The combinatorics of the Jack Parameter and the genus series for topological maps

Michael La Croix

University of Waterloo

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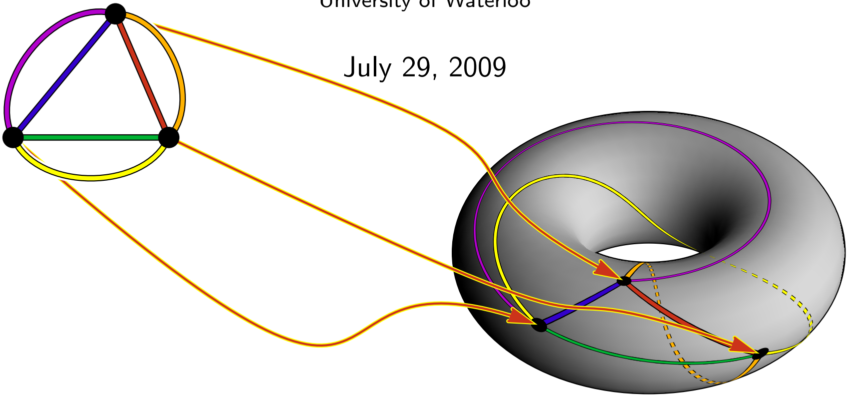

Outline

1. Background
   - The objects
   - An enumerative problem, and two generating series
2. The b-Conjecture
   - An algebraic generalization and the b-Conjecture
   - A family of invariants
   - The invariants resolve a special case
   - Evidence that they are b-invariants
3. The q-Conjecture
   - A remarkable identity and the q-Conjecture
   - A refinement
4. Future Work

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Graphs, Surfaces, and Maps

Definition

A **surface** is a compact 2-manifold without boundary.

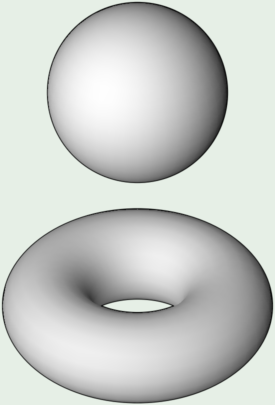
Definition

A **graph** is a finite set of *vertices* together with a finite set of *edges*, such that each edge is associated with either one or two vertices.

Definition

A **map** is a 2-cell embedding of a graph in a surface.

Example



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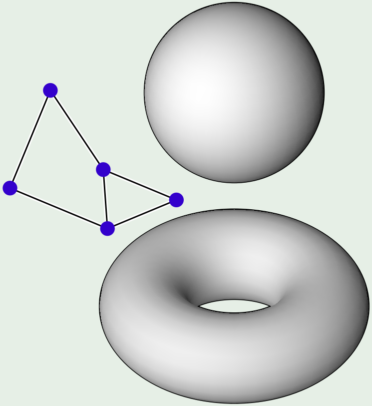

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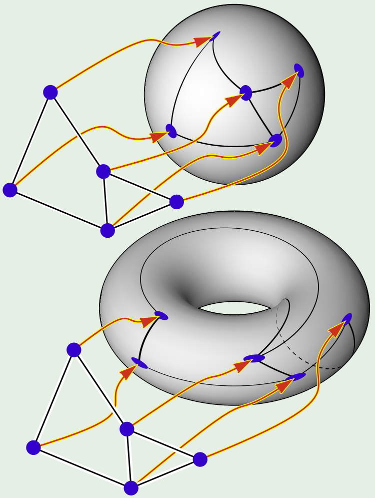

Ribbon Graphs

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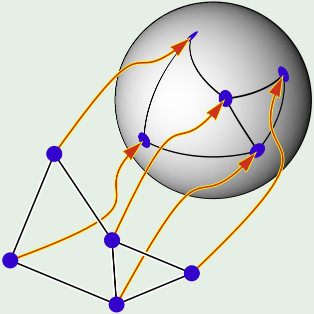

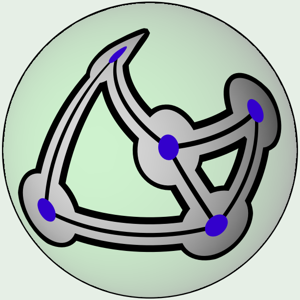

The homeomorphism class of an embedding is determined by a neighbourhood of the graph.

Ribbon Graphs

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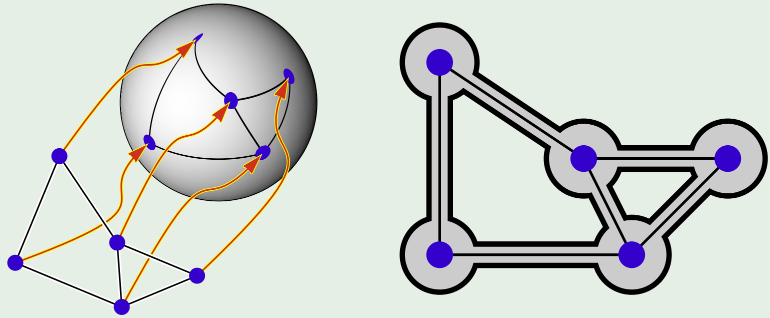

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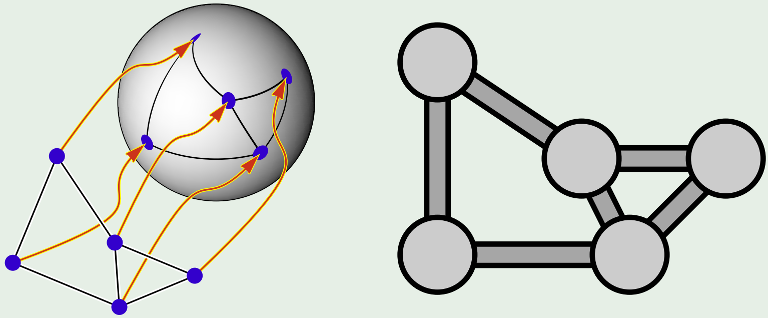
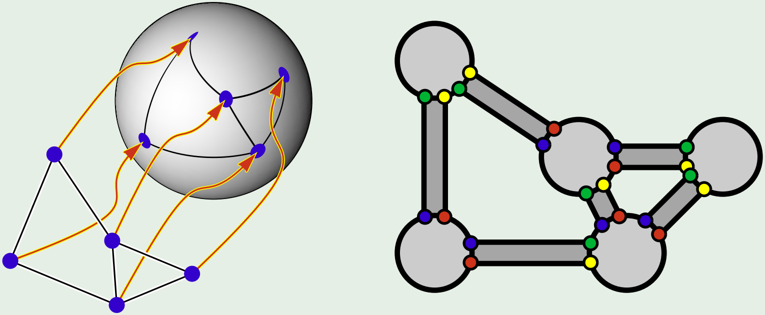


Neighbourhoods of vertices and edges can be replaced by discs and ribbons to form a ribbon graph. Extra Examples

Flags

Example



The boundaries of ribbons determine flags.

Flags

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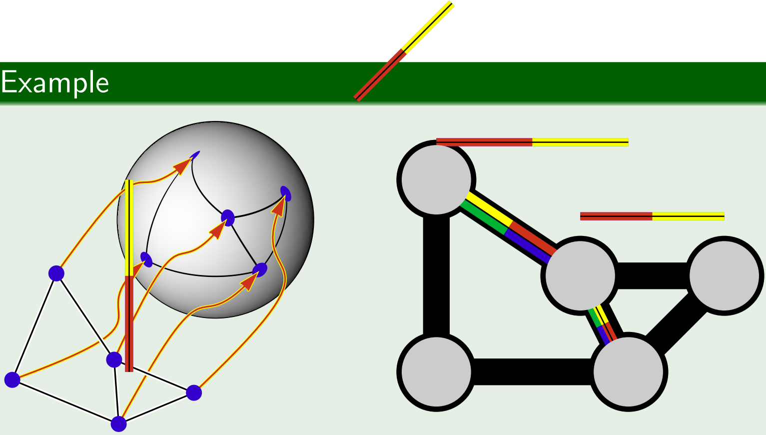


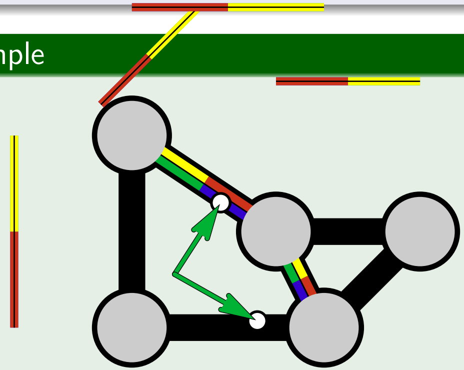
The boundaries of ribbons determine flags, and these can be associated with quarter edges.

Rooted Maps

Definition

A **rooted map** is a map together with a distinguished orbit of flags under the action of its automorphism group.

