An enumerative relationship between maps and 4-regular maps

Michael La Croix

April 9, 2008

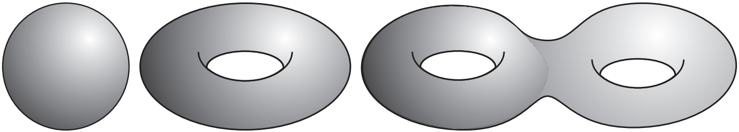
Outline

Outline

Surfaces

Definition

A **Surface** is a compact connected 2-manifold without boundary.

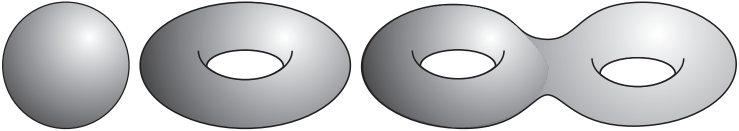


This talk will focus on orientable surfaces.

Surfaces

Theorem (Classification Theorem)

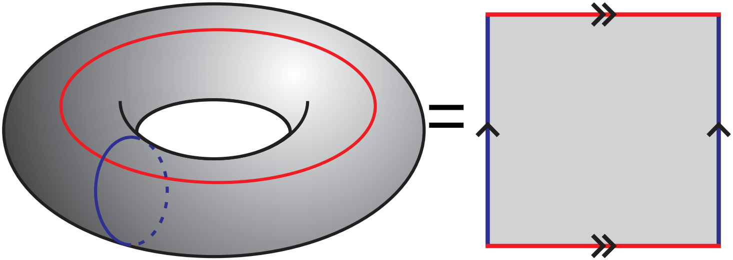
Every orientable surface is an n-torus for some $n \geq 0$.



n is the genus of the surface.

Polygonal Representations

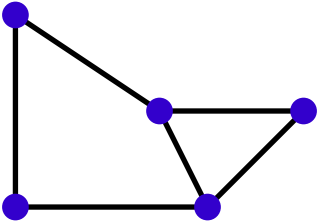
Surfaces can be represented by polygons with sides identified.



Maps

Definition

A **map** is a 2-cell embedding of a multigraph in a surface.



The graph is necessarily connected.

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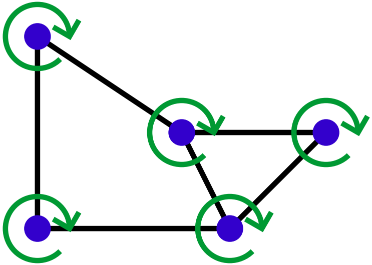

The embedding provides a cyclic order to edges at each vertex.

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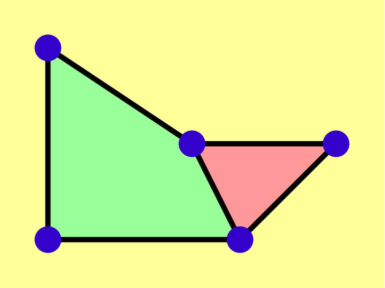

The embedding also defines faces.

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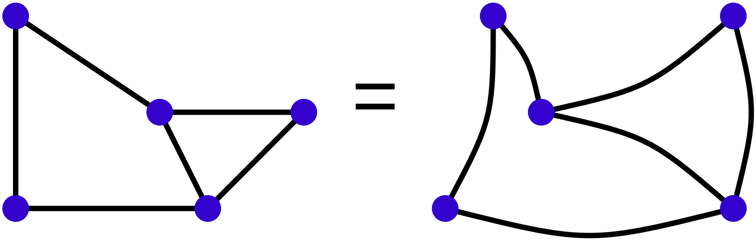

Maps are considered up to topological deformations.

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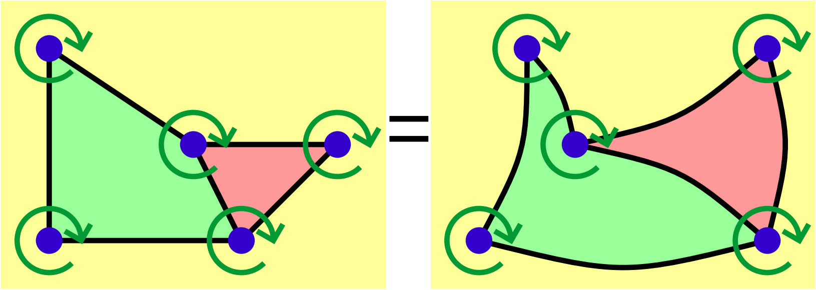
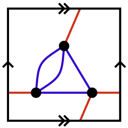
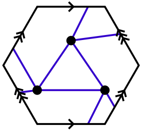


Deformations preserve faces and cyclic orders.

