces, and Maps

Definition

A **surface** is a compact 2-manifold without boundary.

Definition

A **graph** is a finite set of *vertices* together with a finite set of *edges*, such that each edge is associated with either one or two vertices.

Definition

A **map** is a 2-cell embedding of a graph in a surface.

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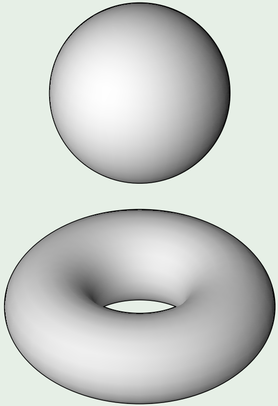

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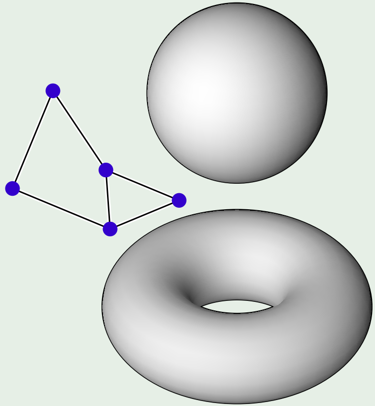

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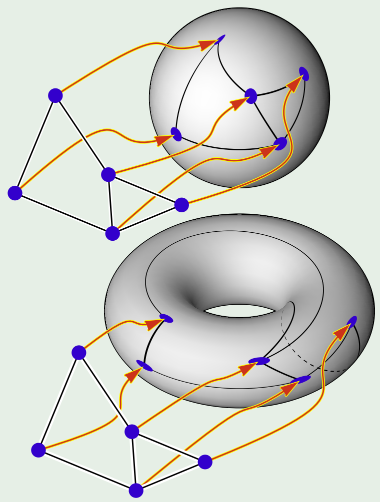
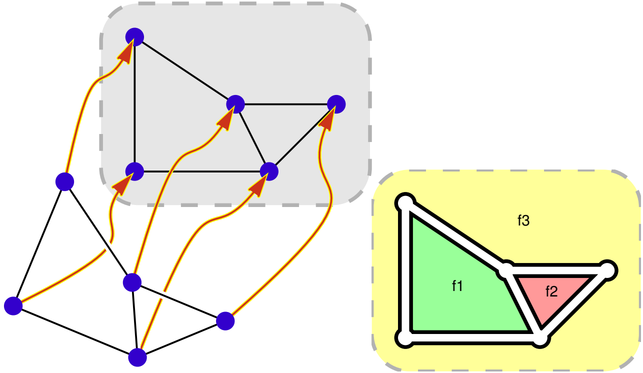


Maps and Faces

Once a graph is drawn, the unused portion of the paper is split into faces.



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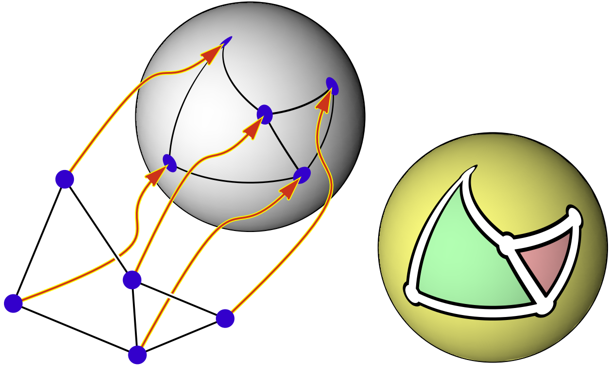

For symmetry, the outer face is thought of as part of a sphere.

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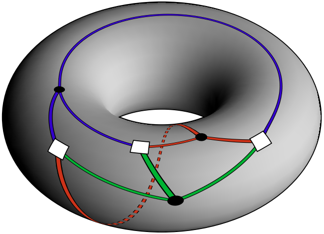
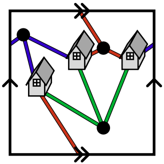
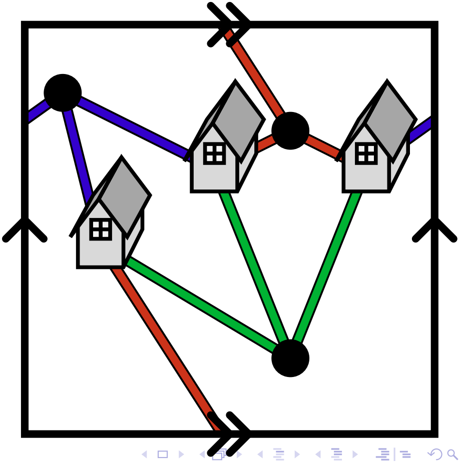
Tiling the Representation

The faces of a map can be made more evident by tessellating the tile that represents the surface.







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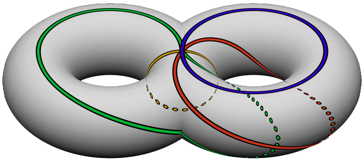

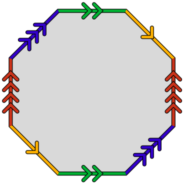

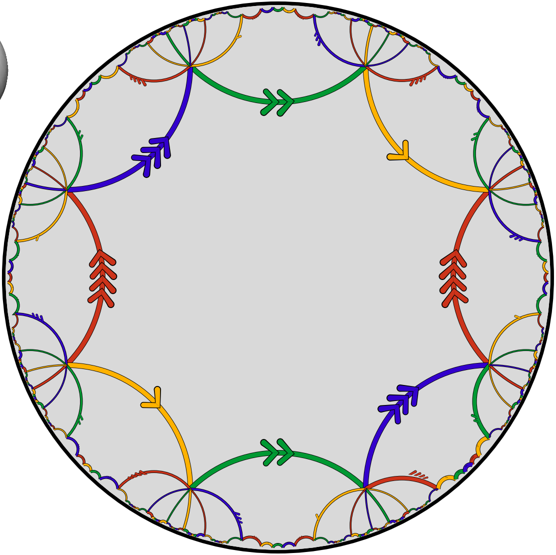

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Ribbon Graphs

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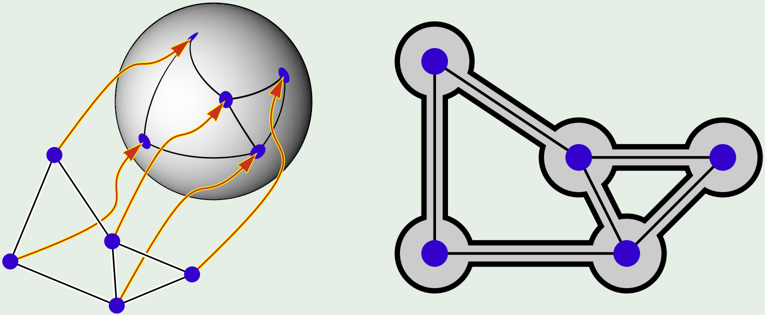

The homeomorphism class of an embedding is determined by a neighbourhood of the graph.