Example



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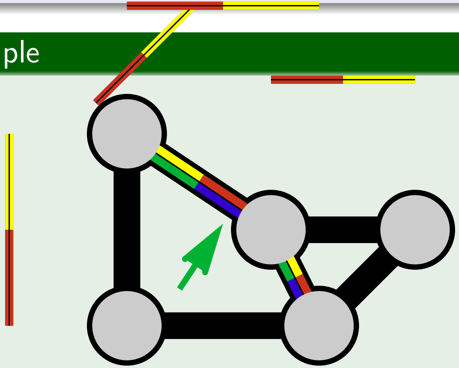

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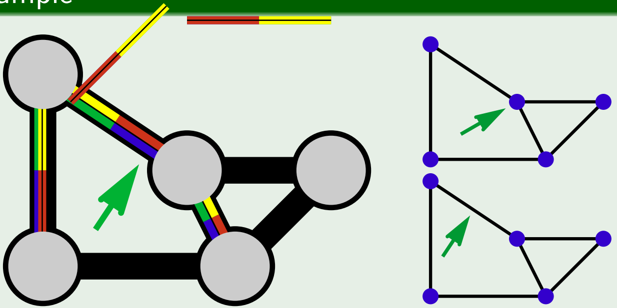

Outline

The Map Series

An enumerative problem associated with maps is to determine the number of rooted maps with specified vertex- and face- degree partitions.

Definition

The **map series** for a set $\mathcal{M}$ of rooted maps is the combinatorial sum

$$M(\mathbf{x}, \mathbf{y}, z) := \sum_{\mathfrak{m} \in \mathcal{M}} \mathbf{x}^{\nu(\mathfrak{m})} \mathbf{y}^{\varphi(\mathfrak{m})} z^{|E(\mathfrak{m})|}$$

where $\nu(\mathfrak{m})$ and $\varphi(\mathfrak{m})$ are the the vertex- and face-degree partitions of $\mathfrak{m}$.

Example

Rootings of are enumerated by $(x_2^3\, x_3^2)\,(y_3\, y_4\, y_5)\, z^6$.

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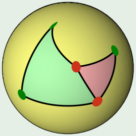
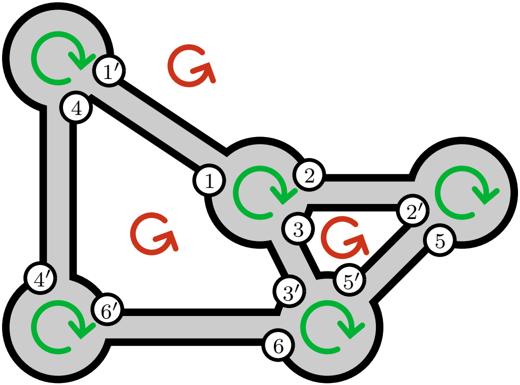
Rootings of  are enumerated by $(\boldsymbol{x_2^3}\,\boldsymbol{x_3^2})\,(y_3\,y_4\,y_5)\,z^6$.

Encoding Orientable Maps

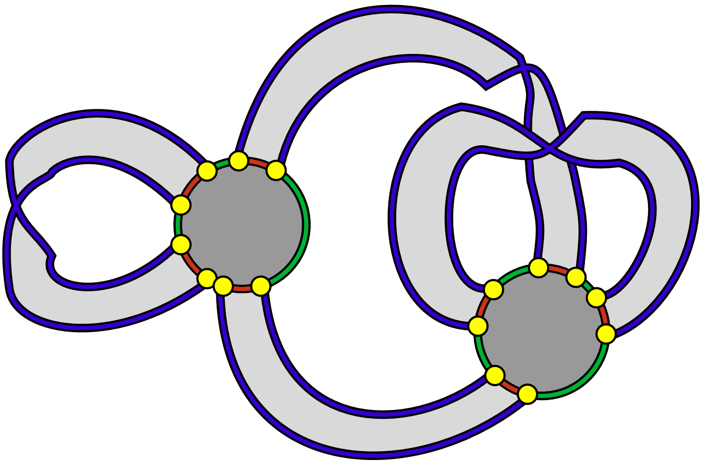
1. Orient and label the edges.
2. This induces labels on flags.
3. Clockwise circulations at each vertex determine ν.
4. Face circulations are the cycles of $\epsilon\nu$.

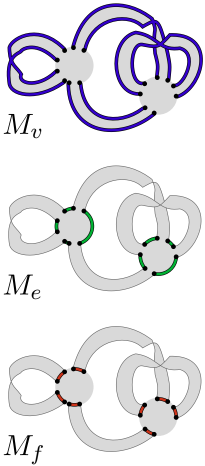


$$\epsilon = (1\ 1')(2\ 2')(3\ 3')(4\ 4')(5\ 5')(6\ 6')$$
$$\nu = (1\ 2\ 3)(1'\ 4)(2'\ 5)(3'\ 5'\ 6)(4'\ 6')$$
$$\epsilon\nu = \varphi = (1\ 4\ 6'\ 3')(1'\ 2\ 5\ 6\ 4')(2'\ 3\ 5')$$

Details

Encoding Locally Orientable Maps





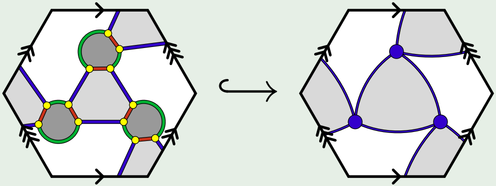
Ribbon boundaries determine 3 perfect matchings of flags. Details

Hypermaps

Generalizing the combinatorial encoding, an arbitrary triple of perfect matchings determines a **hypermap** when the triple induces a connected graph, with cycles of $M_e \cup M_f$, $M_e \cup M_v$, and $M_v \cup M_f$ determining vertices, hyperfaces, and hyperedges. ▸ Example

Hypermaps both **specialize** and generalize maps.

Example



Hypermaps can be represented as face-bipartite maps.

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Hypermaps both specialize and **generalize** maps.

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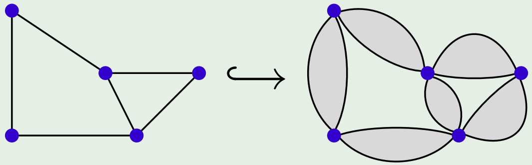

Maps can be represented as hypermaps with $\epsilon = [2^n]$.

The Hypermap Series

Definition

The **hypermap series** for a set $\mathcal{H}$ of hypermaps is the combinatorial sum

$$H(\mathbf{x},\mathbf{y},\mathbf{z}) := \sum_{\mathfrak{h}\in\mathcal{H}} \mathbf{x}^{\nu(\mathfrak{h})}\mathbf{y}^{\varphi(\mathfrak{h})}\mathbf{z}^{\epsilon(\mathfrak{h})}$$

where $\nu(\mathfrak{h})$, $\varphi(\mathfrak{h})$, and $\epsilon(\mathfrak{h})$ are the vertex-, hyperface-, and hyperedge-degree partitions of $\mathfrak{h}$. ▸ Example

Note

$$M(\mathbf{x},\mathbf{y},z) = H(\mathbf{x},\mathbf{y},\mathbf{z})\Big|_{z_i=z\delta_{i,2}}$$

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Explicit Formulae

The hypermap series can be computed explicitly when $\mathcal{H}$ consists of all orientable or locally orientable hypermaps.

Theorem (Jackson and Visentin)

When $\mathcal{H}$ is the set of orientable hypermaps,

$$H\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}); 0\Big) = t\frac{\partial}{\partial t} \ln \left(\sum_{\theta \in \mathscr{P}} H_\theta s_\theta(\mathbf{x}) s_\theta(\mathbf{y}) s_\theta(\mathbf{z}) \right)\Bigg|_{t=0}.$$

Theorem (Goulden and Jackson)

When $\mathcal{H}$ is the set of locally orientable hypermaps,

$$H\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}); 1\Big) = 2t\frac{\partial}{\partial t} \ln \left(\sum_{\theta \in \mathscr{P}} \frac{1}{H_{2\theta}} Z_\theta(\mathbf{x}) Z_\theta(\mathbf{y}) Z_\theta(\mathbf{z}) \right)\Bigg|_{t=0}.$$

Outline

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Jack Symmetric Functions

Jack symmetric functions, ▸ Definition, are a one-parameter family, denoted by $\{J_\theta(\alpha)\}_\theta$, that generalizes both Schur functions and zonal polynomials.

Proposition (Stanley)

Jack symmetric functions are related to Schur functions and zonal polynomials by:

$$J_\lambda(1) = H_\lambda s_\lambda, \qquad \langle J_\lambda, J_\lambda \rangle_1 = H_\lambda^2,$$

and

$$J_\lambda(2) = Z_\lambda, \qquad \langle J_\lambda, J_\lambda \rangle_2 = H_{2\lambda},$$

where 2λ is the partition obtained from λ by multiplying each part by two.

A Generalized Series

b-Conjecture (Goulden and Jackson)

The generalized series,

$$H\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}); b\Big)$$
$$:= (1+b)t\frac{\partial}{\partial t}\ln\left(\sum_{\theta\in\mathscr{P}}\frac{J_\theta(\mathbf{x};1+b)J_\theta(\mathbf{y};1+b)J_\theta(\mathbf{z};1+b)}{\langle J_\theta, J_\theta\rangle_{1+b}}\right)\Bigg|_{t=0}$$
$$=\sum_{n\geq 0}\sum_{\nu,\varphi,\epsilon\vdash n} c_{\nu,\varphi,\epsilon}(b)p_\nu(\mathbf{x})p_\varphi(\mathbf{y})p_\epsilon(\mathbf{z}),$$

has an combinatorial interpretation involving hypermaps. In particular $c_{\nu,\varphi,\epsilon}(b) = \sum_{\mathfrak{h}\in\mathcal{H}_{\nu,\varphi,\epsilon}} b^{\beta(\mathfrak{h})}$ *for some invariant* β *of rooted hypermaps.*

b is ubiquitous

The many lives of b

	$b = 0$		$b = 1$
Hypermaps	Orientable	**?**	Locally Orientable
Symmetric Functions	s_θ	$J_\theta(b)$	Z_θ
Matrix Integrals	Hermitian	**?**	Real Symmetric
Moduli Spaces	over $\mathbb{C}$	**?**	over $\mathbb{R}$
Matching Systems	Bipartite	**?**	All

A b-Invariant

The b-Conjecture assumes that $c_{\nu,\varphi,\epsilon}(b)$ is a polynomial, and numerical evidence suggests that its degree is the genus of the hypermaps it enumerates. A b-invariant must:

1. be zero for orientable hypermaps,
2. be positive for non-orientable hypermaps, and
3. depend on rooting.