Maps on the Torus

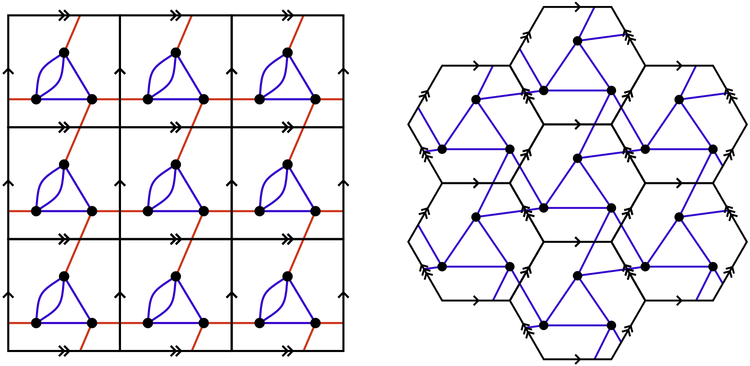
Polygonal representations obfuscate structure.





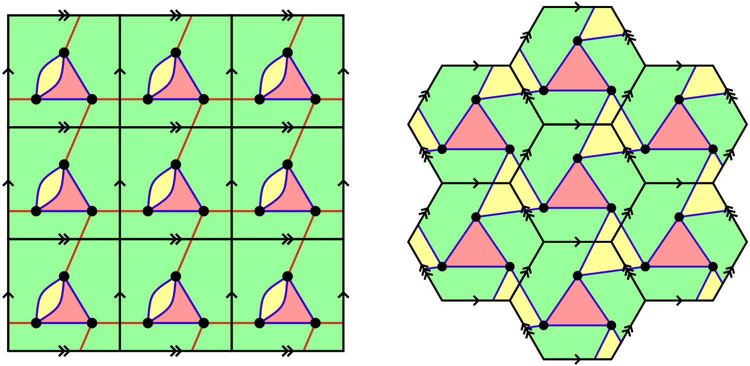
Maps on the Torus

Tiling the fundamental domain produces the universal cover,



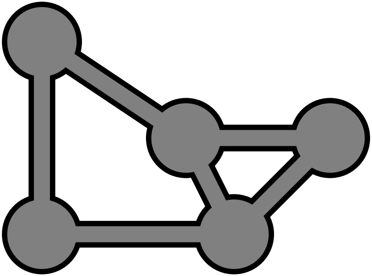
Maps on the Torus

and reveals face structure.



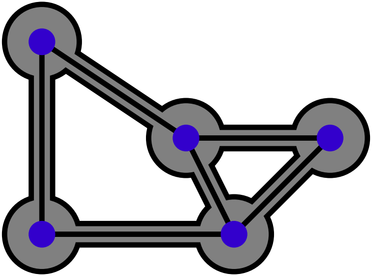
Ribbon Graphs and Flags

The neighbourhood of a map defines a ribbon graph.



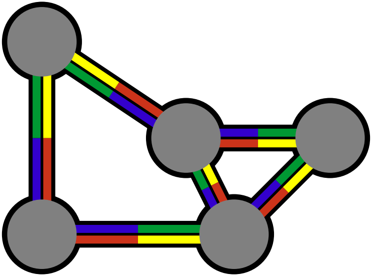
Ribbon Graphs and Flags

A ribbon graph determines the surface and embedding.



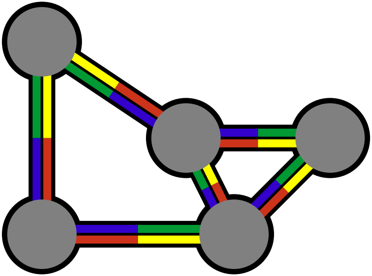
Ribbon Graphs and Flags

Vertex-edge intersections define flags.



Ribbon Graphs and Flags

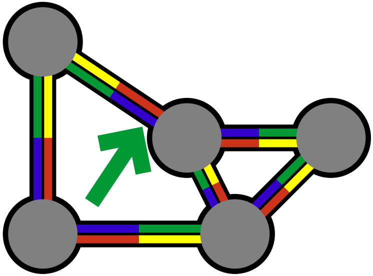
Flags are permuted by map automorphisms.



Rooted Maps

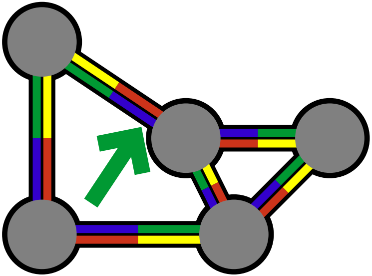
Definition

A **rooted map** is a map together with a distinguished orbit of flags under the action of its automorphism group.



Rooted Maps

Rootings are indicated with arrows.



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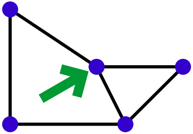
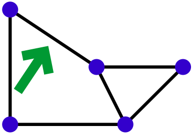


or



Note: A map with no edges has a single rooting.