Ribbon Graphs

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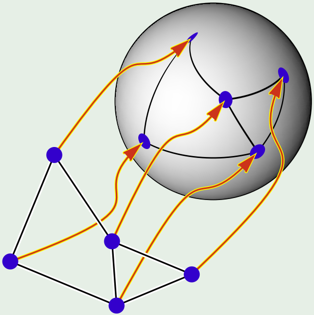

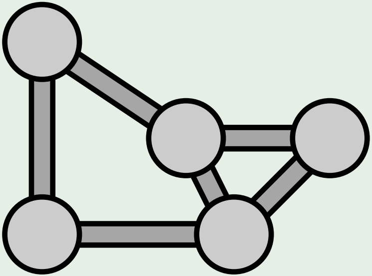

Neighbourhoods of vertices and edges can be replaced by discs and ribbons to form a ribbon graph.

Flags

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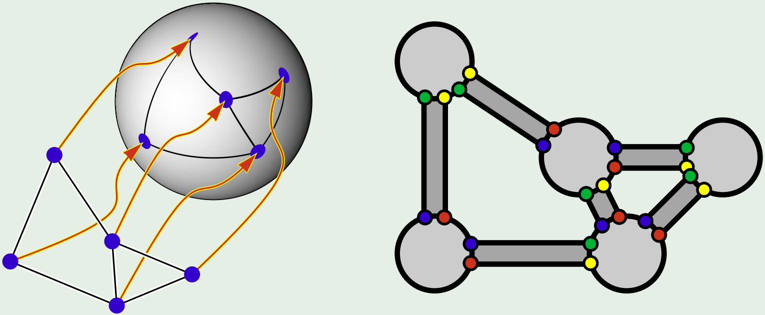

The boundaries of ribbons determine flags.

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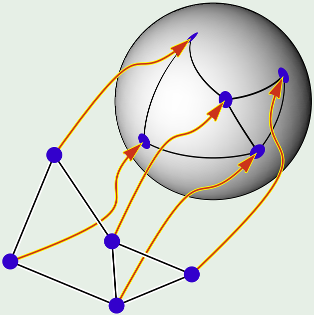

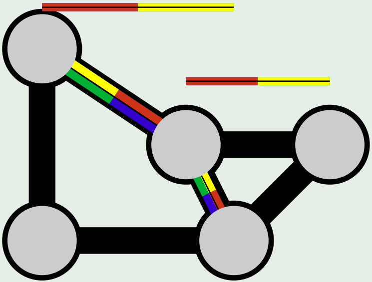

The boundaries of ribbons determine flags, and these can be associated with quarter edges.

Rooted Maps

Definition

A **rooted map** is a map together with a distinguished orbit of flags under the action of its automorphism group.

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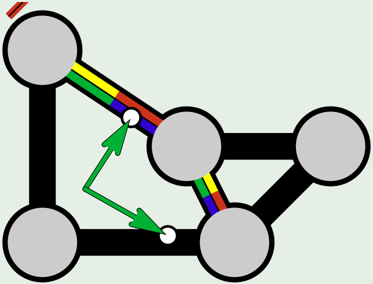

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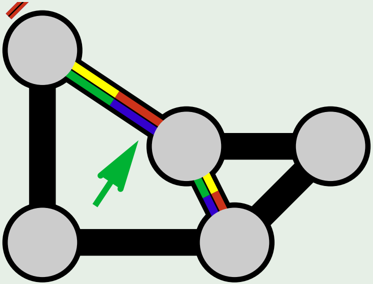

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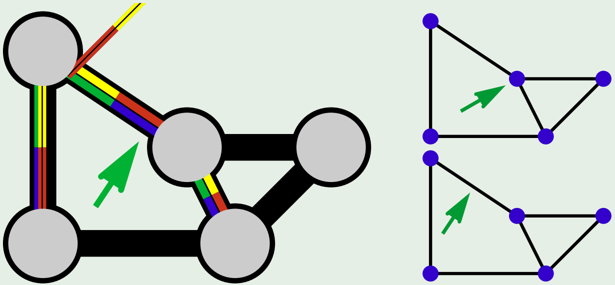

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2-dimensional Quantum Gravity

Two models of 2-dimensional quantum gravity are analyzed by enumerating rooted orientable maps.

- The Penner Model involves all smooth maps.
- $\phi - 4$ model involves only 4-regular maps.

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The models have the same behaviour.

An algebraic explanation - A remarkable identity

Theorem (Jackson and Visentin)

$$Q(u^2, x, y, z) = \tfrac{1}{2}M(4u^2, y+u, y, xz^2) + \tfrac{1}{2}M(4u^2, y-u, y, xz^2)$$
$$= \mathsf{bis}_{\textit{even}\ u}\ M(4u^2, y+u, y, xz^2)$$

M is the genus series for rooted orientable maps, and
Q is the corresponding series for 4-regular maps.

$$M(u^2, x, y, z) := \sum_{\mathfrak{m}\in\mathcal{M}} u^{2g(\mathfrak{m})} x^{v(\mathfrak{m})} y^{f(\mathfrak{m})} z^{e(\mathfrak{m})}$$
$$Q(u^2, x, y, z) := \sum_{\mathfrak{m}\in\mathbb{Q}} u^{2g(\mathfrak{m})} x^{v(\mathfrak{m})} y^{f(\mathfrak{m})} z^{e(\mathfrak{m})}$$

$g(\mathfrak{m})$, $v(\mathfrak{m})$, $f(\mathfrak{m})$, and $e(\mathfrak{m})$ are genus, #vertices, #faces, and #edges

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$$= \mathsf{bis}_{\mathit{even}\ u}\ M(4u^2, y+u, y, xz^2)$$

The right hand side is a generating series for a set $\bar{\mathcal{M}}$ consisting of elements of $\mathcal{M}$ with

- each handle decorated independently in one of 4 ways, and
- an even subset of vertices marked.