Example

Rootings of precisely three maps are enumerated by $c_{[4],[4],[2^2]}(b) = 1 + b + 3b^2$.

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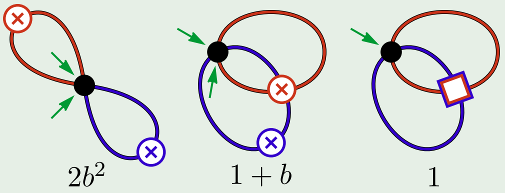

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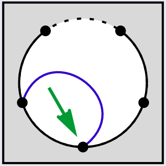
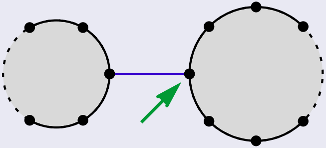
A root-edge classification

There are four possible types of root edges in a map.

Borders



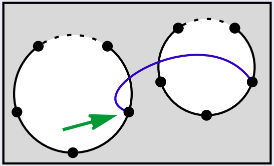
Bridges



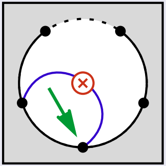
Example

A handle

Handles



Cross-Borders



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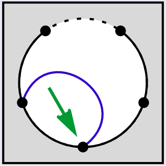

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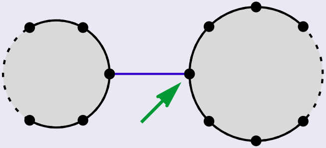

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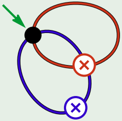

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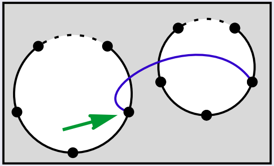

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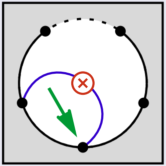

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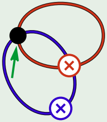

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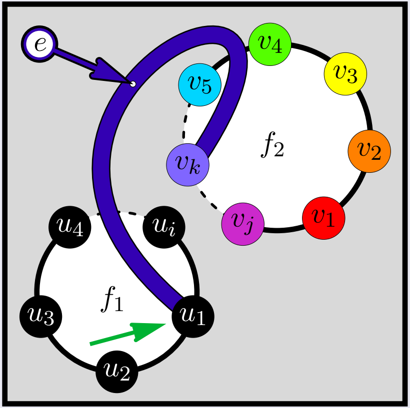
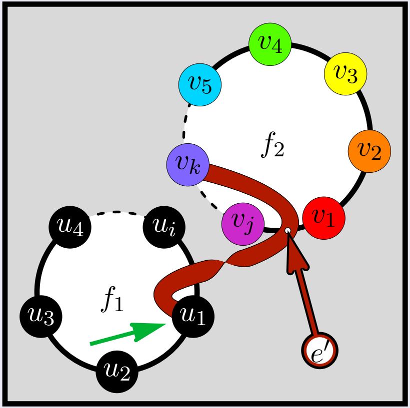
Handles occur in pairs



Untwisted



Twisted

A family of invariants

The invariant η

- Iteratively deleting the root edge assigns a type to each edge in a map.
- An invariant, η, is given by
$$\eta(\mathfrak{m}) := (\#\text{ of cross-borders}) + (\#\text{ of twisted handles})\,.$$
- Different handle twisting determines a different invariant. Example

Example

Handle
Cross-Border
Border

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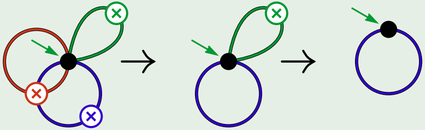

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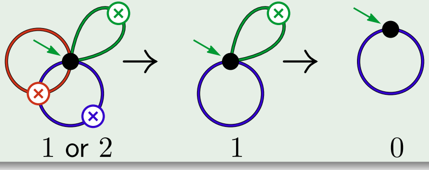

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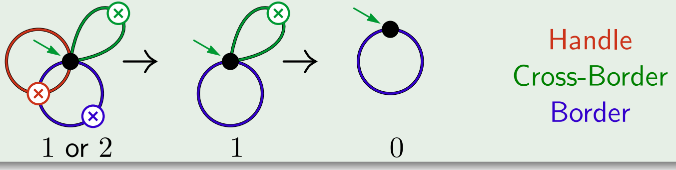

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Main result (marginal b-invariants exist)

Theorem (La Croix)

If φ partitions $2n$ and η is a member of the family of invariants then,

$$d_{v,\varphi}(b) := \sum_{\ell(\nu)=v} c_{\nu,\varphi,[2^n]}(b) = \sum_{\mathfrak{m}\in\mathcal{M}_{v,\varphi}} b^{\eta(\mathfrak{m})}.$$

Corollary

$M(x,\mathbf{y},z;b) = \sum_{\mathfrak{m}\in\mathcal{M}} x^{|V(\mathfrak{m})|}\mathbf{y}^{\varphi(\mathfrak{m})}z^{|E(\mathfrak{m})|}$ *is an element of* $\mathbb{Z}_+[x,\mathbf{y},b][[z]]$.

Corollary

There is an uncountably infinite family of marginal b-invariants.

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There is an uncountably infinite family of marginal b-invariants.

Main result (marginal b-invariants exist)

Theorem (La Croix)

If φ partitions $2n$ and η is a member of the family of invariants then,

$$d_{v,\varphi}(b) := \sum_{\ell(\nu)=v} c_{\nu,\varphi,[2^n]}(b) = \sum_{\mathfrak{m}\in\mathcal{M}_{v,\varphi}} b^{\eta(\mathfrak{m})}.$$

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Proof (sketch).

- Distinguish between root and non-root faces in the generating series.
- Show that this series satisfies a PDE with a unique solution.
- Predict an expression for the corresponding algebraic refinement.
- Show that the refined series satisfies the same PDE. □

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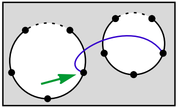
Implications of the proof

- $d_{v,\varphi}(b) = \sum_{0\le i\le g/2} h_{v,\varphi,i} b^{g-2i}(1+b)^i$ is an element of $\mathrm{span}_{\mathbb{Z}_+}(B_g)$.
- The degree of $d_{v,\varphi}(b)$ is the genus of the maps it enumerates.
- The top coefficient, $h_{v,\varphi,0}$, enumerates **unhandled** maps.
- η and root-face degree are independent among maps with given φ.

Finding a partial differential equation

Root-edge type	Schematic	Contribution to M
Cross-border		$z\sum_{i\geq 0}(i+1)br_{i+2}\frac{\partial}{\partial r_i}M$
Border		$z\sum_{i\geq 0}\sum_{j=1}^{i+1} r_j y_{i-j+2}\frac{\partial}{\partial r_i}M$
Handle		$z\sum_{i,j\geq 0}(1+b)jr_{i+j+2}\frac{\partial^2}{\partial r_i \partial y_j}M$
Bridge		$z\sum_{i,j\geq 0} r_{i+j+2}\left(\frac{\partial}{\partial r_i}M\right)\left(\frac{\partial}{\partial r_j}M\right)$

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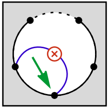
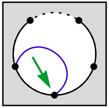
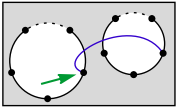
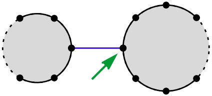
Cross-border		$z\sum_{i\geq 0}(i+1)br_{i+2}\frac{\partial}{\partial r_i}M$
Border		$z\sum_{i\geq 0}\sum_{j=1}^{i+1}r_j y_{i-j+2}\frac{\partial}{\partial r_i}M$
Handle		$z\sum_{i,j\geq 0}(1+b)jr_{i+j+2}\frac{\partial^2}{\partial r_i\partial y_j}M$
Bridge		$z\sum_{i,j\geq 0}r_{i+j+2}\left(\frac{\partial}{\partial r_i}M\right)\left(\frac{\partial}{\partial r_j}M\right)$

An integral expression for $M(N, \mathbf{y}, z; b)$

Define the expectation operator $\langle \cdot \rangle$ by

$$\langle f \rangle := \int_{\mathbb{R}^N} \left| V(\boldsymbol{\lambda}) \right|^{\frac{2}{1+b}} f(\boldsymbol{\lambda}) \exp\left(-\tfrac{1}{2(1+b)} p_2(\boldsymbol{\lambda}) \right) \, \mathrm{d}\boldsymbol{\lambda}.$$

Theorem (Goulden, Jackson, Okounkov)

$$M(N, \mathbf{y}, z; b) = (1+b) 2z \frac{\partial}{\partial z} \ln \left\langle \exp\left(\tfrac{1}{1+b} \sum_{k \geq 1} \tfrac{1}{k} y_k p_k(\boldsymbol{\lambda}) \sqrt{z}^{\,k} \right) \right\rangle$$

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Predict that replacing $2z \dfrac{\partial}{\partial z}$ with $\displaystyle\sum_{j \geq 1} j r_j \frac{\partial}{\partial y_j}$ gives the refinement.

