Combinatorial Models for Random Matrices with Gaussian Entries

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Collaborators

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Outline

Random Matrices

A **Random matrix** is a matrix with random elements.

Often:

- questions involve eigenvalues,
- entries are mostly independent,
- entries are real or complex.

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ A_{21} & A_{22} & \dots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nm} \end{bmatrix}$$

Some Combinatorial Random Matrices

Ginibre	Hermite (Gaussian)	Laguerre (Wishart)
A	$H = \frac{1}{2}(A + A^*)$	$W = AA^*$

Less Combinatorial Random Matrices

Jacobi / MANOVA	Unitary	Orthogonal	Circular

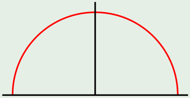
A typical theorem about random matrices:

1. Shows dependence on entry distribution is weak.
2. Establishes the result when entries are i.i.d. Gaussian.

We'll focus on step 2.

Example (Wigner's Semicircle Law)

$$E(\mathrm{tr}(H^{2k})) = n^{k-1}\frac{1}{k+1}\binom{2k}{k} + O(n^{k-2})$$



Gaussian Random Variables Have Combinatorial Moments

Real Gaussians, X, with mean 0 and variance 1

$$E(X^n) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^n \, \mathrm{e}^{-\frac{x^2}{2}} \, \mathrm{d}x = \begin{cases} m!! & \text{if } n = 2m \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Complex Gaussians, Z, with mean 0 and variance 1

$$E(Z^k \overline{Z}^l) = \frac{1}{\pi} \int_{\mathbb{C}} z^k \overline{z}^l \, \mathrm{e}^{-|z|^2} \, \mathrm{d}z = \begin{cases} k! & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

Gaussian Ensembles

For $\beta \in \{1,2,4\}$ an element of the β-Gaussian ensemble is constructed as

$$A = G + G^*$$

where G is $n \times n$ with i.i.d. Guassian entries selected from $\{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.

Motivating Question

What is the value of $E(f(A))$, when f is a symmetric function of the eigenvalues of its argument?

Example

$$E(\operatorname{tr}(A^5)\operatorname{tr}(A^3)) = \begin{cases} 510n + 720n^2 + 360n^3 + 90n^4 & \beta = 1 \\ 60n^2 + 45n^4 & \beta = 2 \\ -\frac{255}{16}n + 45n^2 - 45n^3 + \frac{45}{2}n^4 & \beta = 4 \end{cases}$$

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Example

$$E(\mathrm{tr}(A^4)) = \begin{cases} 5n + 5n^2 + 2n^3 & \beta = 1 \\ n + 2n^3 & \beta = 2 \\ \frac{5}{4}n - \frac{5}{2}n^2 + 2n^3 & \beta = 4 \end{cases}$$

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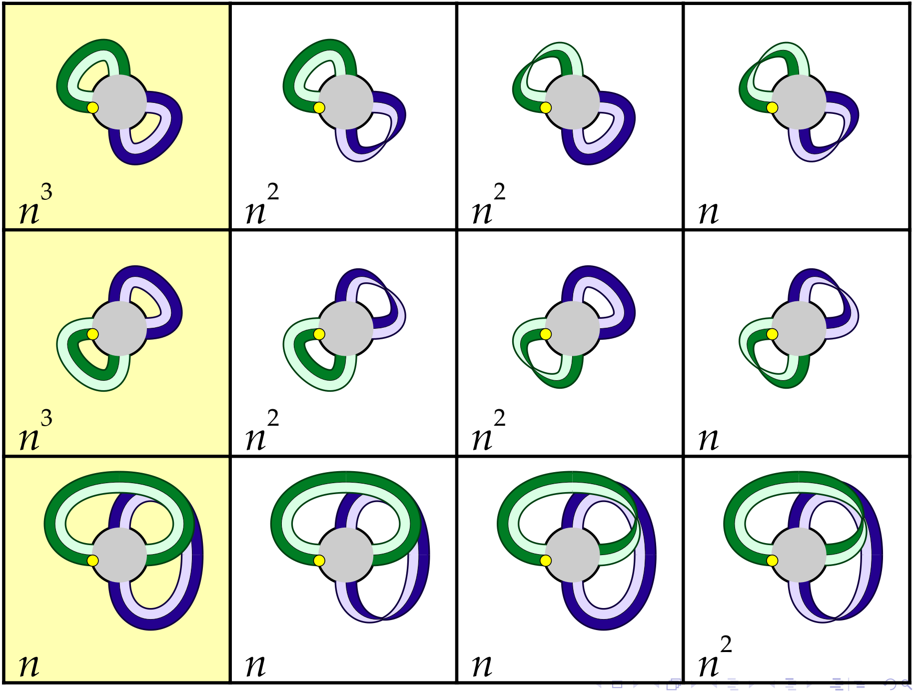
Example

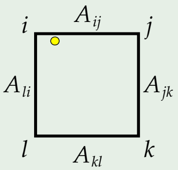
$$E(\mathrm{tr}(A^4)) = \begin{cases} 5n + 5n^2 + 2n^3 & \beta = 1 \\ n + 2n^3 & \beta = 2 \\ (1 + b + 3b^2)n + 5bn^2 + 2n^3 & \beta = \frac{2}{1+b} \end{cases}$$


n^3
n^2
n^2
n
n^3
n^2
n^2
n
n
n
n
n^2

What's being counted?