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q-Conjecture (Jackson and Visentin)

The identity is explained by a **natural** *bijection ϕ from $\bar{\mathcal{M}}$ to $\mathcal{Q}$.*

A decorated map with

- v *vertices*
- $2k$ *marked vertices*
- e *edges*
- f *faces*
- *genus* g

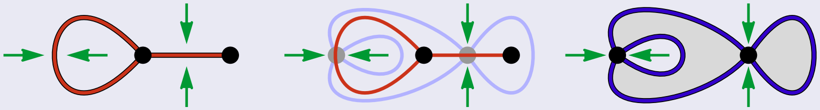
$$\xrightarrow{\phi}$$

A 4-regular map with

- e *vertices*
- $2e$ *edges*
- $f+v-2k$ *faces*
- *genus* $g+k$

Two Clues

The radial construction for undecorated maps



One extra image of ϕ

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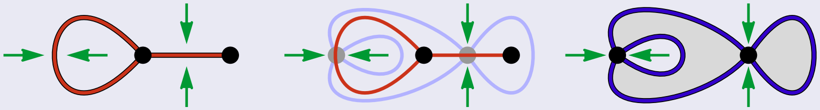

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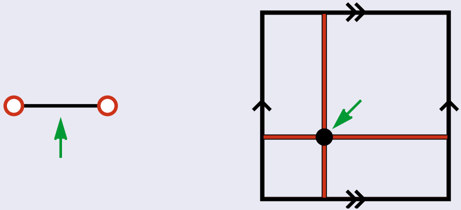

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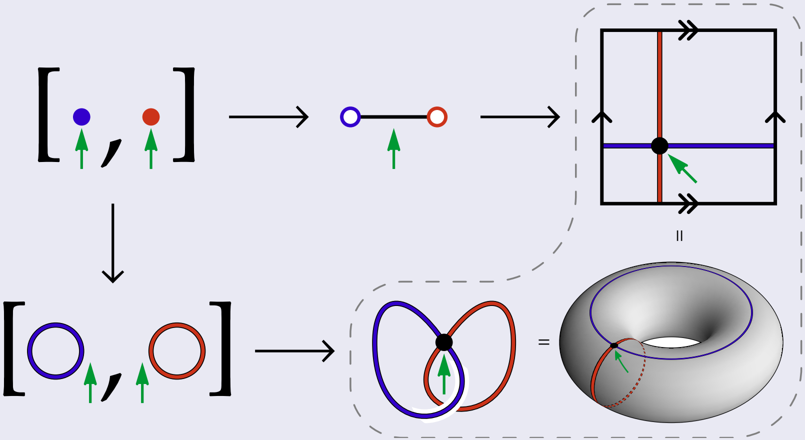


A refined q-Conjecture

Conjecture (La Croix)

There is a natural bijection ϕ from $\bar{\mathcal{M}}$ to $\mathcal{Q}$ such that:

A decorated map with

- v *vertices*
- $2k$ *marked vertices*
- e *edges*
- f *faces*
- *genus* g

$$\xrightarrow{\phi}$$

A 4-regular map with

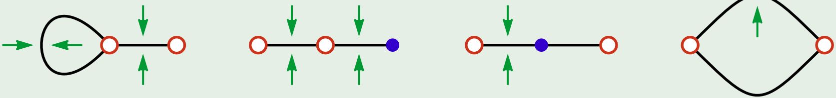
- e *vertices*
- $2e$ *edges*
- $f + v - 2k$ *faces*
- *genus* $g + k$

and

the root edge of $\phi(\mathfrak{m})$ is face-separating *if and only if* *the root vertex of $\mathfrak{m}$ is not decorated.*

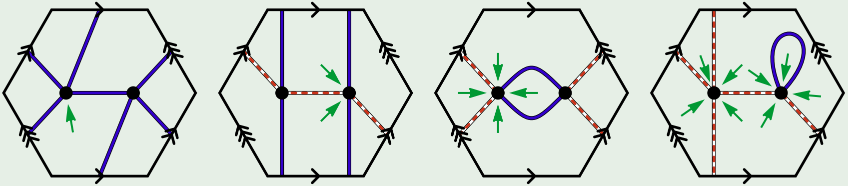
Root vertices in $\bar{\mathcal{M}}$ are related to root edges in $\mathcal{Q}$

Example (planar maps with 2 edges and 2 decorated vertices)



Nine of eleven rooted maps have a decorated root vertex.

Example (4-regular maps on the torus with two vertices)



Nine of fifteen rooted maps have face-non-separating root edges.

Outline

The Map Series

Extra symmetry makes it easier to work with a more refined series.
An enumerative problem associated with maps is to determine the number of rooted maps with specified vertex- and face- degree partitions.

Definition

The **map series** for a set $\mathcal{M}$ of rooted maps is the combinatorial sum

$$M(\mathbf{x}, \mathbf{y}, z) := \sum_{\mathfrak{m} \in \mathcal{M}} \mathbf{x}^{\nu(\mathfrak{m})} \mathbf{y}^{\phi(\mathfrak{m})} z^{|E(\mathfrak{m})|}$$

where $\nu(\mathfrak{m})$ and $\phi(\mathfrak{m})$ are the the vertex- and face-degree partitions of $\mathfrak{m}$.

Example

Rootings of are enumerated by $(x_2^3\, x_3^2)\,(y_3\, y_4\, y_5)\, z^6$.

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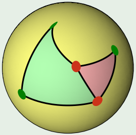
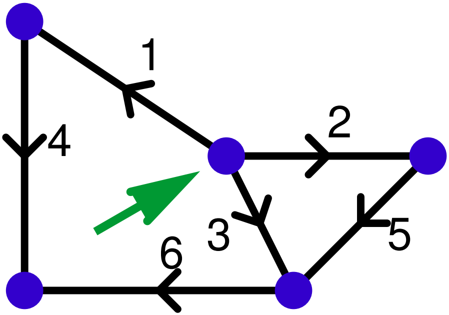
Rootings of  are enumerated by $(x_2^3\, x_3^2)\,(y_3\, y_4\, y_5)\, z^6$.