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Verify the guess using the following lemma.

Lemma (La Croix)

If N is a fixed positive integer, then

$$\langle p_{j+2} p_\theta \rangle = (j+1) b \langle p_j p_\theta \rangle + (1+b) \sum_{i \in \theta} i \, m_i(\theta) \langle p_{j+i} p_{\theta \setminus i} \rangle + \sum_{i=0}^{j} \langle p_i p_{j-i} p_\theta \rangle .$$

Outline

The basis B_g

Is $c_{\nu,\varphi,\epsilon}(b)$ in $\operatorname{span}_{\mathbb{Z}_+}(B_g)$

- The sum $\displaystyle\sum_{\ell(\nu)=v} c_{\nu,\varphi,[2^n]}(b)$ is.
- If so, then $c_{\nu,\varphi,\epsilon}(b)$ satisfies a functional equation.
- This has been verified.
- For polynomials Ξ_g equals $\operatorname{span}_{\mathbb{Z}}(B_g)$.

$$B_g := \left\{\, b^{g-2i}(1+b)^i \colon 0 \le i \le g/2 \,\right\}$$
$$\Xi_g := \left\{\, p \colon p(b-1) = (-b)^g p\left(\tfrac{1}{b} - 1\right) \,\right\}$$

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The basis B_g

$c_{\nu,\varphi,\epsilon}(b) \in \mathrm{span}_{\mathbb{Z}_+}(B_g)$

$c_{\nu,\varphi,\epsilon}(b) \in \mathbb{Z}_+[b]$

$c_{\nu,\varphi,\epsilon}(b) \in \mathrm{span}_{\mathbb{Z}}(B_g)$

$c_{\nu,\varphi,\epsilon}(b) \in \mathbb{Q}_+[\![b]\!]$

$c_{\nu,\varphi,\epsilon}(b) \in \mathrm{span}_{\mathbb{Q}}(B_g)$

$c_{\nu,\varphi,\epsilon}(b) \in \mathbb{Q}[b]$

$c_{\nu,\varphi,\epsilon}(b) \in \Xi_g$

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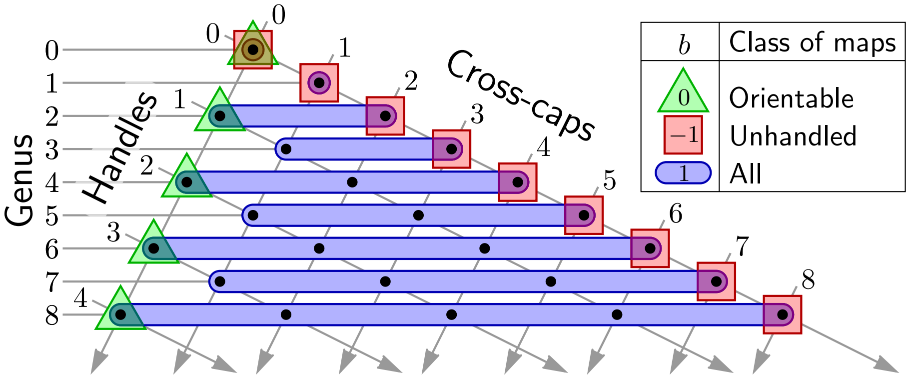
Low genus coefficients can be verified

Each dot represents a coefficient of $c_{\nu,\varphi,[2^n]}(b)$ with respect to B_g.



Shaded sums can be obtained by evaluating M at special values of b.

Possible extensions

Genus	Edges	Vertices	What is needed?
≤ 1	any number	any number	❶
≤ 2	any number	≤ 3	❶
≤ 2	and number	any number	❶ and ❸
≤ 4	any number	≤ 2	❶ and ❷
≤ 4	any number	any number	❶, ❷, and ❸
any genus	≤ 4	any number	Verified
any genus	≤ 5	any number	❸ or ❹
any genus	≤ 6	any number	❶ and ❹
any genus	any number	1	Verified

❶ $c_{\nu,\varphi,[2^n]}(b)$ is a polynomial

❷ $M(-\mathbf{x}, -\mathbf{y}, -z; -1)$ enumerates unhandled maps

❸ Combinatorial sums are in $\mathrm{span}(B_g)$

❹ An analogue of duality

Outline

Outline

A remarkable identity

Theorem (Jackson and Visentin)

$$Q(u^2, x, y, z) = \tfrac{1}{2} M(4u^2, y+u, y, xz^2) + \tfrac{1}{2} M(4u^2, y-u, y, xz^2)$$
$$= \mathrm{bis}_{\textit{even } u}\ M(4u^2, y+u, y, xz^2)$$

M is the genus series for rooted orientable maps, and
Q is the corresponding series for 4-regular maps.

$$M(u^2, x, y, z) := \sum_{\mathfrak{m} \in \mathcal{M}} u^{2g(\mathfrak{m})} x^{v(\mathfrak{m})} y^{f(\mathfrak{m})} z^{e(\mathfrak{m})}$$
$$Q(u^2, x, y, z) := \sum_{\mathfrak{m} \in \mathbb{Q}} u^{2g(\mathfrak{m})} x^{v(\mathfrak{m})} y^{f(\mathfrak{m})} z^{e(\mathfrak{m})}$$

$g(\mathfrak{m})$, $v(\mathfrak{m})$, $f(\mathfrak{m})$, and $e(\mathfrak{m})$ are genus, #vertices, #faces, and #edges

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$$= \mathrm{bis}_{\textsf{even}\ u}\ M(4u^2,y+u,y,xz^2)$$

The right hand side is a generating series for a set $\overline{\mathcal{M}}$ consisting of elements of $\mathcal{M}$ with

- each handle decorated independently in one of 4 ways, and
- an even subset of vertices marked.

A remarkable identity

Theorem (Jackson and Visentin)

$$Q(u^2, x, y, z) = \tfrac{1}{2}M(4u^2, y + u, y, xz^2) + \tfrac{1}{2}M(4u^2, y - u, y, xz^2)$$

q-Conjecture (Jackson and Visentin)

The identity is explained by a **natural** *bijection* φ *from* $\overline{\mathcal{M}}$ *to* $\mathcal{Q}$.

A decorated map with

- v *vertices*
- $2k$ *marked vertices*
- e *edges*
- f *faces*
- *genus* g

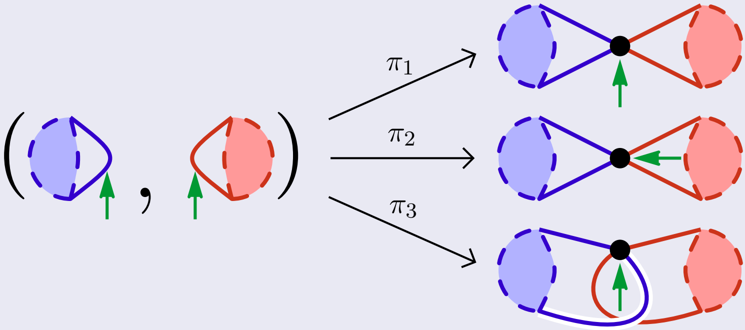
$\xrightarrow{\varphi}$

A 4-regular map with

- e *vertices*
- $2e$ *edges*
- $f + v - 2k$ *faces*
- *genus* $g + k$

Two special cases suggest comparing products on $\overline{\mathcal{M}}$ and $\mathcal{Q}$. ▸ Details

Products acting on $\mathcal{Q}$

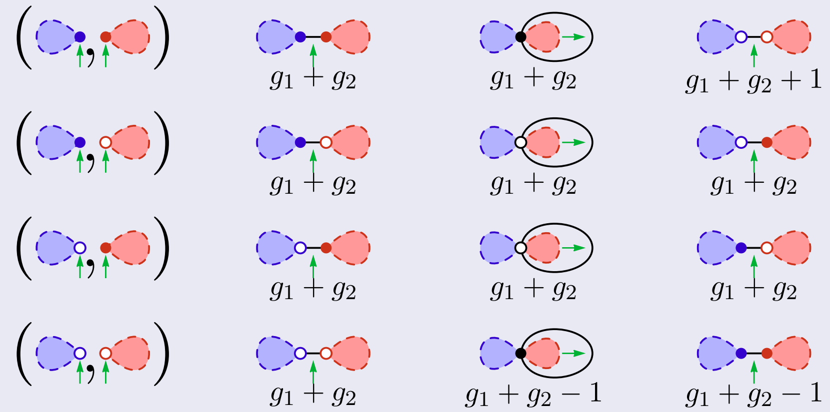


Products of rooted maps

Two special cases suggest comparing products on $\overline{\mathcal{M}}$ and $\mathcal{Q}$. ▸ Details

Products acting on $\overline{\mathcal{M}}$

$(\mathfrak{m}_1, \mathfrak{m}_2)$	$\rho_1(\mathfrak{m}_1, \mathfrak{m}_2)$	$\rho_2(\mathfrak{m}_1, \mathfrak{m}_2)$	$\rho_3(\mathfrak{m}_1, \mathfrak{m}_2)$



Outline

A refined q-Conjecture

Conjecture (La Croix)