Example (for $\beta \in \{1, 2, 4\}$)

$$\mathrm{tr}(A^4) = \sum_{i,j,k,l} A_{ij} A_{jk} A_{kl} A_{li}$$



$$\begin{aligned}\mathrm{E}\left(\mathrm{tr}(A^4)\right) &= (n)_1 \mathrm{E}(A_{11}A_{11}A_{11}A_{11}) + (n)_2 \mathrm{E}(2A_{11}A_{12}A_{22}A_{21}) \\ &+ (n)_2 \mathrm{E}(4A_{11}A_{11}A_{12}A_{21}) + (n)_4 \mathrm{E}(A_{12}A_{23}A_{34}A_{41}) \\ &+ (n)_2 \mathrm{E}(A_{12}A_{21}A_{12}A_{21}) + (n)_3 \mathrm{E}(4A_{11}A_{12}A_{23}A_{31}) \\ &+ (n)_3 \mathrm{E}(2A_{12}A_{21}A_{13}A_{31})\end{aligned}$$

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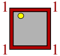

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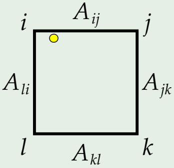

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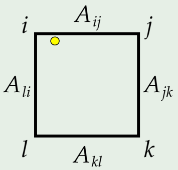

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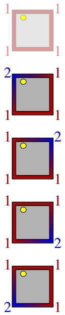

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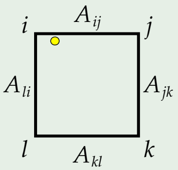

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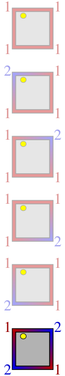

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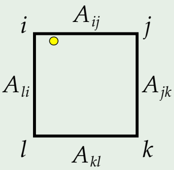

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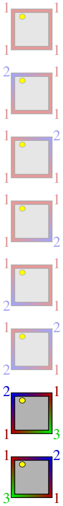

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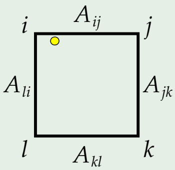

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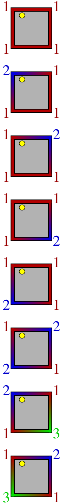

Expectations as Sums

Since the entries of A are centred Gaussians, Wick's lemma applies

$$E(A_{i_1j_1}A_{i_2j_2}\dots A_{i_kj_k}) = \sum_m \prod_{(u,v)\in m} E(A_uA_v)$$

summed over perfect matchings of the multiset $\{i_1j_1, i_2j_2, \dots, i_kj_k\}$

Example

$$\begin{aligned} E(A_uA_vA_wA_x) &= E(A_uA_v)E(A_wA_x) \\ &\quad + E(A_uA_w)E(A_vA_x) \\ &\quad + E(A_uA_x)E(A_vA_w) \end{aligned}$$

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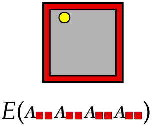

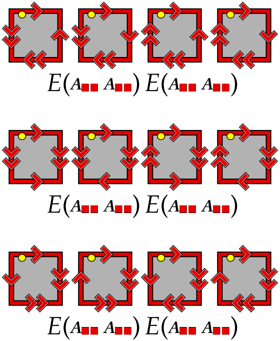

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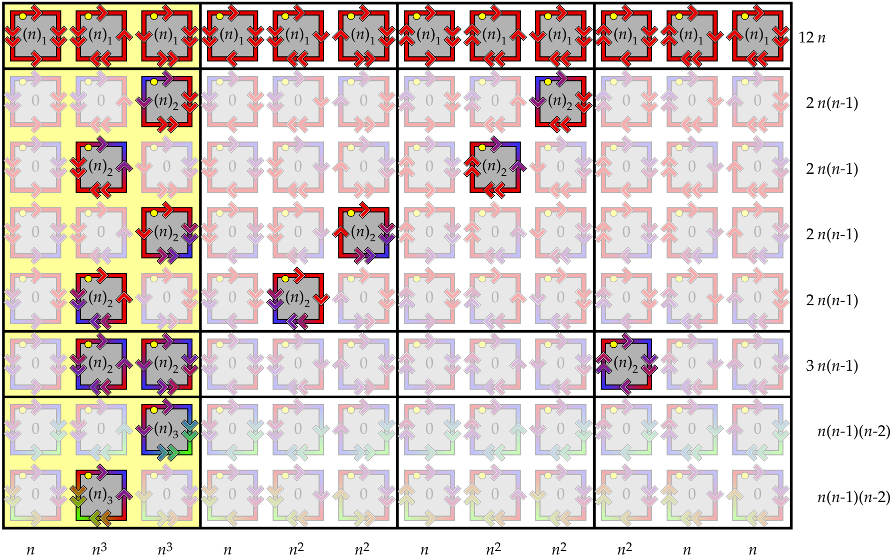
summed over perfect matchings of the multiset $\{i_1 j_1, i_2 j_2, \dots, i_k j_k\}$

$$\sum_{p \text{ a painting}} \#\{\text{pairings consistent with } p\}$$

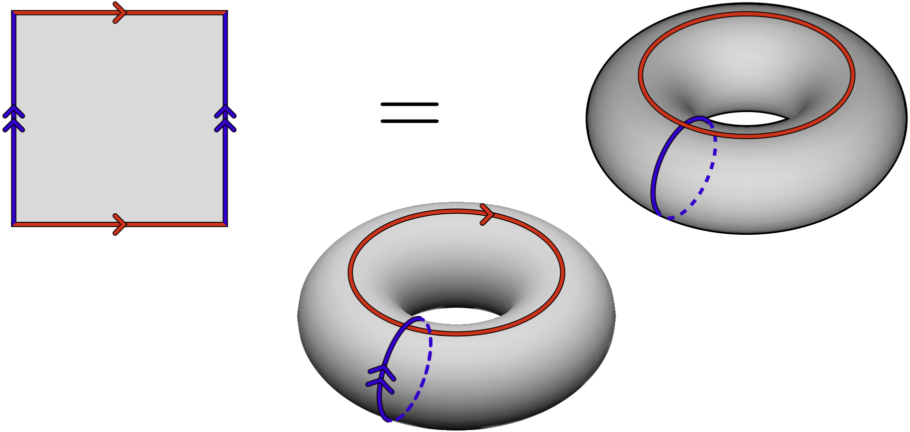
$$= \sum_{m \text{ a matching}} \#\{\text{paintings consistent with } m\}$$