Outline

How Many Maps are There?

Denote the set of rooted orientable maps by $\mathcal{M}$.

- How many elements of $\mathcal{M}$ have genus g, v vertices, f faces, and e edges?

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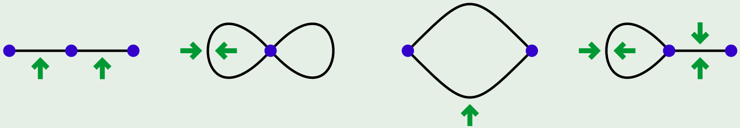
Example

Of the planar rooted maps with 2 edges, two have 3 vertices, five have 2 vertices, and two have 1 vertex.



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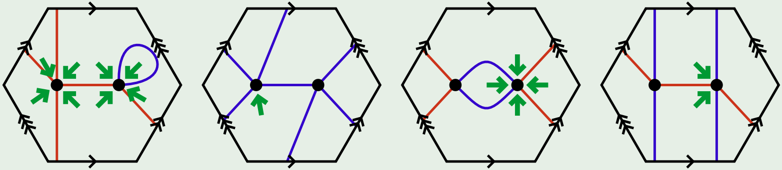
Example

There are 15 maps rooted maps that are 4-regular with 2 vertices, 4 edges, 2 faces, and genus 1.



Generating Series

The genus series for rooted orientable maps is

$$M(u^2, x, y, z) = \sum_{\mathfrak{m} \in \mathcal{M}} u^{2g(\mathfrak{m})} x^{v(\mathfrak{m})} y^{f(\mathfrak{m})} z^{e(\mathfrak{m})}.$$

The weights $g(\mathfrak{m})$, $v(\mathfrak{m})$, $f(\mathfrak{m})$, and $e(\mathfrak{m})$ are the genus, number of vertices, number of faces, and number of edges of $\mathfrak{m}$.

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The corresponding series for 4-regular maps is

$$Q(u^2, x, y, z) = \sum_{\mathfrak{m} \in \mathcal{Q}} u^{2g(\mathfrak{m})} x^{v(\mathfrak{m})} y^{f(\mathfrak{m})} z^{e(\mathfrak{m})}.$$

The weights $g(\mathfrak{m})$, $v(\mathfrak{m})$, $f(\mathfrak{m})$, and $e(\mathfrak{m})$ are the genus, number of vertices, number of faces, and number of edges of $\mathfrak{m}$.

A Remarkable Identity

Jackson and Visentin derived the functional relation

$$
\begin{aligned}
Q(u^2, x, y, z) &= \frac{1}{2} M(4u^2, y+u, y, xz^2) + \frac{1}{2} M(4u^2, y-u, y, xz^2) \\
&= \underset{\text{even}}{\text{bis}}\ _u\ M(4u^2, y+u, y, xz^2).
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A Combinatorial Interpretation

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The right hand side is a generating series for a set $\bar{\mathcal{M}}$.

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They conjectured that this bijection has a natural interpretation.

Conjecture (The q-Conjecture)

There is a natural bijection ϕ from $\bar{\mathcal{M}}$ to $\mathcal{Q}$.

$$\phi \colon \bar{\mathcal{M}} \to \mathcal{Q}$$

A decorated map with
- v vertices
- $2k$ marked vertices
- e edges
- f faces
- genus g

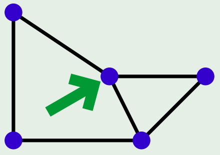
$\xrightarrow{\phi}$

A 4-regular map with
- e vertices
- $2e$ edges
- $f + v - 2k$ faces
- genus $g + k$

Deriving the Identity

Jackson and Visentin proved the identity indirectly.

Example (Encoding a Map)



Begin with a rooted map.

Deriving the Identity

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- Maps are decorated with edge labels and orientations.

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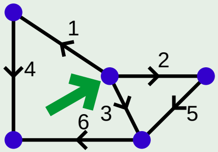

$$\epsilon = (1\ 1')(2\ 2')(3\ 3')(4\ 4')(5\ 5')$$

Decorate the edges.

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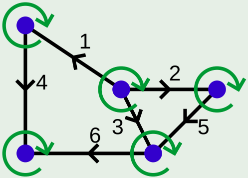

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The labels and cyclic orders give a vertex permutation.

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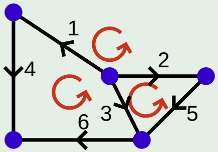

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Multiplying produces the face permutation.

Deriving the Identity

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- The permutations are enumerated using character sums.

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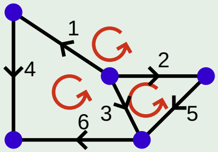

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Deriving the Identity

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- The permutations are enumerated using character sums.
- Maps can be recovered using standard techniques.