There is a natural bijection φ from $\overline{\mathcal{M}}$ to $\mathcal{Q}$ such that:

A decorated map with

- v *vertices*
- $2k$ *marked vertices*
- e *edges*
- f *faces*
- *genus* g

$\xrightarrow{\varphi}$

A 4-regular map with

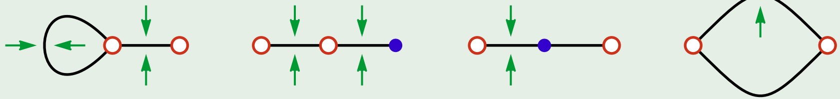
- e *vertices*
- $2e$ *edges*
- $f + v - 2k$ *faces*
- *genus* $g + k$

and

the root edge of $\varphi(\mathfrak{m})$ is face-separating *if and only if* *the root vertex of $\mathfrak{m}$ is not decorated.*

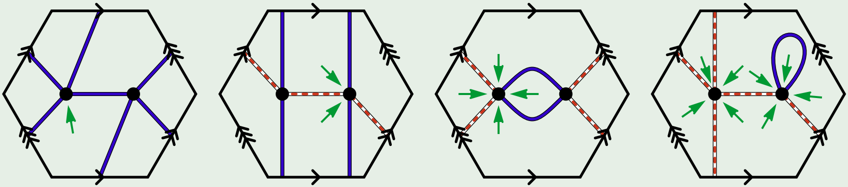
Root vertices in $\overline{\mathcal{M}}$ are related to root edges in $\mathcal{Q}$

Example (planar maps with 2 edges and 2 decorated vertices)



Nine of eleven rooted maps have a decorated root vertex.

Example (4-regular maps on the torus with two vertices)



Nine of fifteen rooted maps have face-non-separating root edges.

Testing the refined conjecture

The refined conjecture has been tested numerically for images of maps with at most 20 edges by expressing the relevant generating series as linear combination of Q and the generating series for $(3,1)$-pseudo-4-regular maps.

An analytic reformulation

The existence of an appropriate bijection, modulo the definition of 'natural', is equivalent to the following conjectured identity:

$$\left\langle (p_4 + p_1 p_3) \mathrm{e}^{p_4 x} \right\rangle_{(N)} \left\langle \mathrm{e}^{p_4 x} \right\rangle_{(N+1)} = - \left\langle m_{[1,3]} \mathrm{e}^{p_4 x} \right\rangle_{(N+1)} \left\langle \mathrm{e}^{p_4 x} \right\rangle_{(N)} .$$

for every positive integer N.

Outline

1. Background
   - The objects
   - An enumerative problem, and two generating series
2. The b-Conjecture
   - An algebraic generalization and the b-Conjecture
   - A family of invariants
   - The invariants resolve a special case
   - Evidence that they are b-invariants
3. The q-Conjecture
   - A remarkable identity and the q-Conjecture
   - A refinement
4. Future Work

Future Work

On the b-Conjecture

- Show that $c_{\nu,\varphi,\epsilon}(b)$ is a polynomial for every ν, φ, and ϵ.
- Show that the generating series for maps is an element of $\mathrm{span}(B_g)$.
- Explicitly compute the generating series for unhandled maps.
- Extend the analysis to hypermaps.

On the q-Conjecture

- Verify one of the algebraic or analytic properties that characterizes the refinement.
- Use the refinement to determine additional structure of the bijection.

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The End

Thank You

Appendices

Jack Symmetric Functions

With respect to the inner product defined by

$$\langle p_\lambda(\mathbf{x}), p_\mu(\mathbf{x})\rangle = \delta_{\lambda,\mu}\frac{|\lambda|!}{|C_\lambda|}\alpha^{\ell(\lambda)},$$

Jack symmetric functions are the unique family satisfying:

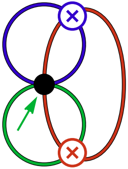
(P1) (Orthogonality) If $\lambda \neq \mu$, then $\langle J_\lambda, J_\mu\rangle_\alpha = 0$.

(P2) (Triangularity) $J_\lambda = \sum_{\mu \preccurlyeq \lambda} v_{\lambda\mu}(\alpha) m_\mu$, where $v_{\lambda\mu}(\alpha)$ is a rational function in α, and '$\preccurlyeq$' denotes the natural order on partitions.

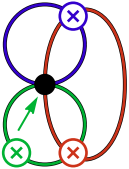
(P3) (Normalization) If $|\lambda| = n$, then $v_{\lambda,[1^n]}(\alpha) = n!$.

Return

Computing η



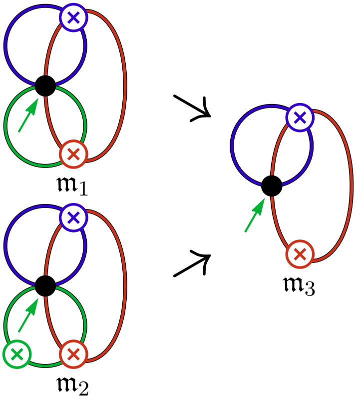
$\mathfrak{m}_1$



$\mathfrak{m}_2$

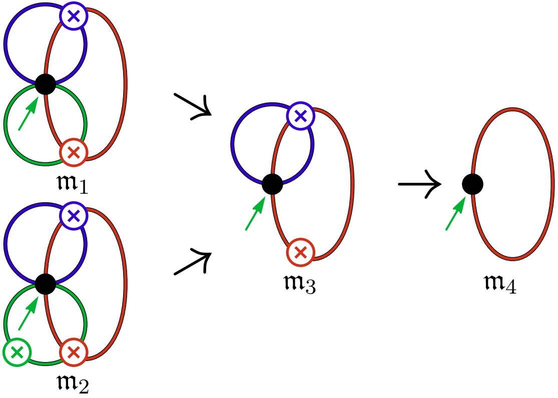
Return

Computing η



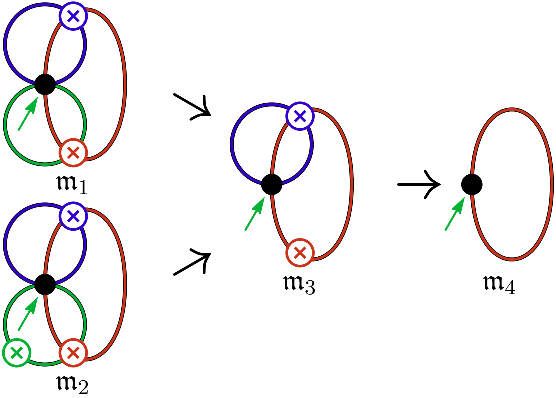
Return

Computing η



Return

Computing η



$$\eta(\mathfrak{m}_4) = 0$$

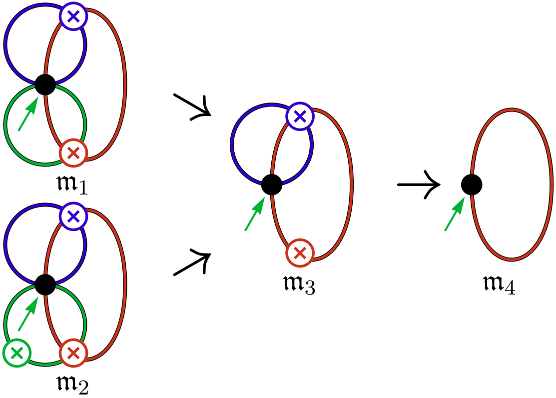
$$\eta(\mathfrak{m}_3) = \eta(\mathfrak{m}_4) + 1 = 1$$

$$\eta(\mathfrak{m}_2) = \eta(\mathfrak{m}_3) + 1 = 2$$

$$\eta(\mathfrak{m}_1) = \eta(\mathfrak{m}_3) = 1$$

Return

Computing η



$$\eta(\mathfrak{m}_4) = 0$$

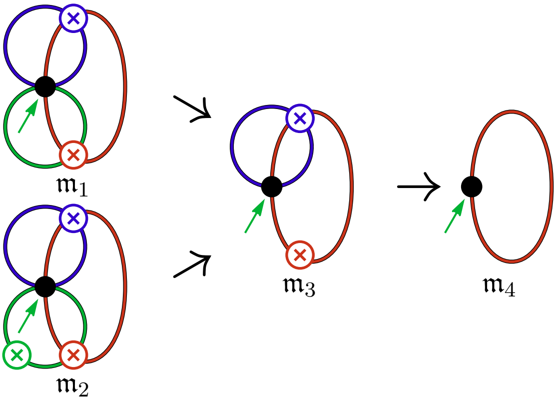
$$\eta(\mathfrak{m}_3) = \eta(\mathfrak{m}_4) + 1 = 1$$

$$\eta(\mathfrak{m}_2) = \eta(\mathfrak{m}_3) + 1 = 2$$

$$\eta(\mathfrak{m}_1) = \eta(\mathfrak{m}_3) = 1$$

Return

Computing η



$$\eta(\mathfrak{m}_4) = 0$$
$$\eta(\mathfrak{m}_3) = \eta(\mathfrak{m}_4) + 1 = 1$$
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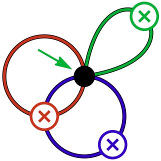
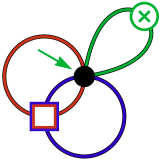
Return