Count the polygon glueings in 2 different ways



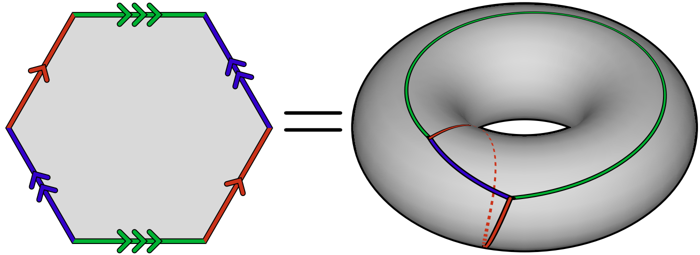
Polygon Glueings = Maps

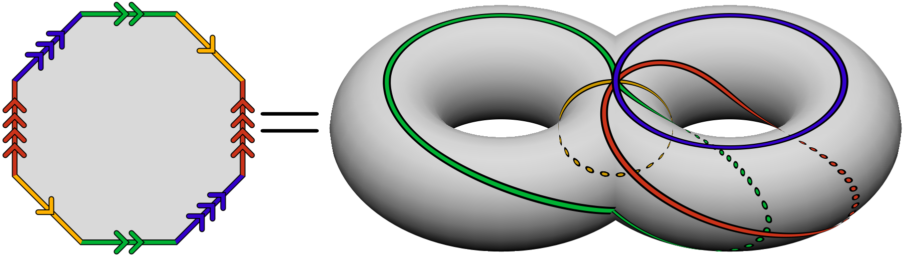
Identifying the edges of a polygon creates a surface.



Its boundary is a graph embedded in the surface.

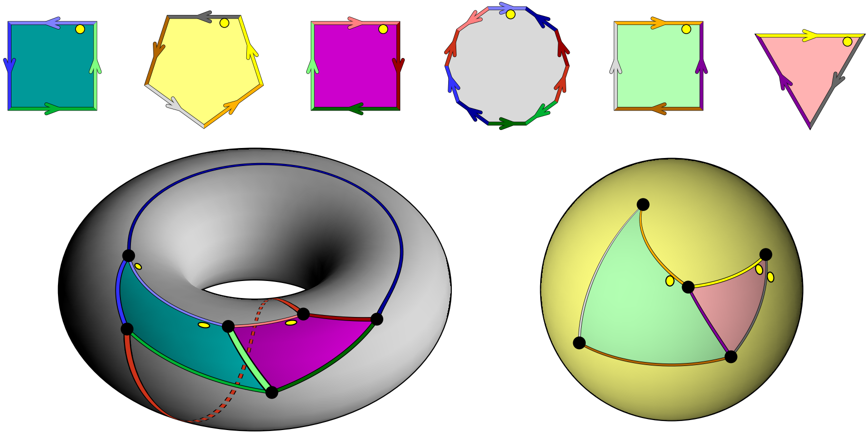
Polygon Glueings = Maps





Polygon Glueings = Maps

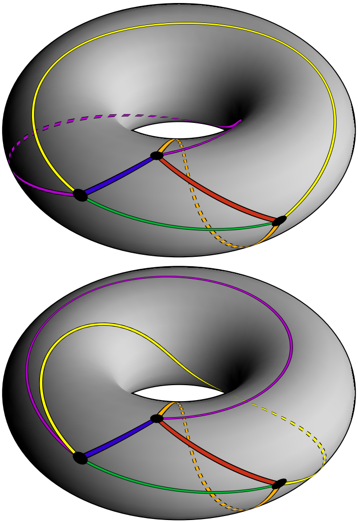
Extra polygons give extra faces (and possibles extra components)

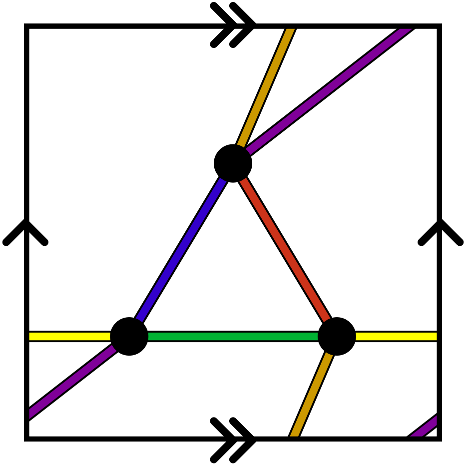


This glueing contributes n^{11} to $E\big(p_{3,4^3,5,10}(A)\big)$

Equivalence of Maps

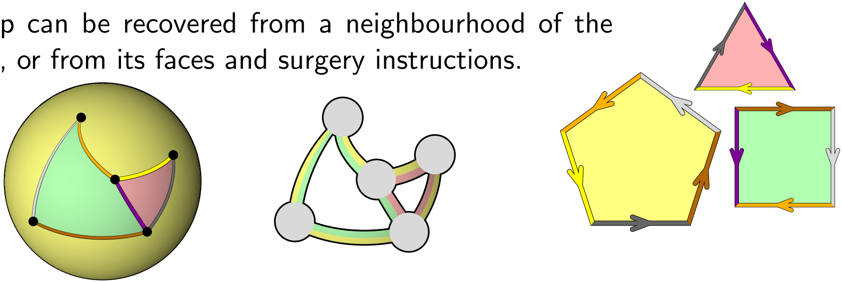
Two maps are equivalent if the embeddings are homeomorphic.



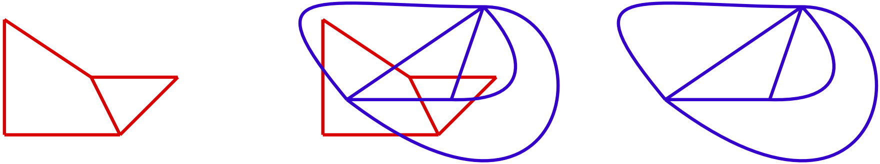


Vertices and Face

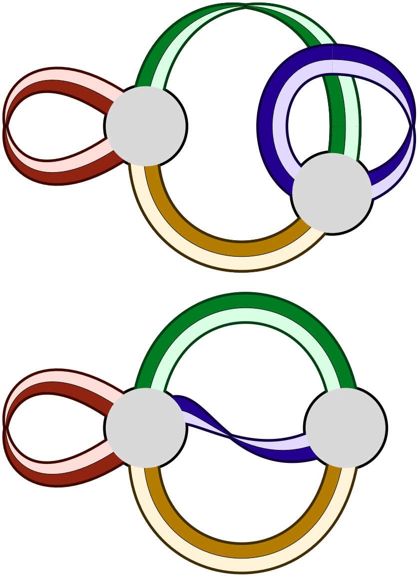
A map can be recovered from a neighbourhood of the graph, or from its faces and surgery instructions.