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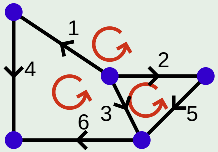

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Deriving the Identity

Using this encoding,

$$M(u^2, x, y, z) = 2u^2 z \frac{\partial}{\partial z} \ln R\left(\frac{x}{u}, \frac{y}{u}, \frac{zu}{2}\right)$$
$$Q(u^2, x, y, z) = 2u^2 z \frac{\partial}{\partial z} \ln R_4\left(\frac{x}{u}, \frac{y}{u}, \frac{zu}{2}\right)$$

where R and R_4 are exponential generating series for edge-labelled not-necessarily-connected maps. The proof involved factoring R_4.

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where R and R_4 are exponential generating series for edge-labelled not-necessarily-connected maps. The proof involved factoring R_4.

$$R_4(x, y, z) = R\left(\frac{1}{2}x, \frac{1}{2}(x+1), 4z^2 y\right) \cdot R\left(\frac{1}{2}x, \frac{1}{2}(x-1), 4z^2 y\right)$$

An Interpretive Bottleneck

It is difficult to interpret the factorization in terms of maps.

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- the proof requires more refinement than the identity it proves,
- it uses character sums.

The Planar Case

Evaluating the series at $u = 0$ restricts the sums to planar maps and gives

$$Q(0, x, y, z) = M(0, y, y, xz^2).$$

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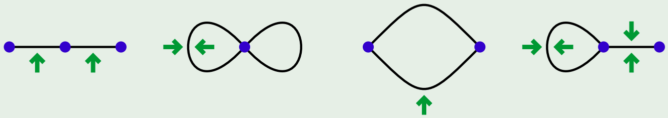
Combinatorially, the number of 4-regular planar maps with n vertices is equal to the number of planar maps with n edges. Tutte's medial construction explains this bijectively.

The Medial Construction

Tutte's medial construction explains the planar case.

Example



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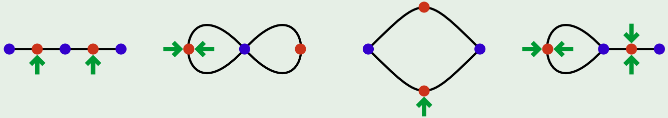

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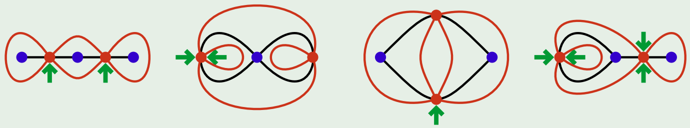

The Medial Construction

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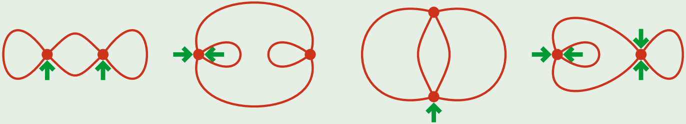
- Place a vertex on each edge.
- Join edges that are incident around a vertex circulation.
- The medials of planar duals are the same map.

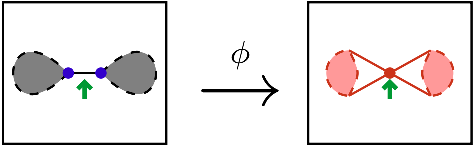
Example



Properties of the Medial Construction

The construction has several properties that make it natural.

- Cut edges become cut vertices.



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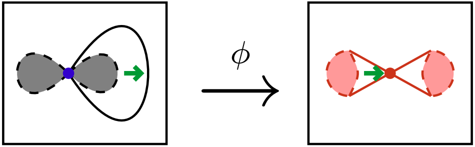

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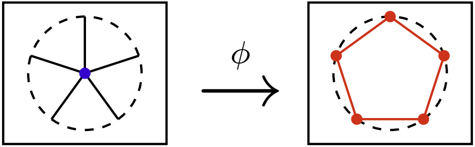
- Faces and vertices of degree k become faces of degree k.



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- Cut edges become cut vertices.
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- Faces and vertices of degree k become faces of degree k.
- Duality in $\mathcal{M}$ corresponds to reflection in $\mathcal{Q}$.

The Medial Construction at Higher Genus

The medial construction extends to all surfaces.

- It produces all face-bipartite 4-regular maps.
- It preserves genus.

This gives an injection from undecorated maps to 4-regular maps.

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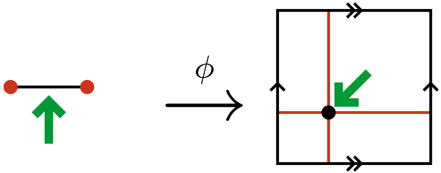
Conjecture

The medial construction is the restriction of ϕ to $\mathcal{M}$.

What Else do we know?

There is only one 4-regular map with one vertex on the torus.