Computing η



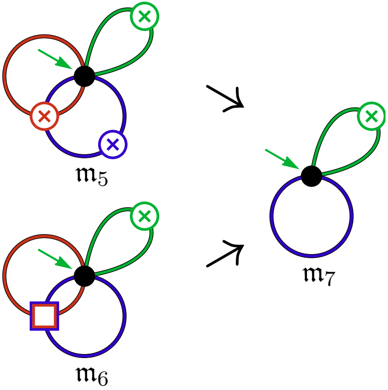
$\mathfrak{m}_5$



$\mathfrak{m}_6$

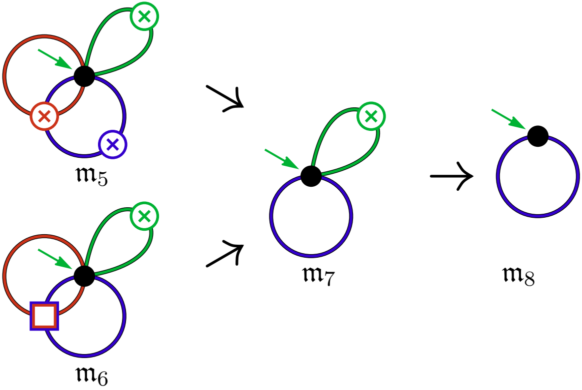
Return

Computing η



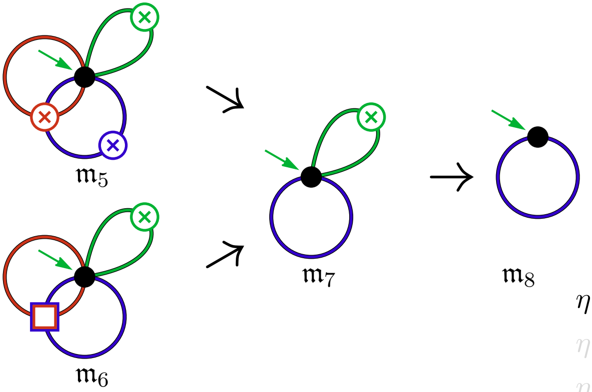
◂ Return

Computing η



Return

Computing η



$\eta(\mathfrak{m}_8) = 0$

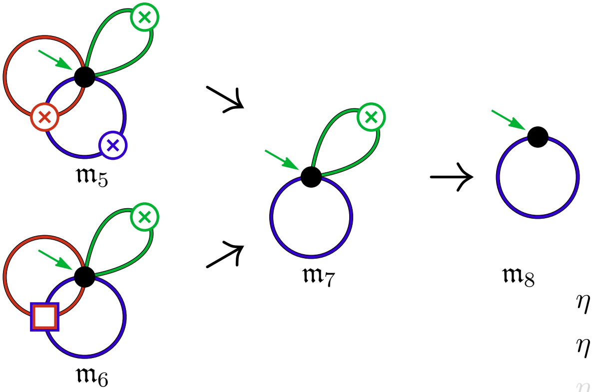
$\eta(\mathfrak{m}_7) = \eta(\mathfrak{m}_8) + 1 = 1$

$\eta(\mathfrak{m}_6) = \eta(\mathfrak{m}_7)$ or $\eta(\mathfrak{m}_7) + 1$

$\eta(\mathfrak{m}_5) = \eta(\mathfrak{m}_7) + 1$ or $\eta(\mathfrak{m}_7)$

Return

$\{\eta(\mathfrak{m}_5), \eta(\mathfrak{m}_6)\} = \{1, 2\}$



$$\eta(\mathfrak{m}_8) = 0$$

$$\eta(\mathfrak{m}_7) = \eta(\mathfrak{m}_8) + 1 = 1$$

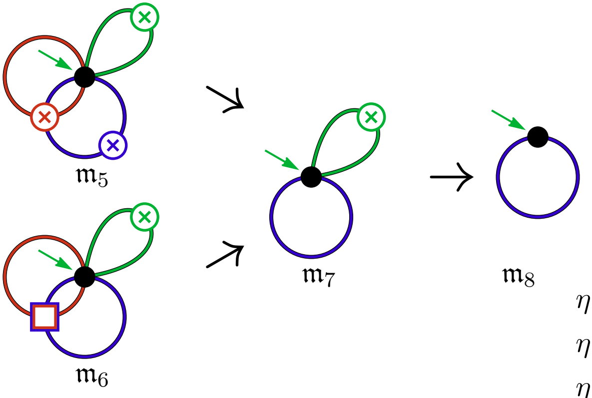
$$\eta(\mathfrak{m}_6) = \eta(\mathfrak{m}_7) \text{ or } \eta(\mathfrak{m}_7) + 1$$

$$\eta(\mathfrak{m}_5) = \eta(\mathfrak{m}_7) + 1 \text{ or } \eta(\mathfrak{m}_7)$$

Return

$$\{\eta(\mathfrak{m}_5), \eta(\mathfrak{m}_6)\} = \{1, 2\}$$

Computing η



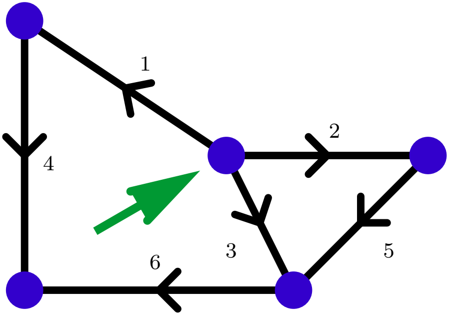
$$\eta(\mathfrak{m}_8) = 0$$
$$\eta(\mathfrak{m}_7) = \eta(\mathfrak{m}_8) + 1 = 1$$
$$\eta(\mathfrak{m}_6) = \eta(\mathfrak{m}_7) \text{ or } \eta(\mathfrak{m}_7) + 1$$
$$\eta(\mathfrak{m}_5) = \eta(\mathfrak{m}_7) + 1 \text{ or } \eta(\mathfrak{m}_7)$$

$$\{\eta(\mathfrak{m}_5), \eta(\mathfrak{m}_6)\} = \{1, 2\}$$

Return

Encoding Orientable Maps

1. Orient and label the edges.
2. This induces labels on flags.
3. Clockwise circulations at each vertex determine ν.
4. Face circulations are the cycles of $\epsilon\nu$.



$$\epsilon = (1\ 1')(2\ 2')(3\ 3')(4\ 4')(5\ 5')(6\ 6')$$

$$\nu = (1\ 2\ 3)(1'\ 4)(2'\ 5)(3'\ 5'\ 6)(4'\ 6')$$

$$\epsilon\nu = \varphi = (1\ 4\ 6'\ 3')(1'\ 2\ 5\ 6\ 4')(2'\ 3\ 5')$$

Return

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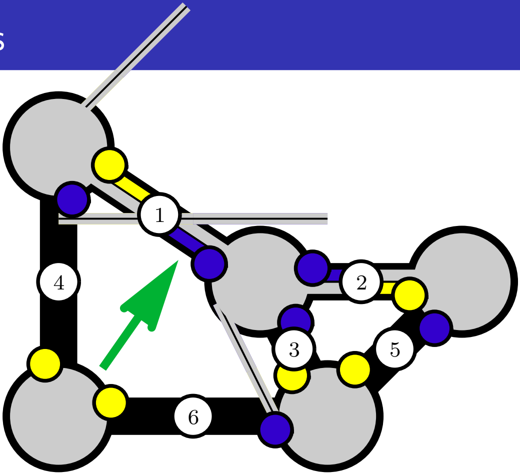

$$\epsilon = (1\ 1')(2\ 2')(3\ 3')(4\ 4')(5\ 5')(6\ 6')$$

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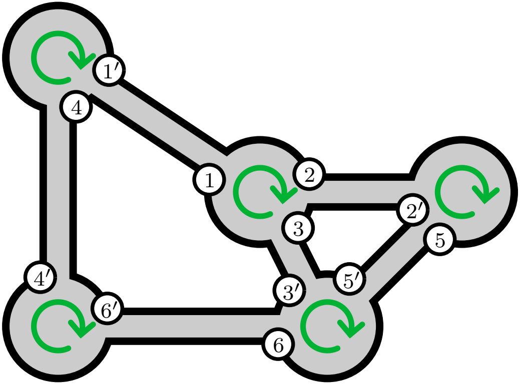

$$\epsilon = (1\ 1')(2\ 2')(3\ 3')(4\ 4')(5\ 5')(6\ 6')$$

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Return

1. Orient and label the edges.
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3. Clockwise circulations at each vertex determine ν.
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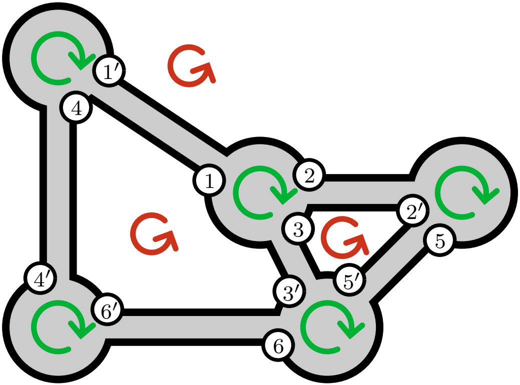
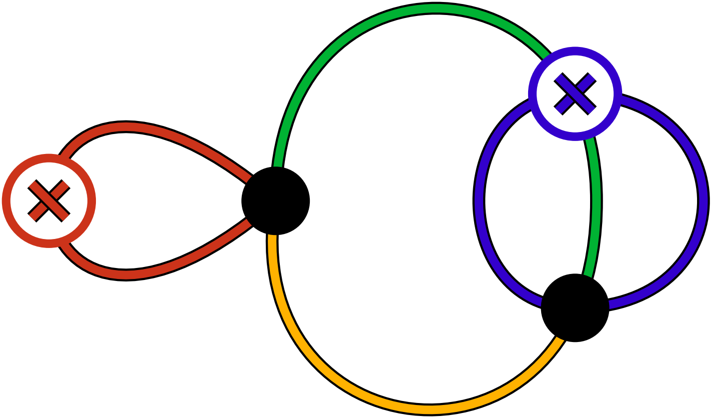