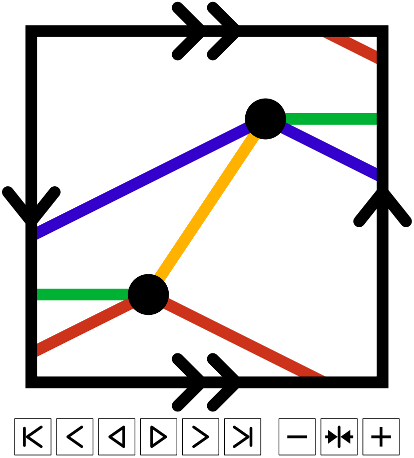


Vertex and face degrees are interchanged by duality.



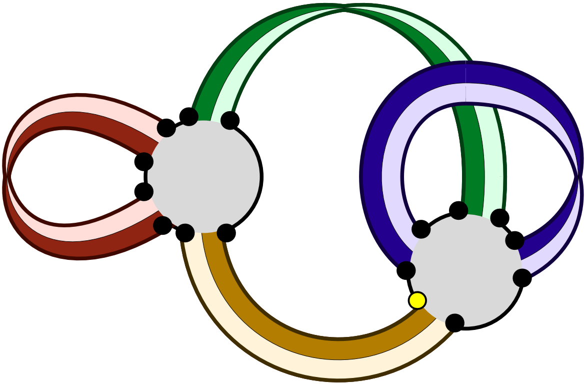
Duality

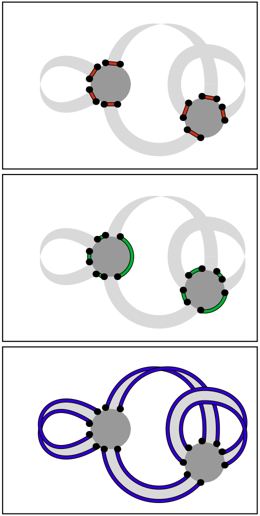




Outline

Three natural involutions reroot a map.





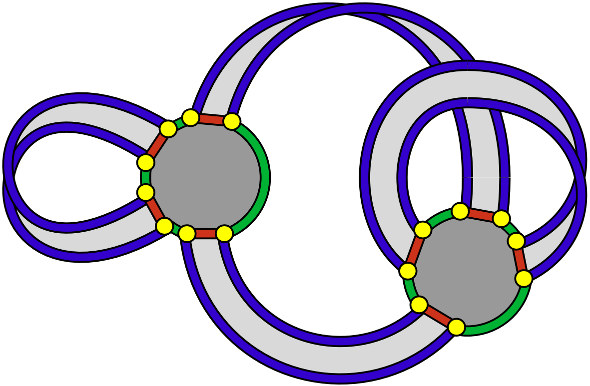
Across Edge

Around Vertex

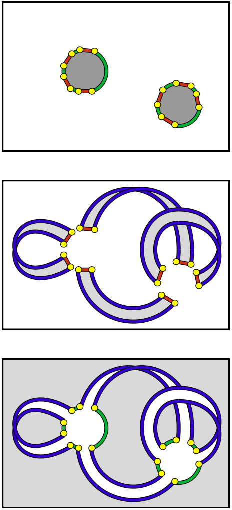
Along Edge

The Matchings Encode the Map

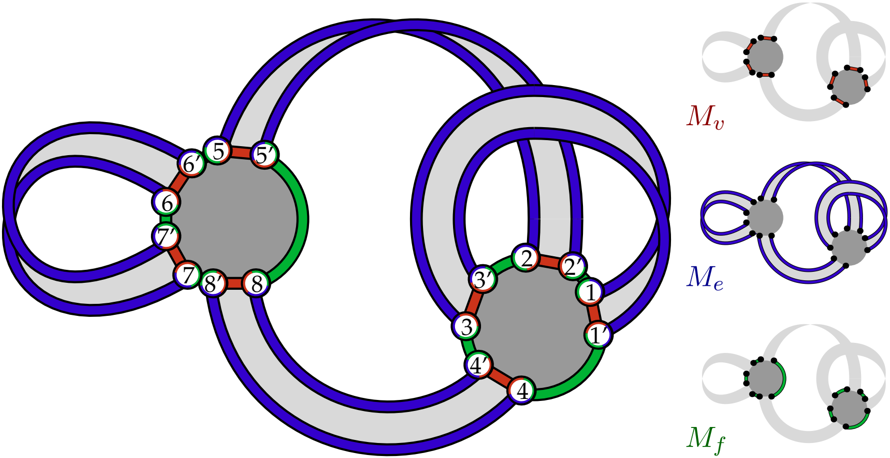
Each involution gives a perfect matching of flags.



Pairs of matchings recover vertices, edges, and faces.