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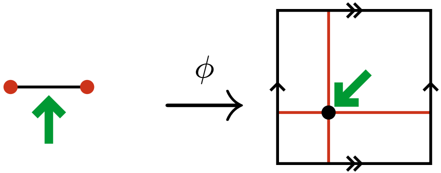

It is impossible to construct ϕ such that it preserves face degrees.

Outline

A Differential Equation

By considering root deletion, a refinement of M can be shown to satisfy a combinatorially significant differential equation.

$$M(1, x, \vec{y}, z, \vec{r}) = r_0 x + z \sum_{i \geq 0} \sum_{j=1}^{i+1} r_j y_{i-j+2} \frac{\partial}{\partial r_i} M + z \sum_{i,j \geq 0} j r_{i+j+2} \frac{\partial^2}{\partial r_i \partial y_j} M + z \sum_{i,j \geq 0} r_{i+j+2} \left(\frac{\partial}{\partial r_i} M \right) \left(\frac{\partial}{\partial r_j} M \right).$$

Here y_i marks non-root faces of degree i and r_i marks a root face of degree i.

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Both M and Q are evaluations of this series.

The differential equations allows a proof of the following theorem within the realm of connected maps.

Theorem

With N a positive integer and $\langle\cdot\rangle_e$ defined by

$$\langle f\rangle_e = \frac{\int_{\mathbb{R}^N} |V(\lambda)|^2 f(\lambda) \exp\left(\sum_{k\geq 1}\frac{1}{k}x_k p_k \sqrt{z}^k\right) e^{-\frac{1}{2}p_2(\lambda)}\mathrm{d}\lambda}{\int_{\mathbb{R}^N} |V(\lambda)|^2 \exp\left(\sum_{k\geq 1}\frac{1}{k}x_k p_k \sqrt{z}^k\right) e^{-\frac{1}{2}p_2(\lambda)}\mathrm{d}\lambda},$$

evaluations of the map series are given by

$$M(1,\vec{x},N,z) = \sum_{k=0}^{\infty} x_k \sqrt{z}^k \langle p_k\rangle_e .$$

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This suggests an inductive approach to identifying ϕ. All that remains (!) is to determine how $\phi(\mathfrak{m})$ and $\phi(\mathfrak{m}\backslash e)$ differ when e is a root edge of each type.

Cut-Edges

For decorated maps, root edges come in two forms:

- Even cut edge have an even number of decorated vertices on each side of the cut.

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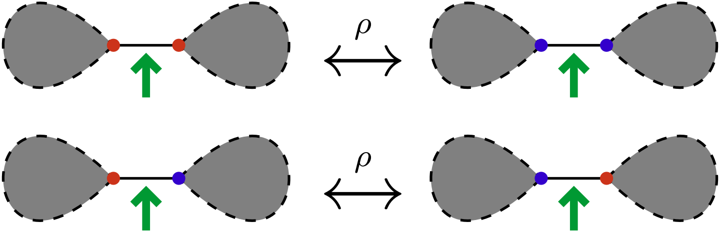
An involution ρ switches the form.



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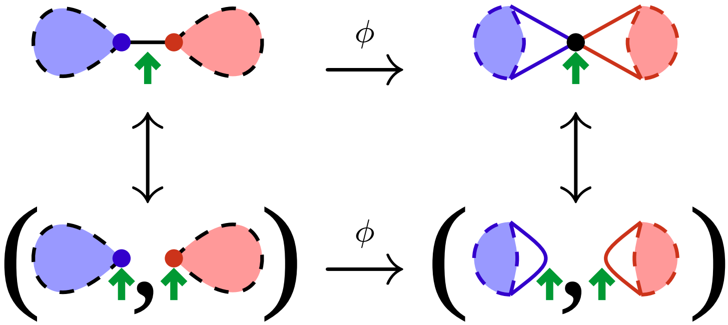
The action of ϕ, when the root edge is an even cut-edge, can speculated from the following commutative diagram.



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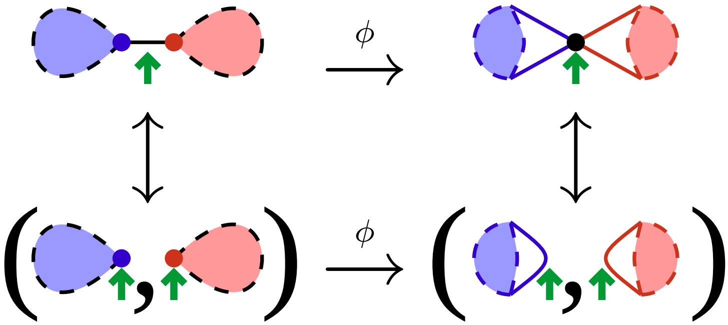

The induced product on $\mathcal{Q} \times \mathcal{Q}$ is genus additive.

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ϕ and ρ induce a product π.

$$\pi \colon \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}$$
$$(\mathfrak{q}_1, \mathfrak{q}_2) \mapsto \mathfrak{q}$$

The Product π

π is nearly genus additive.