Encoding Orientable Maps

1. Orient and label the edges.
2. This induces labels on flags.
3. Clockwise circulations at each vertex determine ν.
4. Face circulations are the cycles of $\epsilon\nu$.



$$\epsilon = (1\,1')(2\,2')(3\,3')(4\,4')(5\,5')(6\,6')$$

$$\nu = (1\,2\,3)(1'\,4)(2'\,5)(3'\,5'\,6)(4'\,6')$$

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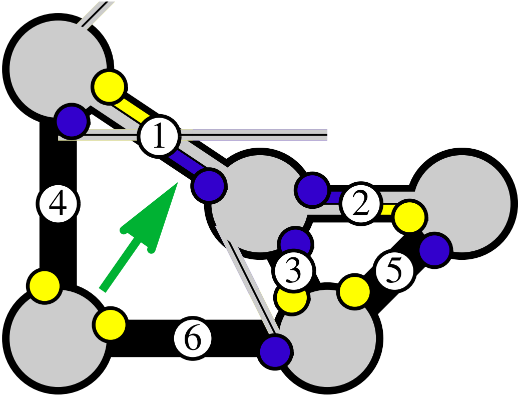

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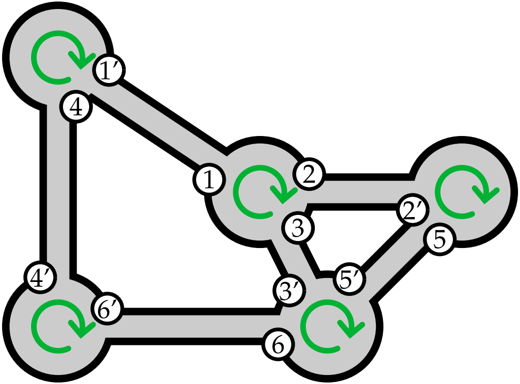

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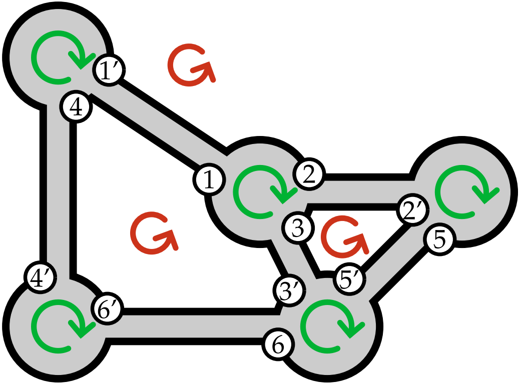

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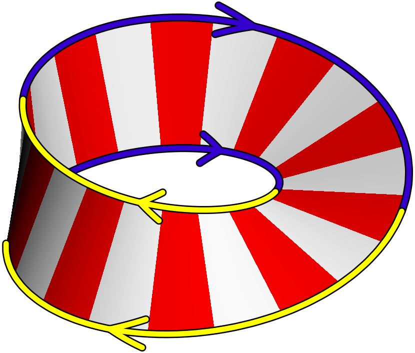
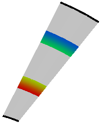
A Möbius Strip

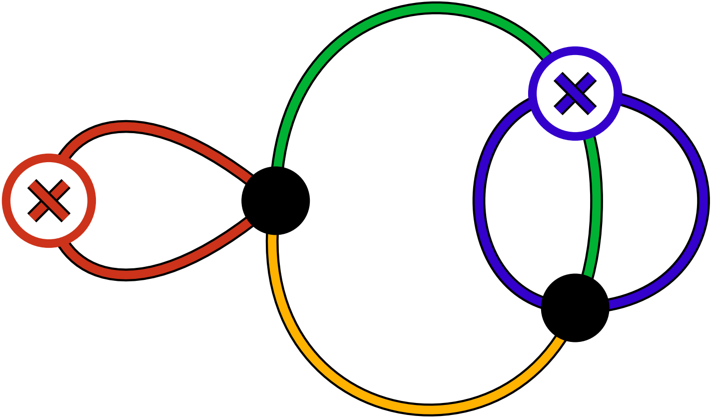
Maps can also be drawn in surfaces that contain Möbius strips.

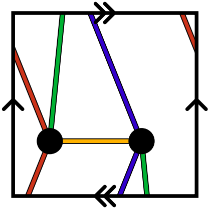




Encoding Locally Orientable Maps

A new encoding is needed to record twisting.





Start with a ribbon graph.

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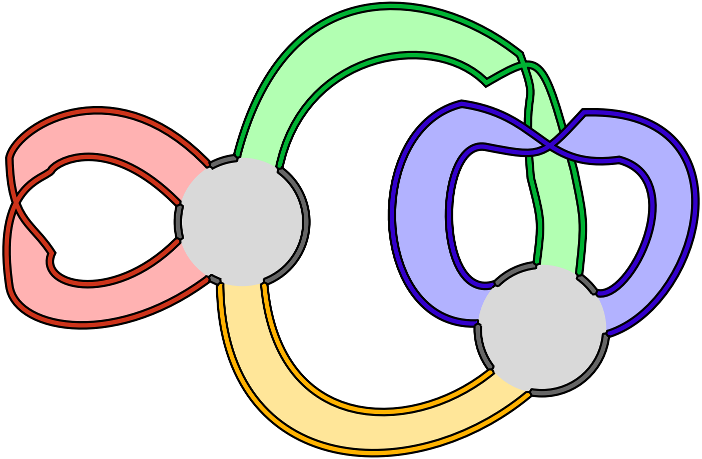