Encodinge Non-oriented Maps



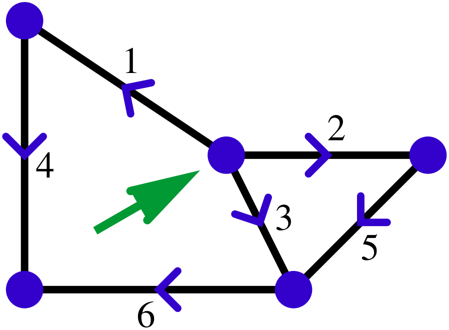
$$M_v = \{\{1,3\},\{1',3'\},\{2,5\},\{2',5'\},\{4,8'\},\{4',8\},\{6,7\},\{6',7'\}\}$$

$$M_e = \{\{1,2'\},\{1',4\},\{2,3'\},\{3,4'\},\{5,6'\},\{5',8\},\{6,7'\},\{7,8'\}\}$$

$$M_f = \{\{1,1'\},\{2,2'\},\{3,3'\},\{4,4'\},\{5,5'\},\{6,6'\},\{7,7'\},\{8,8'\}\}$$

Encoding Oriented Maps ($\beta = 2$)

1. Orient and label the edges.
2. This induces labels on flags.
3. Clockwise circulations at each vertex determine ν.
4. Face circulations are the cycles of $\epsilon\nu$.



$$\epsilon = (1\ 1')(2\ 2')(3\ 3')(4\ 4')(5\ 5')(6\ 6')$$
$$\nu = (1\ 2\ 3)(1'\ 4)(2'\ 5)(3'\ 5'\ 6)(4'\ 6')$$
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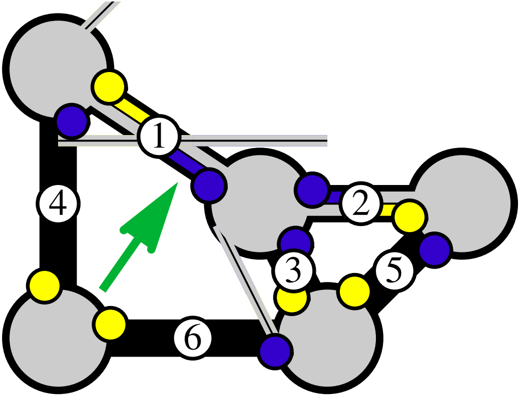

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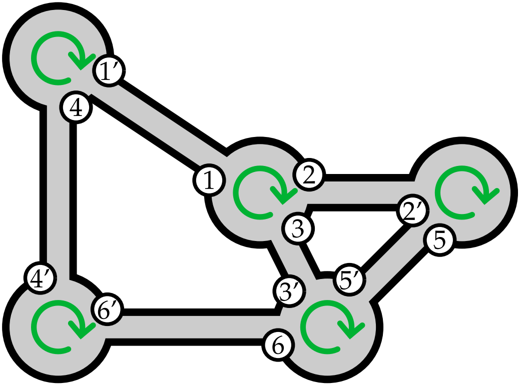

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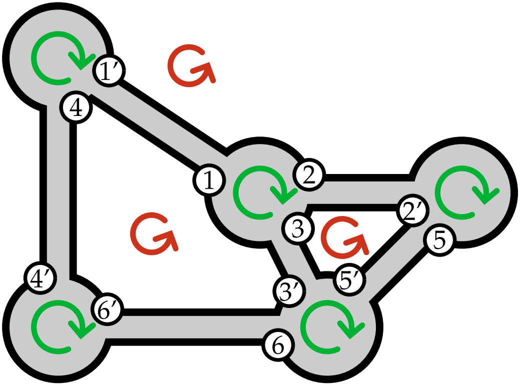


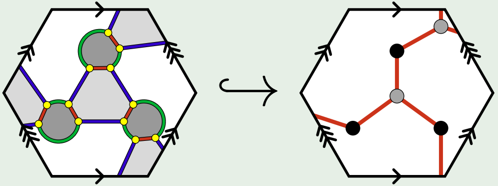
$$\epsilon = (1\ 1')(2\ 2')(3\ 3')(4\ 4')(5\ 5')(6\ 6')$$
$$\nu = (1\ 2\ 3)(1'\ 4)(2'\ 5)(3'\ 5'\ 6)(4'\ 6')$$
$$\epsilon\nu = \phi = (1\ 4\ 6'\ 3')(1'\ 2\ 5\ 6\ 4')(2'\ 3\ 5')$$

Hypermaps

An **arbitrary** triple of perfect matchings determines a **hypermap** with cycles of $M_e \cup M_f$, $M_e \cup M_v$, and $M_v \cup M_f$ determining vertices, faces, and **hyperedges**. ▸ Example

Hypermaps both **specialize** and generalize maps.

Example



A hypermap can be represented as a bipartite map.

Hypermaps

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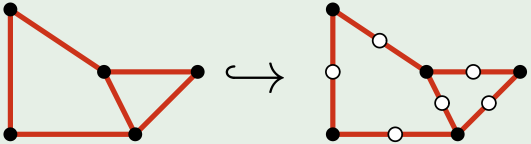
Hypermaps both specialize and **generalize** maps.

Example



Subdivide edges to get a hypermap from a map.

The Hypermap Series

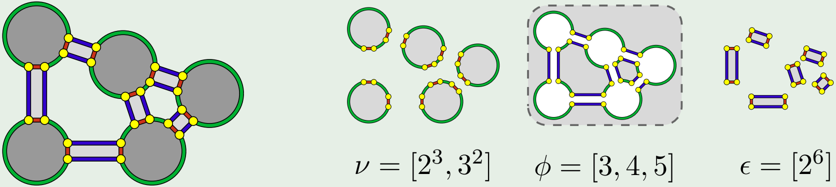
Definition

The **hypermap series** for a set $\mathcal{H}$ of hypermaps is the combinatorial sum

$$H(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \sum_{\mathfrak{h} \in \mathcal{H}} \mathbf{x}^{\nu(\mathfrak{h})} \mathbf{y}^{\phi(\mathfrak{h})} \mathbf{z}^{\epsilon(\mathfrak{h})}$$

$\nu(\mathfrak{h})$, $\phi(\mathfrak{h})$, and $\epsilon(\mathfrak{h})$ are vertex-, hyperface-, and hyperedge- degrees.

Example



$\nu = [2^3, 3^2]$ $\quad \phi = [3, 4, 5]$ $\quad \epsilon = [2^6]$

contributes $12\left(x_2^3\, x_3^2\right)\left(y_3\, y_4\, y_5\right)\left(z_2^6\right)$.