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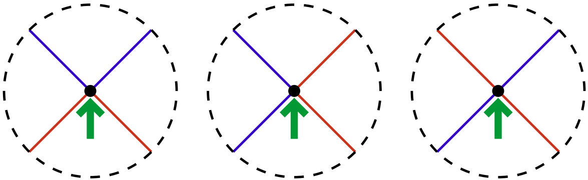
A Candidate For π

In arbitrary genus, the root vertex of a 4-regular map can be a cut-vertex in three distinct ways.



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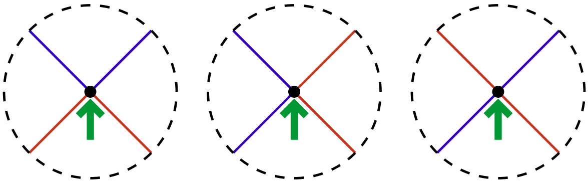
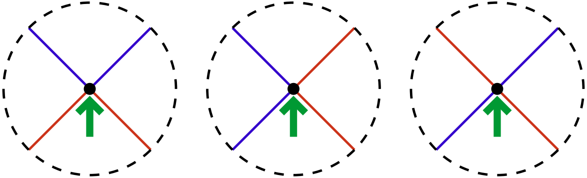
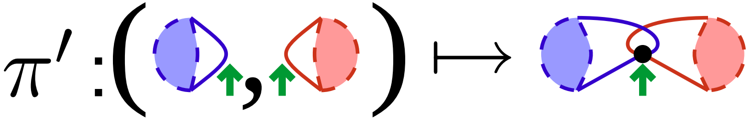


The first two cuts correspond to genus additive products.

A Candidate For π



The third corresponds to the product:



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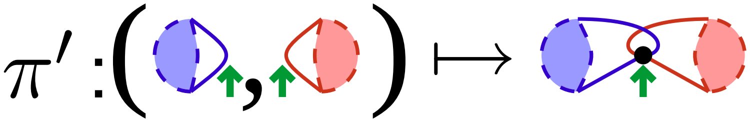

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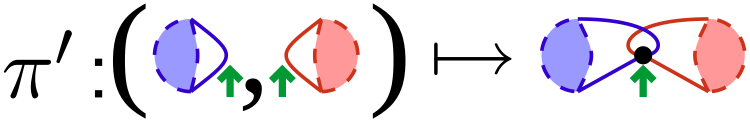

π' is nearly genus additive. The correction term depends on how many factors have root edges that are face-separating, but π' is never subadditive with respect to genus.

A Hidden Relationship?

The qualitative similarities between π' and π suggest a relationship between decorated maps with a decorated root-vertex and 4-regular maps with a face-non-separating root-edge.

A Numerical Surprise!

Constructing all maps with up to 5 edges, and all 4-regular maps with up to 5 vertices suggests that the sets are bijective.

	$v=1$	$v=2$	$v=3$	$v=4$	$v=5$	$v=6$
$g=0$	42	386	1030	1030	386	42
$g=1$	420	1720	1720	420		
$g=2$	483	483				

5-edge maps

	Total	Non-Sep	Sep
$g=0$	2916	0	2916
$g=1$	31266	7290	23976
$g=2$	56646	28674	27972
$g=3$	9450	9450	0

5-vertex, 4-regular maps

$$2916 = 42 + 386 + 1030 + 1030 + 386 + 42$$
$$23979 = \binom{2}{2}1030 + \binom{3}{2}1030 + \binom{4}{2}386 + \binom{5}{2}42 + 4(420 + 1720 + 1720 + 420)$$
$$27972 = \binom{4}{4}386 + \binom{5}{4}42 + 4\left(\binom{2}{2}1720 + \binom{3}{2}420\right) + 16(483 + 483)$$
$$7920 = \binom{1}{1}386 + \binom{2}{1}1030 + \binom{3}{1}1030 + \binom{4}{1}386 + \binom{5}{1}42$$
$$28674 = \binom{3}{3}1030 + \binom{4}{3}386 + \binom{5}{3}42 + 4\left(\binom{1}{1}1720 + \binom{2}{1}1720 + \binom{3}{1}420\right)$$
$$9450 = \binom{1}{1}42 + 4\binom{1}{1}420 + 16\binom{1}{1}483$$

Conjecture (Refined q-Conjecture)

If $\mathcal{Q}_1$ is the restriction of $\mathcal{Q}$ to maps rooted on face-separating edges, and $\hat{\mathcal{M}}_1$ is the restriction of $\hat{\mathcal{M}}$ to maps with undecorated root vertices, then

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In terms of generating series

$$Q_1(u^2, x, y, z) = \operatorname*{bis}_{even}{}_u \frac{y}{y+u} M\left(4u^2, y+u, y, xz^2\right), \textit{ and}$$

$$Q_2(u^2, x, y, z) = \operatorname*{bis}_{even}{}_u \frac{u}{y+u} M\left(4u^2, y+u, y, xz^2\right).$$

Determining Q_1 and Q_2

The integral expression for M does not allow a simultaneous refinement to track root-edge-type and vertex degrees.