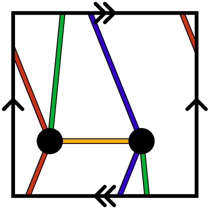

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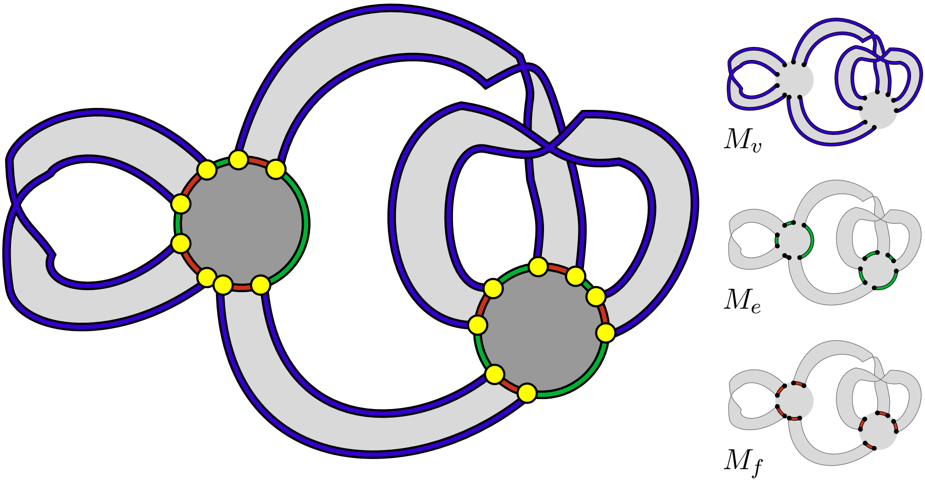

Ribbon boundaries determine 3 perfect matchings of flags.

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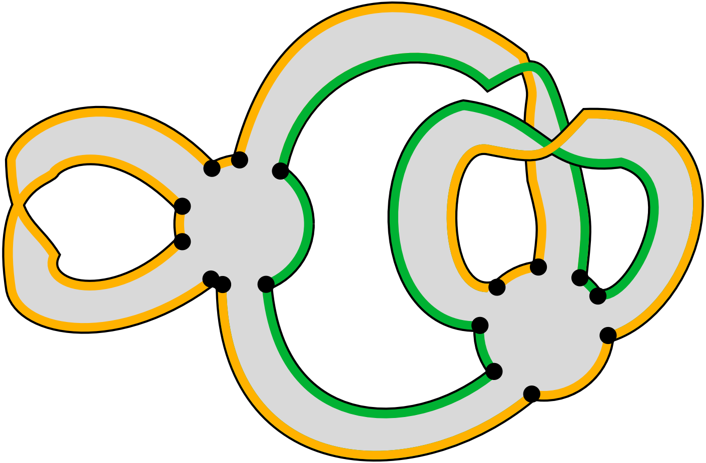
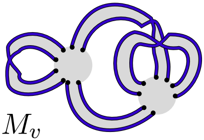
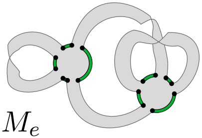




M_v



M_e

Pairs of matchings determine, faces,

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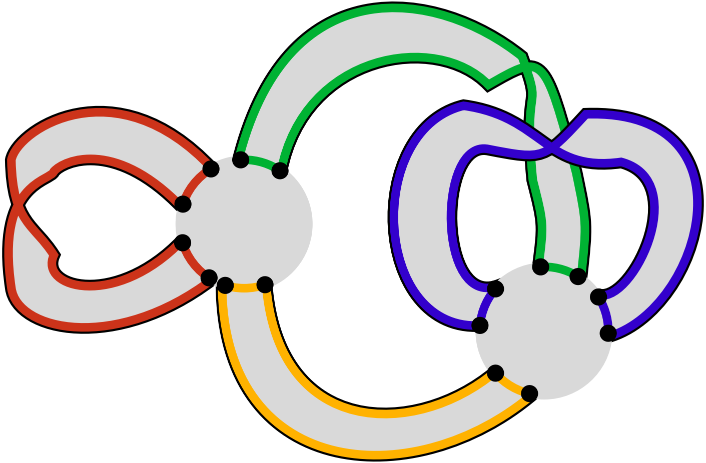

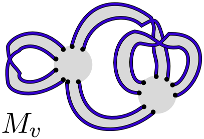

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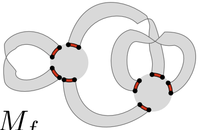

M_f

Pairs of matchings determine, faces, edges,

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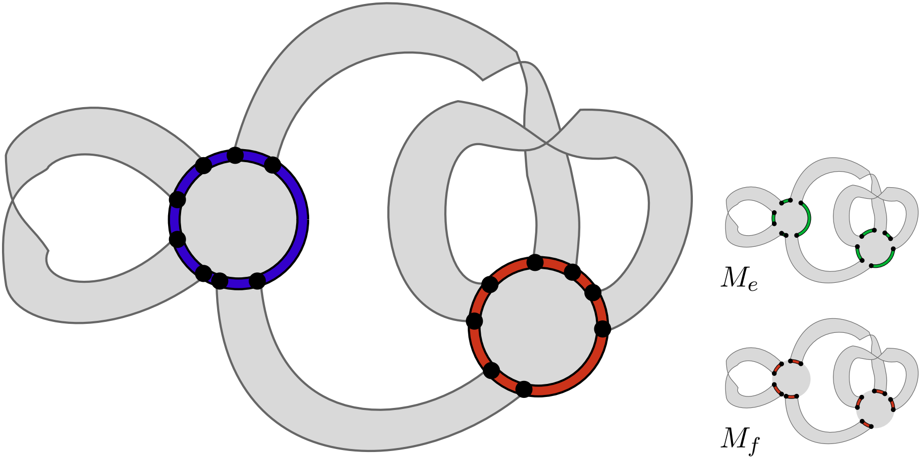

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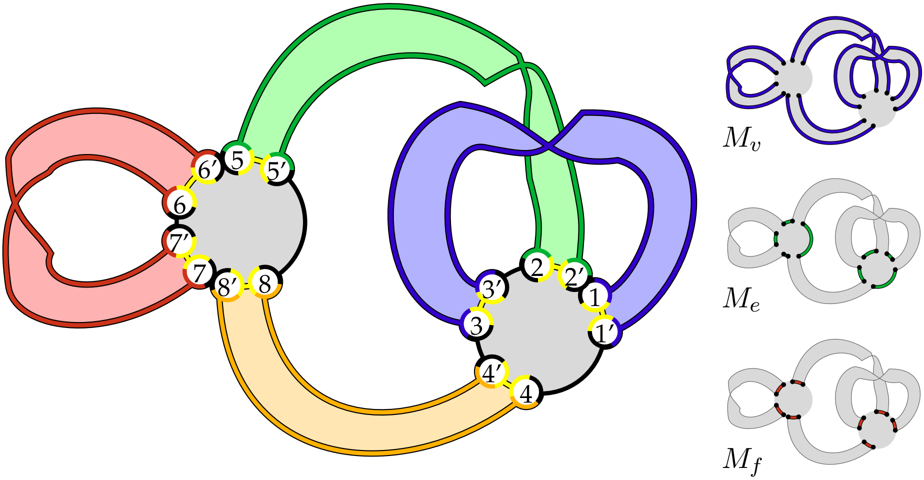


$$M_v = \{\{1,3\},\{1',3'\},\{2,5\},\{2',5'\},\{4,8'\},\{4',8\},\{6,7\},\{6',7'\}\}$$