Generating Series Algebraically

- Instead of counting rooted maps, we can count labelled hypermaps. This adds easily computable multiplicities.
- These in turn are reduced to computing a multiplication table for appropriate algebras.
  - $\mathbb{C}[\mathfrak{S}]$ for oriented hypermaps
  - A double-coset algebra for non-oriented hypermaps.
- These can be evaluated via character theory.
- Appropriate characters appear as coefficients of symmetric functions.
- Standard enumerative techniques restrict the solution to connected maps and remove factors introduced by the labelling.

Explicit Generating Series

Some hypermap series can be computed explicitly.

Theorem (Jackson and Visentin - 1990) ▸ Matrix Derivation

When $\mathcal{H}$ is the set of connected oriented hypermaps,

$$H_{\mathcal{O}}\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}); 0\Big) = t\frac{\partial}{\partial t} \ln \left(\sum_{\theta \in \mathscr{P}} t^{|\theta|} H_\theta s_\theta(\mathbf{x}) s_\theta(\mathbf{y}) s_\theta(\mathbf{z}) \right) \Bigg|_{t=1}.$$

Theorem (Goulden and Jackson - 1996) ▸ Matrix Derivation

When $\mathcal{H}$ is the set of connected non-oriented hypermaps,

$$H_{\mathcal{A}}\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}); 1\Big) = 2t\frac{\partial}{\partial t} \ln \left(\sum_{\theta \in \mathscr{P}} t^{|\theta|} \frac{1}{H_{2\theta}} Z_\theta(\mathbf{x}) Z_\theta(\mathbf{y}) Z_\theta(\mathbf{z}) \right) \Bigg|_{t=1}.$$

Outline

Why do Schur functions and zonal polynomials appear?

This depends on your perspective.

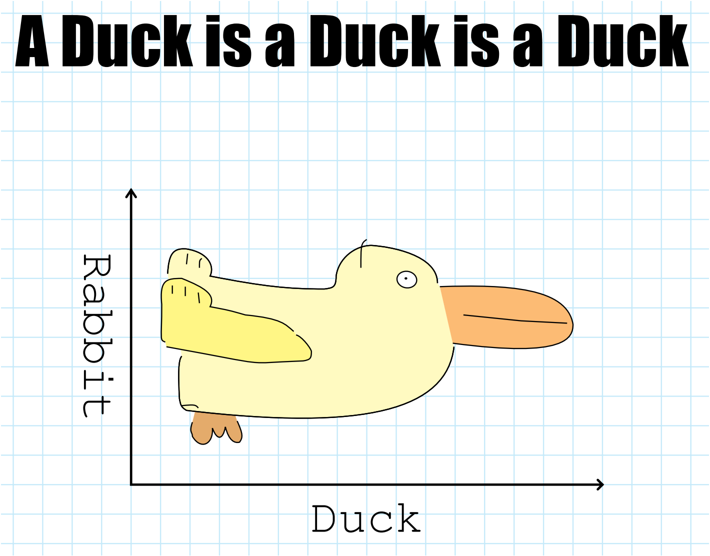
Continuous

Edge labelled maps can be encoded as permutations.
Schur functions come from characters of the symmetric group.

Discrete

Face-Painted maps can be counted by matrix integrals.
Schur functions are characters of Lie groups.

Two Perspectives at Once



An Integration Formula for the GOE

We can also build the generating series from matrix moments.

$$H_{\mathcal{A}}\Big(p(\mathbf{x}), p(\mathbf{y}), \mathbf{z}|_{z_i = z\,\delta_{i,2}}; 0\Big)$$
$$= 2t\frac{\partial}{\partial t}\ln\int \mathrm{e}^{\sum_k \frac{1}{2k}p_k(XM)p_k(Y)\sqrt{z}^k t^k}\mathrm{e}^{-\frac{1}{4}\operatorname{tr}(M^2)}\mathrm{d}M\Big|_{t=1}$$
$$= 2t\frac{\partial}{\partial t}\ln\int\sum_{\theta\in\mathcal{P}}\frac{Z_\theta(XM)Z_\theta(Y)}{\langle Z_\theta, Z_\theta\rangle_2} z^{|\theta|/2}t^{|\theta|}\mathrm{e}^{-\frac{1}{4}\operatorname{tr}(M^2)}\mathrm{d}M\Big|_{t=1}$$

Corollary (Goulden and Jackson - 1997)

When A is taken from GOE_n,

$$\mathrm{E}(Z_\theta)(XA) = Z_\theta(X)\,[p_{2^{|\theta|/2}}]Z_\theta$$

Example Using Zonal Polynomials

$$\begin{aligned}
Z_{[1^4]} &= 1p_{[1^4]} - 6p_{[2,1^2]} + 3p_{[2,2]} + 8p_{[3,1]} - 6p_{[4]} \\
Z_{[2,1^2]} &= 1p_{[1^4]} - p_{[2,1^2]} - 2p_{[2,2]} - 2p_{[3,1]} + 4p_{[4]} \\
Z_{[2^2]} &= 1p_{[1^4]} + 2p_{[2,1^2]} + 7p_{[2,2]} - 8p_{[3,1]} - 2p_{[4]} \\
Z_{[3,1]} &= 1p_{[1^4]} + 5p_{[2,1^2]} - 2p_{[2,2]} + 4p_{[3,1]} - 8p_{[4]} \\
Z_{[4]} &= 1p_{[1^4]} + 12p_{[2,1^2]} + 12p_{[2,2]} + 32p_{[3,1]} + 48p_{[4]}
\end{aligned}$$

θ	$[1^4]$	$[2,1^2]$	$[2^2]$	$[3,1]$	$[4]$
$\langle p_\theta, p_\theta \rangle_2$	$4! \cdot 2^4 = 384$	$2! \cdot 2 \cdot 2^3 = 32$	$2! \cdot 2^2 \cdot 2^2 = 32$	$3 \cdot 2^2 = 12$	$4 \cdot 2 = 8$
$\langle Z_\theta, Z_\theta \rangle_2$	2880	720	2880	2016	40320