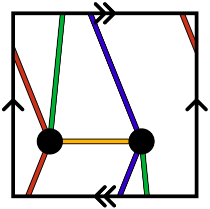
$$\epsilon = (1\ 1')(2\ 2')(3\ 3')(4\ 4')(5\ 5')(6\ 6')$$
$$\nu = (1\ 2\ 3)(1'\ 4)(2'\ 5)(3'\ 5'\ 6)(4'\ 6')$$
$$\epsilon\nu = \varphi = (1\ 4\ 6'\ 3')(1'\ 2\ 5\ 6\ 4')(2'\ 3\ 5')$$

Return

Encoding Locally Orientable Maps

Return

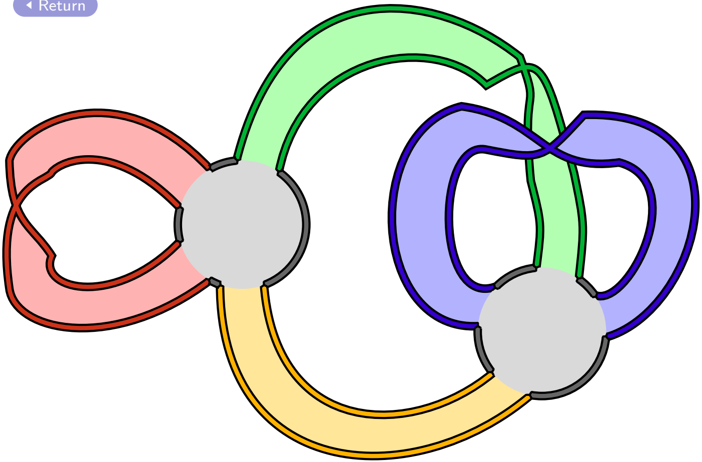


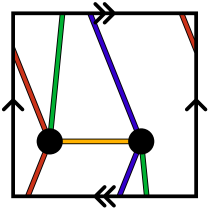


Start with a ribbon graph.

Encoding Locally Orientable Maps

◂ Return

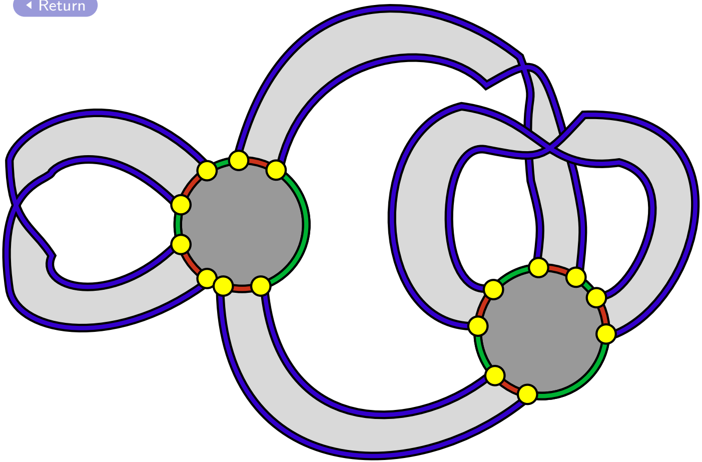


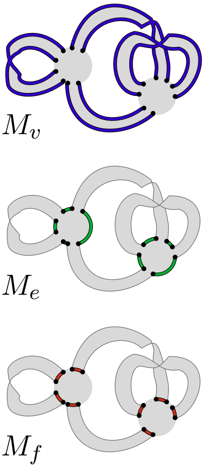


Start with a ribbon graph.

Encoding Locally Orientable Maps

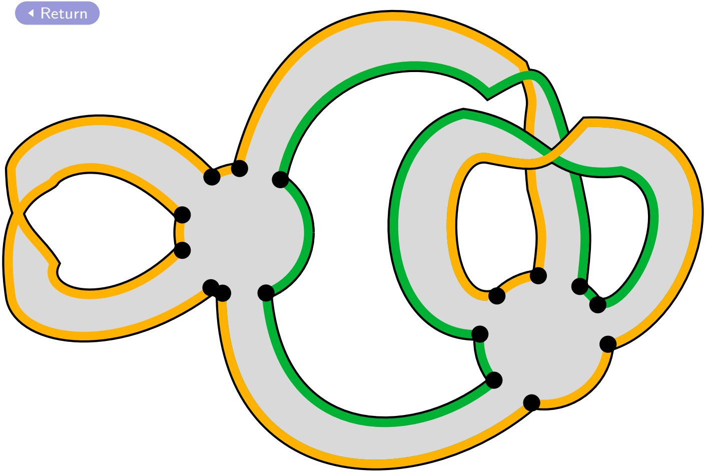
◂ Return

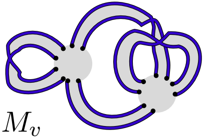




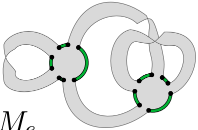
Ribbon boundaries determine 3 perfect matchings of flags.

Return





M_v

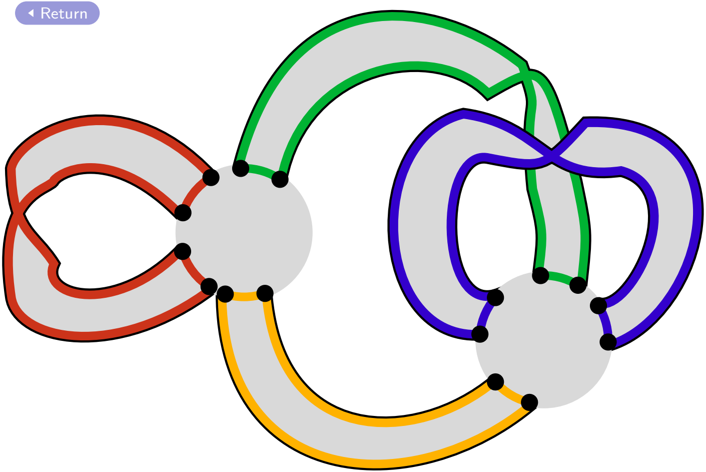


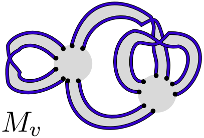
M_e

Pairs of matchings determine, faces,

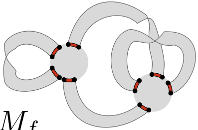
Encoding Locally Orientable Maps

Return





M_v

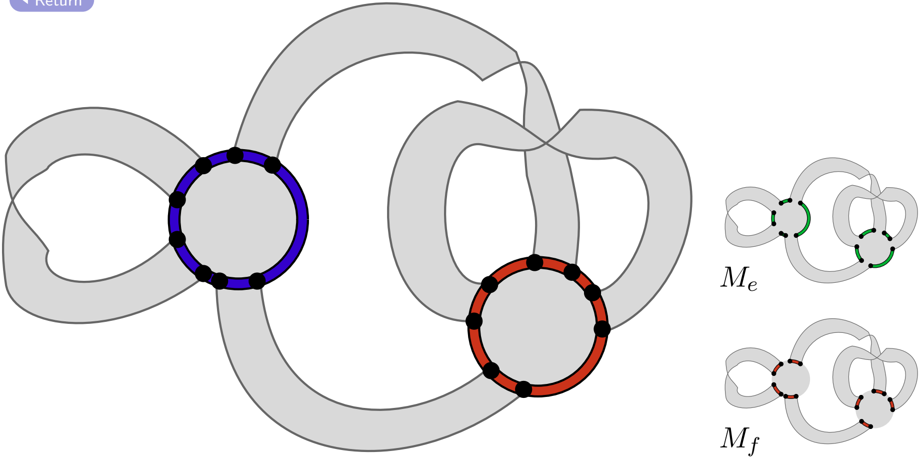


M_f

Pairs of matchings determine, faces, edges,

Encoding Locally Orientable Maps

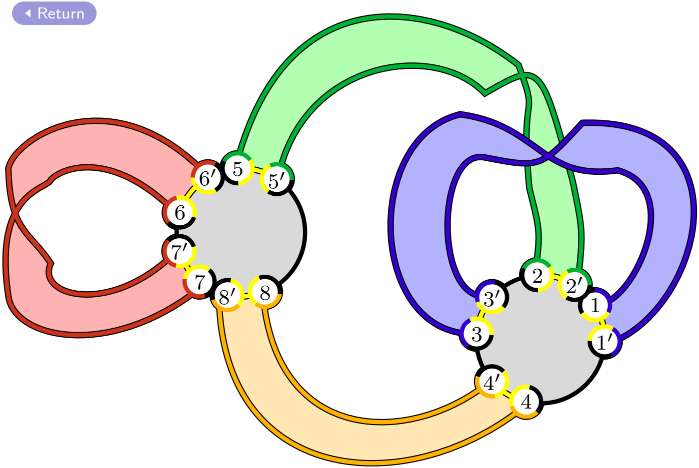
Return

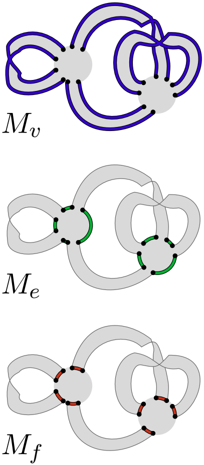


Pairs of matchings determine, faces, edges, and vertices.