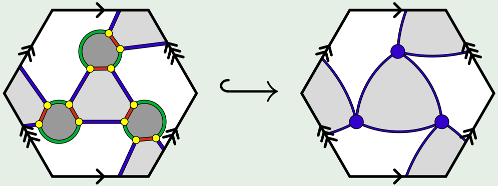
$$M_e = \{\{1,2'\},\{1',4\},\{2,3'\},\{3,4'\},\{5,6'\},\{5',8\},\{6,7'\},\{7,8'\}\}$$

$$M_f = \{\{1,1'\},\{2,2'\},\{3,3'\},\{4,4'\},\{5,5'\},\{6,6'\},\{7,7'\},\{8,8'\}\}$$

Generalizing the combinatorial encoding, an arbitrary triple of perfect matchings determines a **hypermap** when the triple induces a connected graph, with cycles of $M_e \cup M_f$, $M_e \cup M_v$, and $M_v \cup M_f$ determining vertices, hyperfaces, and hyperedges. ▸ Example

Hypermaps both **specialize** and generalize maps.

Example



Hypermaps can be represented as face-bipartite maps.

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Hypermaps both specialize and **generalize** maps.

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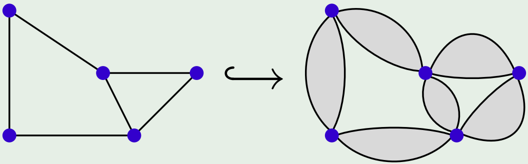

Maps can be represented as hypermaps with $\epsilon = [2^n]$.

The Hypermap Series

Definition

The **hypermap series** for a set $\mathcal{H}$ of hypermaps is the combinatorial sum

$$H(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \sum_{\mathfrak{h} \in \mathcal{H}} \mathbf{x}^{\nu(\mathfrak{h})} \mathbf{y}^{\phi(\mathfrak{h})} \mathbf{z}^{\epsilon(\mathfrak{h})}$$

where $\nu(\mathfrak{h})$, $\phi(\mathfrak{h})$, and $\epsilon(\mathfrak{h})$ are the vertex-, hyperface-, and hyperedge-degree partitions of $\mathfrak{h}$. ▸ Example

Note

$$M(\mathbf{x}, \mathbf{y}, z) = H(\mathbf{x}, \mathbf{y}, \mathbf{z})\Big|_{z_i = z\delta_{i,2}}$$

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How does this help?

- Instead of counting rooted maps, we can count labelled hypermaps. The numbers are different, but the correction factor is easy.
- Labelled counting problems are turned into problems involving counting factorizations.
- These can be answered via character theory.
- Appropriate characters appear as coefficients of symmetric functions.

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Explicit Formulae

The hypermap series can be computed explicitly when $\mathcal{H}$ consists of all orientable or locally orientable hypermaps.

Theorem (Jackson and Visentin)

When $\mathcal{H}$ is the set of orientable hypermaps,

$$H\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}); 0\Big) = t\frac{\partial}{\partial t} \ln \left(\sum_{\theta \in \mathscr{P}} H_\theta s_\theta(\mathbf{x}) s_\theta(\mathbf{y}) s_\theta(\mathbf{z}) \right)\Bigg|_{t=0}.$$

Theorem (Goulden and Jackson)

When $\mathcal{H}$ is the set of locally orientable hypermaps,

$$H\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}); 1\Big) = 2t\frac{\partial}{\partial t} \ln \left(\sum_{\theta \in \mathscr{P}} \frac{1}{H_{2\theta}} Z_\theta(\mathbf{x}) Z_\theta(\mathbf{y}) Z_\theta(\mathbf{z}) \right)\Bigg|_{t=0}.$$

Outline

Parallel Problems

We started with two similar problems, applied similar techniques, and found similar looking solutions.

The natural question is, "Could we have solved both problems at once?"

Jack Symmetric Functions

Jack symmetric functions, ▸ Definition, are a one-parameter family, denoted by $\{J_\theta(\alpha)\}_\theta$, that generalizes both Schur functions and zonal polynomials.

Proposition (Stanley)

Jack symmetric functions are related to Schur functions and zonal polynomials by:

$$J_\lambda(1) = H_\lambda s_\lambda, \qquad \langle J_\lambda, J_\lambda\rangle_1 = H_\lambda^2,$$

and

$$J_\lambda(2) = Z_\lambda, \qquad \langle J_\lambda, J_\lambda\rangle_2 = H_{2\lambda},$$

where 2λ is the partition obtained from λ by multiplying each part by two.

A Generalized Series

b-Conjecture (Goulden and Jackson)

The generalized series,

$$\begin{aligned} H\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}); b\Big) \\ &:= (1+b)t\frac{\partial}{\partial t}\ln\left(\sum_{\theta\in\mathscr{P}}\frac{J_\theta(\mathbf{x};1+b)J_\theta(\mathbf{y};1+b)J_\theta(\mathbf{z};1+b)}{\langle J_\theta, J_\theta\rangle_{1+b}}\right)\Bigg|_{t=0} \\ &= \sum_{n\geq 0}\sum_{\nu,\phi,\epsilon\vdash n} c_{\nu,\phi,\epsilon}(b)p_\nu(\mathbf{x})p_\phi(\mathbf{y})p_\epsilon(\mathbf{z}), \end{aligned}$$

has an combinatorial interpretation involving hypermaps. In particular $c_{\nu,\phi,\epsilon}(b) = \sum_{\mathfrak{h}\in\mathcal{H}_{\nu,\phi,\epsilon}} b^{\beta(\mathfrak{h})}$ *for some invariant* β *of rooted hypermaps.*

b is ubiquitous

The many lives of b

	$b = 0$		$b = 1$
Hypermaps	Orientable	**?**	Locally Orientable
Symmetric Functions	s_θ	$J_\theta(b)$	Z_θ
Matrix Integrals	Hermitian	**?**	Real Symmetric
Moduli Spaces	over $\mathbb{C}$	**?**	over $\mathbb{R}$
Matching Systems	Bipartite	**?**	All

A b-Invariant

The b-Conjecture assumes that $c_{\nu,\phi,\epsilon}(b)$ is a polynomial, and numerical evidence suggests that its degree is the genus of the hypermaps it enumerates. A b-invariant must:

1. be zero for orientable hypermaps,
2. be positive for non-orientable hypermaps, and
3. depend on rooting.