Determining Q_1 and Q_2

David Jackson indirectly suggested an indirect approach to computing Q_1 and Q_2.

Determining Q_1 and Q_2

M gives an expression for the generating series for $\mathcal{P}$, the set of maps that have a root vertex of degree 3, a vertex of degree 1, and are otherwise 4-regular.

$$P(1, x, N, 1) = x^2 \frac{\left\langle p_3 p_1 \exp(\frac{1}{4} p_4 x) \right\rangle}{\left\langle \exp(\frac{1}{4} p_4 x) \right\rangle}$$

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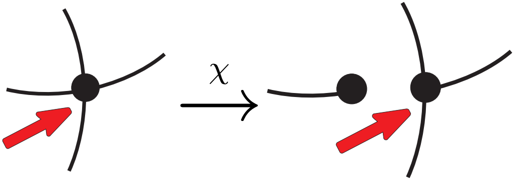
Root-cutting is a bijection from $\mathcal{Q}$ to $\mathcal{P}$.



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$$Q(u^2, x, y, z) = Q_1(u^2, x, y, z) + Q_2(u^2, x, y, z)$$
$$P(u^2, x, y, z) = \frac{x}{y} Q_1(u^2, x, y, z) + \frac{xy}{u^2} Q_2(u^2, x, y, z)$$

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The equations can be solved for Q_1 and Q_2.

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Proving the enumerative portion of the refined q-Conjecture reduces to a factorization problem, similar to the existing proof of Jackson and Visentin.

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A consequence would be the interpretation

$$P(u^2, x, y, z) = \frac{x}{u} \operatorname{bis}_{u \text{ odd}} u\, M(4u^2, y+u, y, xz^2).$$

A Special Case

As a special case of the refined conjecture, we get the concrete statement:

Conjecture

The bijection ϕ specializes to a bijection from planar maps with a decorated non-root vertex to 4-regular maps on the torus rooted at a face-non-separating edge.

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This case avoids the product of 4-regular maps with face-non-separating root-edges.

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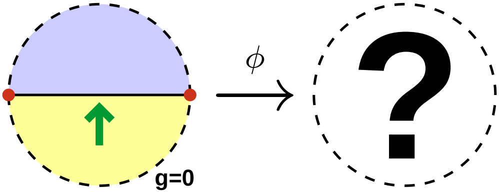

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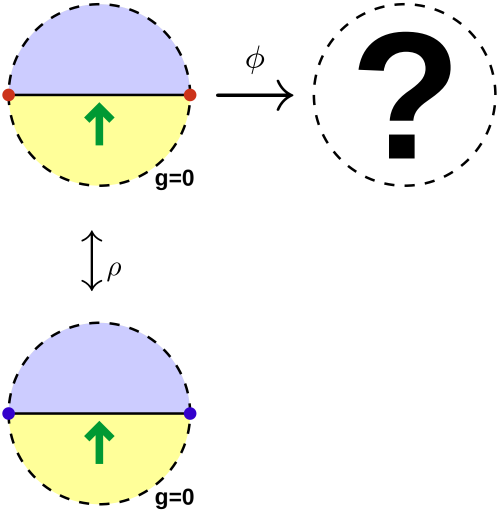

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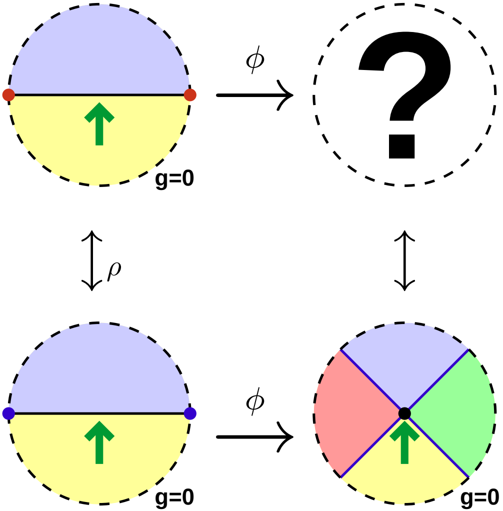

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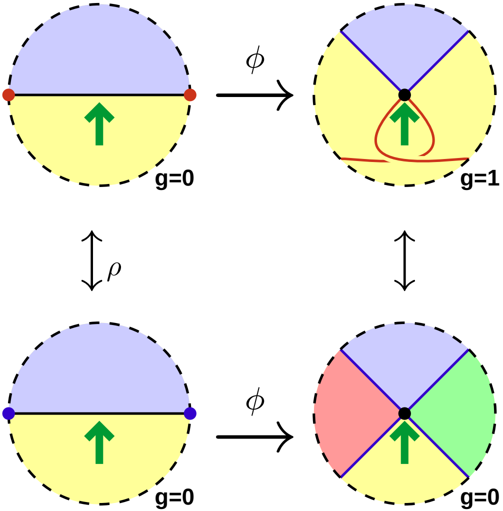