Example

$$\langle Z_{[4]}, Z_{[4]} \rangle = 1^2 \cdot 384 + 12^2 \cdot 32 + 12^2 \cdot 32 + 32^2 \cdot 12 + 48^2 \cdot 8 = 40320$$

Example Using Zonal Polynomials

$$
\begin{aligned}
Z_{[1^4]} &= 1p_{[1^4]} - 6p_{[2,1^2]} + 3p_{[2,2]} + 8p_{[3,1]} - 6p_{[4]} \\
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Z_{[3,1]} &= 1p_{[1^4]} + 5p_{[2,1^2]} - 2p_{[2,2]} + 4p_{[3,1]} - 8p_{[4]} \\
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$$

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$\langle Z_\theta, Z_\theta \rangle_2$	2880	720	2880	2016	40320

Example

$$p_{[4]} = -6\frac{8}{2880}Z_{[1^4]} + 4\frac{8}{720}Z_{[2,1^2]} - 2\frac{8}{2880}Z_{[2^2]} - 8\frac{8}{2016}Z_{[3,1]} + 48\frac{8}{40320}Z_{[4]}$$

Example Using Zonal Polynomials

$$
\begin{aligned}
Z_{[1^4]} &= 1p_{[1^4]} - 6p_{[2,1^2]} + 3p_{[2,2]} + 8p_{[3,1]} - 6p_{[4]} \\
Z_{[2,1^2]} &= 1p_{[1^4]} - p_{[2,1^2]} - 2p_{[2,2]} - 2p_{[3,1]} + 4p_{[4]} \\
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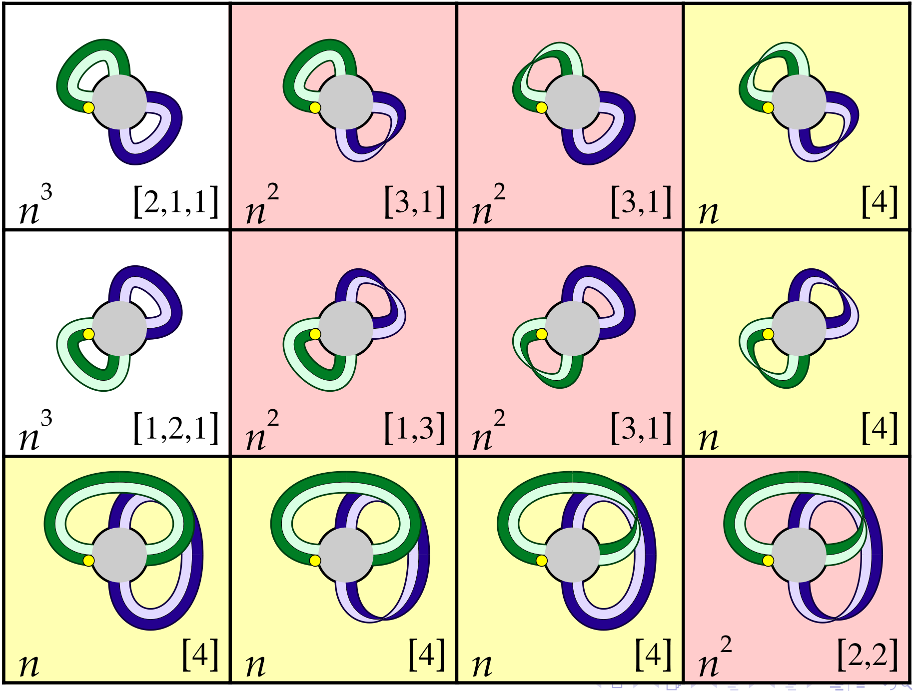
\end{aligned}
$$

Example

$$p_{[4]} = -6\frac{8}{2880}Z_{[1^4]} + 4\frac{8}{720}Z_{[2,1^2]} - 2\frac{8}{2880}Z_{[2^2]} - 8\frac{8}{2016}Z_{[3,1]} + 48\frac{8}{40320}Z_{[4]}$$

$$
\begin{aligned}
\mathrm{E}(p_{[4]}(YA)) = &-6\tfrac{8}{2880}(3)\left(1y_1^4 - 6y_2y_1^2 + 3y_2^2 + 8y_3y_1 - 6y_4\right) + 4\tfrac{8}{720}(-2)\left(1y_1^4 - 1y_2y_1^2 - 2y_2^2 - 2y_3y_1 + 4y_4\right) \\
&- 2\tfrac{8}{2880}(7)\left(1y_1^4 + 2y_2y_1^2 + 7y_2^2 - 8y_3y_1 - 2y_4\right) - 8\tfrac{8}{2016}(-2)\left(1y_1^4 + 5y_2y_1^2 - 2y_2^2 + 4y_3y_1 - 8y_4\right) \\
&+ 48\tfrac{8}{40320}(12)\left(1y_1^4 + 12y_2y_1^2 + 12y_2^2 + 32y_3y_1 + 48y_4\right) \\
= &2y_2y_1^2 + y_2^2 + 4y_3y_1 + 5y_4
\end{aligned}
$$


n^3 [2,1,1]
n^2 [3,1]
n^2 [3,1]
n [4]
n^3 [1,2,1]
n^2 [1,3]
n^2 [3,1]
n [4]
n [4]
n [4]
n [4]
n^2 [2,2]

Outline

A Generalized Hypermap Series

A common generalization involves Jack symmetric functions, Definition.

b-Conjecture (Goulden and Jackson - 1996)

$$H\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}); b\Big)$$

$$:= (1+b)t\frac{\partial}{\partial t}\ln\left(\sum_{\theta\in\mathscr{P}} t^{|\theta|}\frac{J_\theta^{(1+b)}(\mathbf{x})J_\theta^{(1+b)}(\mathbf{y})J_\theta^{(1+b)}(\mathbf{z})}{\langle J_\theta, J_\theta\rangle_{1+b}\,[p_{1^{|\theta|}}]J_\theta}\right)\Bigg|_{t=1}$$

$$= \sum_{n\geq 0}\sum_{\nu,\phi,\epsilon\vdash n} c_{\nu,\phi,\epsilon}(b)p_\nu(\mathbf{x})p_\phi(\mathbf{y})p_\epsilon(\mathbf{z}),$$

enumerates rooted hypermaps with $c_{\nu,\phi,\epsilon}(b) = \sum_{\mathfrak{h}\in\mathcal{H}_{\nu,\phi,\epsilon}} b^{\beta(\mathfrak{h})}$ *for some* β.