Example

Rootings of precisely three maps are enumerated by $c_{[4],[4],[2^2]}(b) = 1 + b + 3b^2$.

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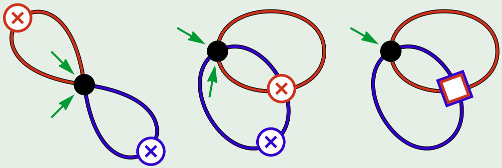
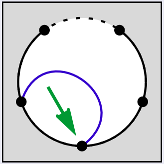


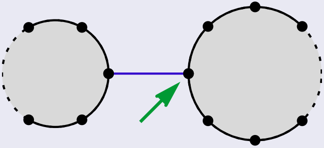
A root-edge classification

There are four possible types of root edges in a map.

Borders



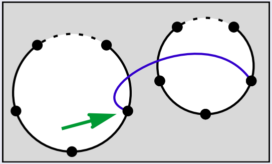
Bridges



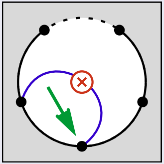
Example

A handle

Handles



Cross-Borders



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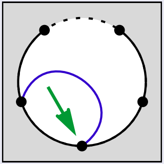

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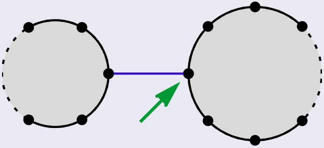

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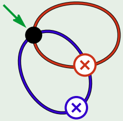
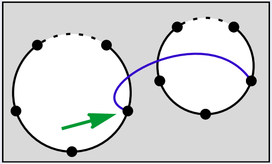
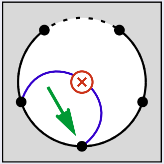


A handle

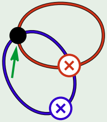
Handles



Cross-Borders



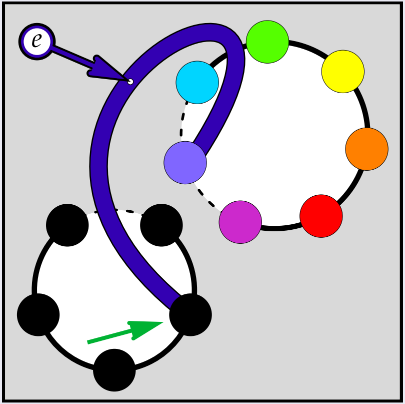
Example



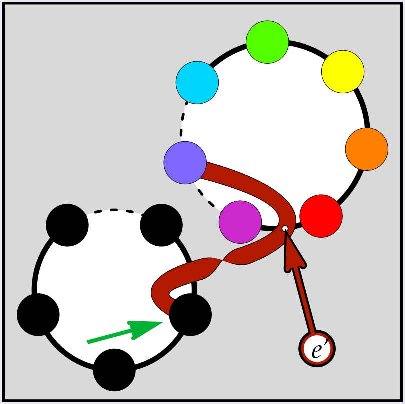
A cross-border

A root-edge classification

Handles occur in pairs



Untwisted



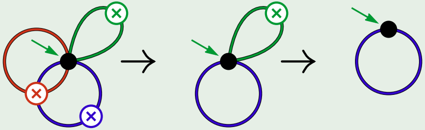
Twisted

A family of invariants

The invariant η

- Iteratively deleting the root edge assigns a type to each edge in a map.
- An invariant, η, is given by
$$\eta(\mathfrak{m}) := (\#\text{ of cross-borders}) + (\#\text{ of twisted handles}).$$
- Different handle twisting determines a different invariant.

Example



Handle
Cross-Border
Border

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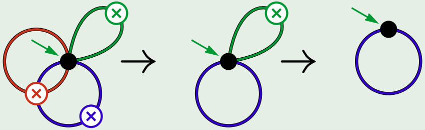

Handle

Cross-Border

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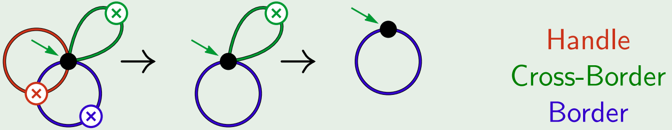

Main result (marginal b-invariants exist)

Theorem (La Croix)

If ϕ partitions $2n$ and η is a member of the family of invariants then,

$$d_{v,\phi}(b) := \sum_{\ell(\nu)=v} c_{\nu,\phi,[2^n]}(b) = \sum_{\mathfrak{m}\in\mathcal{M}_{v,\phi}} b^{\eta(\mathfrak{m})}.$$

Proof (sketch).

- Distinguish between root and non-root faces in the generating series.
- Show that this series satisfies a PDE with a unique solution.
- Predict an expression for the corresponding algebraic refinement.
- Show that the refined series satisfies the same PDE.