Encoding Locally Orientable Maps

Return



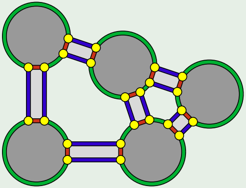


$$M_v = \{\{1,3\},\{1',3'\},\{2,5\},\{2',5'\},\{4,8'\},\{4',8\},\{6,7\},\{6',7'\}\}$$

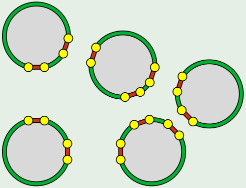
$$M_e = \{\{1,2'\},\{1',4\},\{2,3'\},\{3,4'\},\{5,6'\},\{5',8\},\{6,7'\},\{7,8'\}\}$$

$$M_f = \{\{1,1'\},\{2,2'\},\{3,3'\},\{4,4'\},\{5,5'\},\{6,6'\},\{7,7'\},\{8,8'\}\}$$

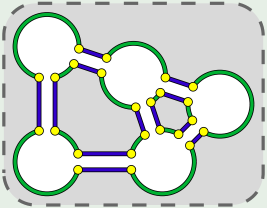
Example



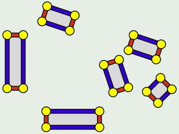
is enumerated by $\left(x_2^3\, x_3^2\right)\left(y_3\, y_4\, y_5\right)\left(z_2^6\right)$.



$\nu = [2^3, 3^2]$



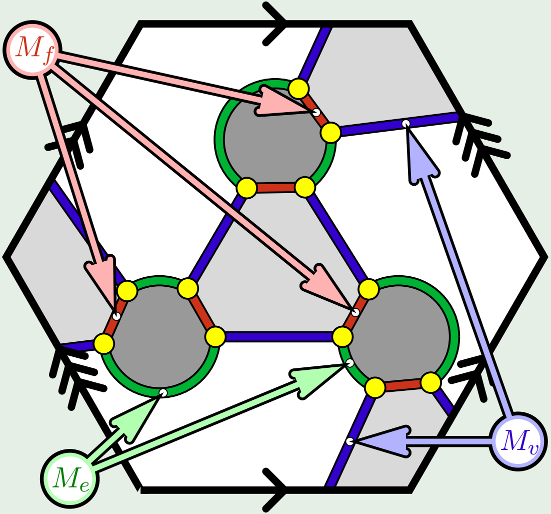
$\varphi = [3, 4, 5]$

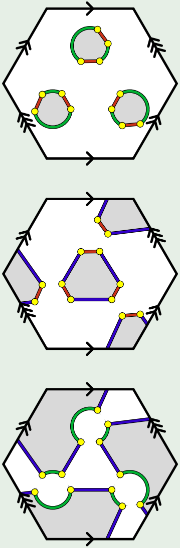


$\epsilon = [2^6]$

Return

Example





$\nu = [2^3]$

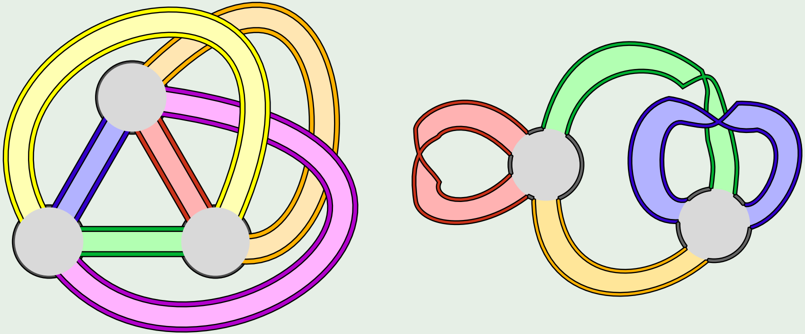
$\epsilon = [3^2]$

$\varphi = [6]$

Return

Ribbon Graphs

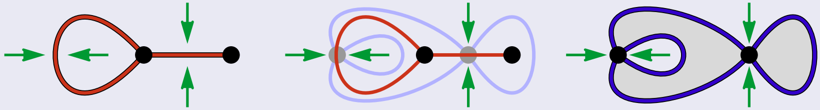
Example



◂ Return

Two Clues

The radial construction for undecorated maps

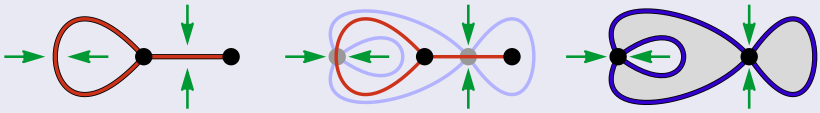


One extra image of φ

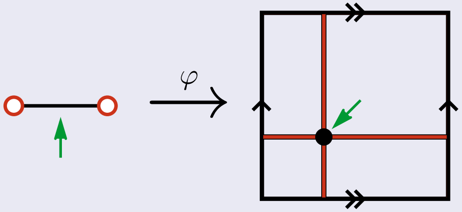
Return to products

Two Clues

The radial construction for undecorated maps



One extra image of φ



Return to products

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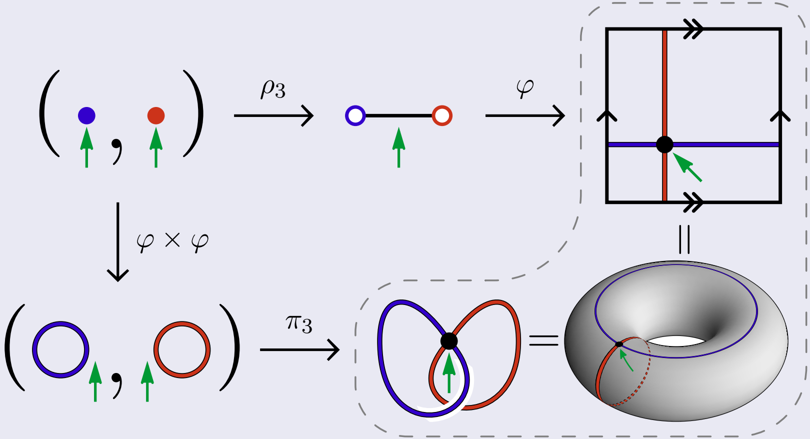
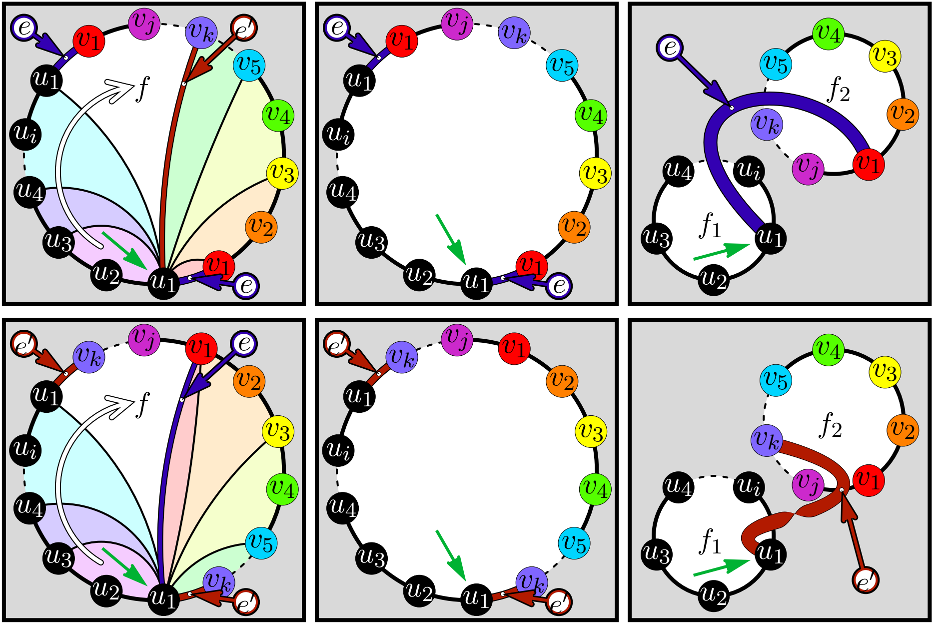
One extra image of φ



Return to products


e
v1
vj
vk
e'
v5
v4
v3
v2
u1
ui
u4
u3
u2
f
f1
f2

The integration formula

Define the expectation operator $\langle \cdot \rangle$ by

$$\langle f \rangle := \int_{\mathbb{R}^N} \left| V(\boldsymbol{\lambda}) \right|^{\frac{2}{1+b}} f(\boldsymbol{\lambda}) \exp\left(-\tfrac{1}{2(1+b)} p_2(\boldsymbol{\lambda}) \right) \, \mathrm{d}\boldsymbol{\lambda}.$$

Lemma (Okounkov)

$$\langle J_\theta(\boldsymbol{\lambda}, 1+b) \rangle = J_\theta(\mathbf{1}_N, 1+b)[p_{[2^n]}] J_\theta \langle 1 \rangle$$