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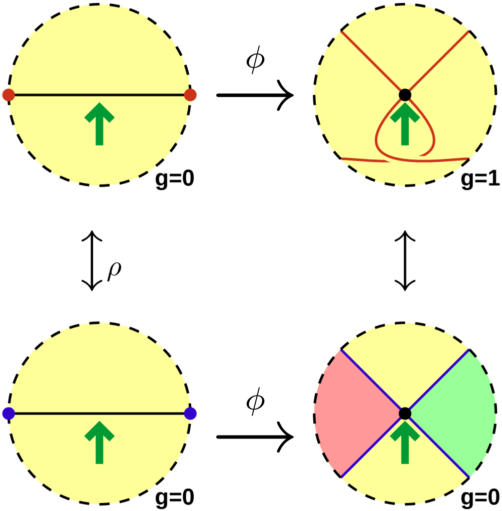

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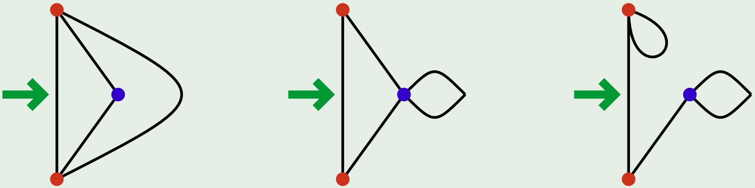
Example



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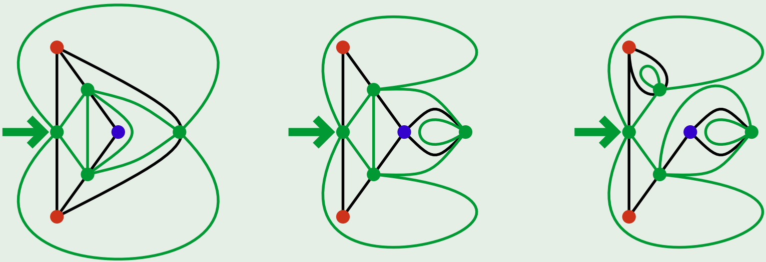

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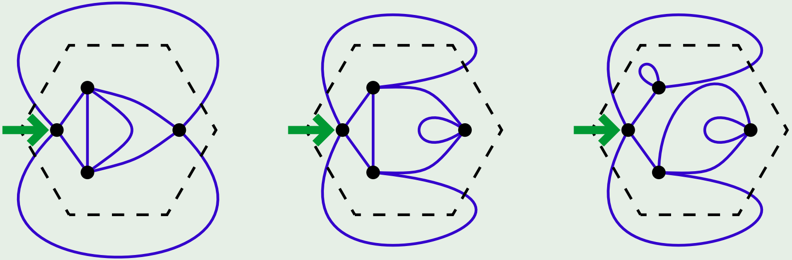

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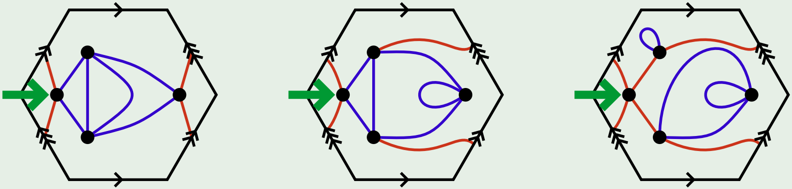

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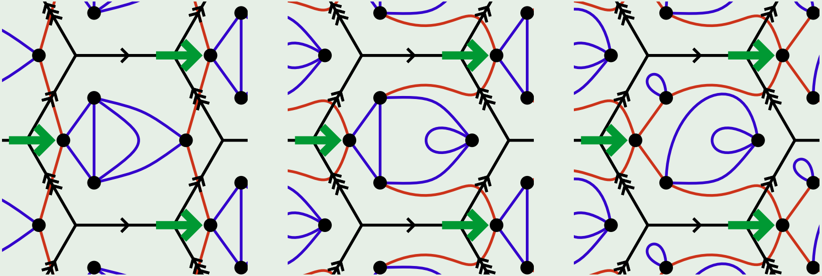

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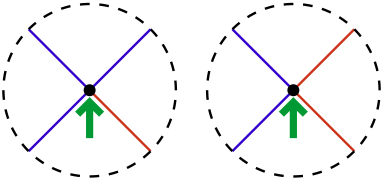
The Missing Case

The remaining maps have images with one of two root configurations.



It should be possible to treat them like contraction.