□

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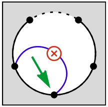
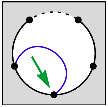
Finding a partial differential equation

Root-edge type	Schematic	Contribution to M
Cross-border		$z \sum_{i \geq 0} (i+1) b r_{i+2} \frac{\partial}{\partial r_i} M$
Border		$z \sum_{i \geq 0} \sum_{j=1}^{i+1} r_j y_{i-j+2} \frac{\partial}{\partial r_i} M$
Handle		$z \sum_{i,j \geq 0} (1+b) j r_{i+j+2} \frac{\partial^2}{\partial r_i \partial y_j} M$
Bridge		$z \sum_{i,j \geq 0} r_{i+j+2} \left(\frac{\partial}{\partial r_i} M \right) \left(\frac{\partial}{\partial r_j} M \right)$

Finding a partial differential equation

Root-edge type	Schematic	Contribution to M
Cross-border		$z\sum_{i\geq 0}(i+1)br_{i+2}\frac{\partial}{\partial r_i}M$
Border		$z\sum_{i\geq 0}\sum_{j=1}^{i+1} r_j y_{i-j+2}\frac{\partial}{\partial r_i}M$
Handle		$z\sum_{i,j\geq 0}(1+b)jr_{i+j+2}\frac{\partial^2}{\partial r_i \partial y_j}M$
Bridge		$z\sum_{i,j\geq 0} r_{i+j+2}\left(\frac{\partial}{\partial r_i}M\right)\left(\frac{\partial}{\partial r_j}M\right)$

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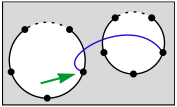
Handle		$z\sum_{i,j\geq 0}(1+b)jr_{i+j+2}\frac{\partial^2}{\partial r_i \partial y_j}M$
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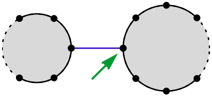
Bridge		$z\sum_{i,j\geq 0} r_{i+j+2}\left(\frac{\partial}{\partial r_i}M\right)\left(\frac{\partial}{\partial r_j}M\right)$

Finding a partial differential equation

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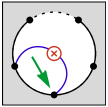
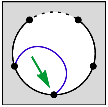
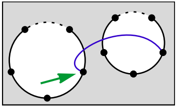
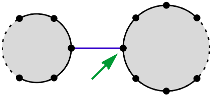
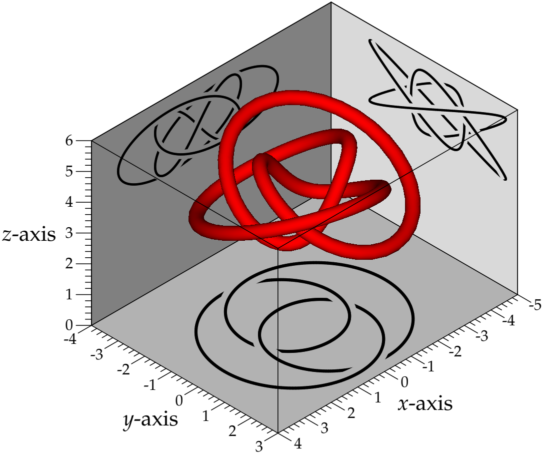
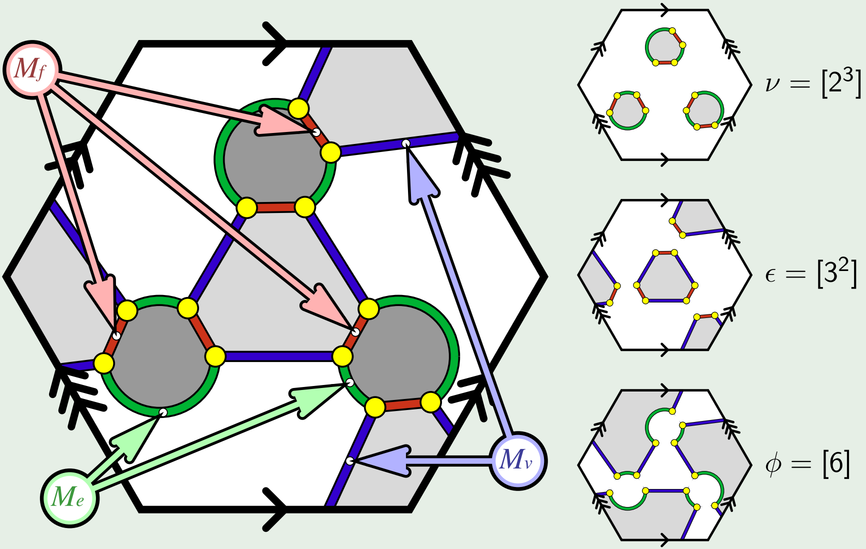
Cross-border		$z\sum_{i\geq 0}(i+1)br_{i+2}\frac{\partial}{\partial r_i}M$
Border		$z\sum_{i\geq 0}\sum_{j=1}^{i+1}r_j y_{i-j+2}\frac{\partial}{\partial r_i}M$
Handle		$z\sum_{i,j\geq 0}(1+b)jr_{i+j+2}\frac{\partial^2}{\partial r_i \partial y_j}M$
Bridge		$z\sum_{i,j\geq 0}r_{i+j+2}\left(\frac{\partial}{\partial r_i}M\right)\left(\frac{\partial}{\partial r_j}M\right)$

The End

Thank You

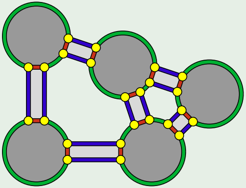


Example

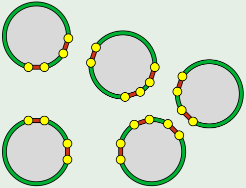


Return

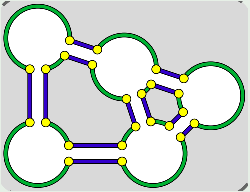
Example



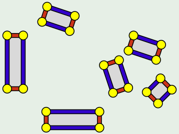
is enumerated by $\left(x_2^3\, x_3^2\right)\left(y_3\, y_4\, y_5\right)\left(z_2^6\right)$.



$\nu = [2^3, 3^2]$



$\phi = [3, 4, 5]$



$\epsilon = [2^6]$

Return

Jack Symmetric Functions

With respect to the inner product defined by

$$\langle p_\lambda(\mathbf{x}), p_\mu(\mathbf{x})\rangle = \delta_{\lambda,\mu} \frac{|\lambda|!}{|C_\lambda|} \alpha^{\ell(\lambda)},$$

Jack symmetric functions are the unique family satisfying:

(P1) (Orthogonality) If $\lambda \neq \mu$, then $\langle J_\lambda, J_\mu \rangle_\alpha = 0$.

(P2) (Triangularity) $J_\lambda = \sum_{\mu \preccurlyeq \lambda} v_{\lambda\mu}(\alpha) m_\mu$, where $v_{\lambda\mu}(\alpha)$ is a rational function in α, and '$\preccurlyeq$' denotes the natural order on partitions.

(P3) (Normalization) If $|\lambda| = n$, then $v_{\lambda,[1^n]}(\alpha) = n!$.

Return