In terms of Jack functions, we get:

Corollary (Goulden and Jackson - 1997)

When A *is taken from* GUE_n,

$$\mathrm{E}(J_\theta^{(1)}(XA)) = J_\theta^{(1)}(X)\ [p_{2^{|\theta|/2}}] J_\theta^{(1)}$$

Corollary (Implicit in Jackson - 1995)

When A *is taken from* GOE_n,

$$\mathrm{E}(J_\theta^{(2)}(XA)) = J_\theta^{(2)}(X)\ [p_{2^{|\theta|/2}}] J_\theta^{(2)}$$

$X = \mathrm{diag}(x_1, x_2, \ldots, x_n)$

For general β, integrate over eigenvalues

We can generalize the eigenvalue density.

Definition

For a function $f\colon \mathbb{R}^n \to \mathbb{R}$, define an expectation operator $\langle\cdot\rangle$ by

$$\langle f\rangle_{1+b} := c_{1+b}\int_{\mathbb{R}^n} |V(\boldsymbol{\lambda})|^{\frac{2}{1+b}} f(\boldsymbol{\lambda}) \mathrm{e}^{-\frac{1}{2(1+b)}p_2(\boldsymbol{\lambda})}\,\mathrm{d}\boldsymbol{\lambda},$$

with c_{1+b} chosen such that $\langle 1\rangle_{1+b} = 1$.

Theorem (Okounkov - 1997)

If n is a positive integer, $1+b>0$, and $\theta \vdash 2n$, then

$$\left\langle J_\theta^{(1+b)}(\boldsymbol{\lambda})\right\rangle_{1+b} = J_\theta^{(1+b)}(I_n)[p_{[2^n]}]J_\theta^{(1+b)}.$$

Expectations of power-sums are polynomials

The eigenvalues of A are all real with joint density proportional to

$$\prod_{1\le i<j\le n} |\lambda_i - \lambda_j|^\beta \exp\left(-\frac{\beta}{2}\sum_{i=1}^{n}\frac{\lambda_i^2}{2}\right)$$

Theorem

For every θ, $E(p_\theta(\boldsymbol{\lambda}))_\beta$ is a polynomial in the variables n and $b = \frac{2}{\beta} - 1$.

Example

$$E(p_4(\boldsymbol{\lambda}))_\beta = (1 + b + 3b^2)n + 5bn^2 + 2n^3$$

Expectations of power-sums are polynomials

The eigenvalues of A are all real with joint density proportional to

$$\prod_{1\leq i<j\leq n} |\lambda_i - \lambda_j|^\beta \exp\left(-\frac{\beta}{2}\sum_{i=1}^{n}\frac{\lambda_i^2}{2}\right)$$

Theorem

For every θ, $E(p_\theta(\boldsymbol{\lambda}))_\beta$ is a polynomial in the variables n and $b = \frac{2}{\beta} - 1$.

Example

$$\begin{aligned} E(p_{5,3}(\boldsymbol{\lambda}))_\beta = {} & (75b + 150b^2 + 180b^3 + 105b^4)n \\ & + (60 + 120b + 300b^2 + 240b^3)n^2 \\ & + (180b + 180b^2)n^3 + (45 + 45b)n^4 \end{aligned}$$

Algebraic and Combinatorial Recurrences agree

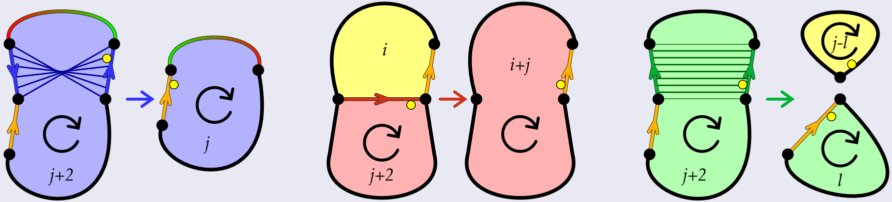
An Algebraic Recurrence

Derivation Example

$$\langle p_{j+2} p_\theta \rangle = b(j+1)\langle p_j p_\theta \rangle + \alpha \sum_{i \in \theta} i m_i(\theta) \langle p_{i+j} p_{\theta \setminus i} \rangle + \sum_{l=0}^{j} \langle p_l p_{j-l} p_\theta \rangle .$$

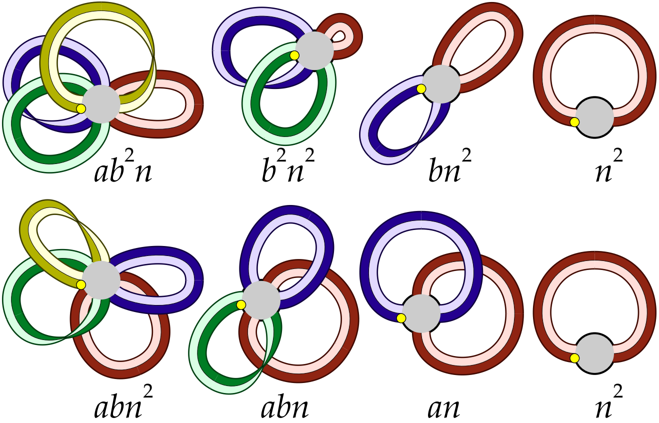
A Combinatorial Recurrence

It corresponds to a combinatorial recurrence for counting polygon glueings.

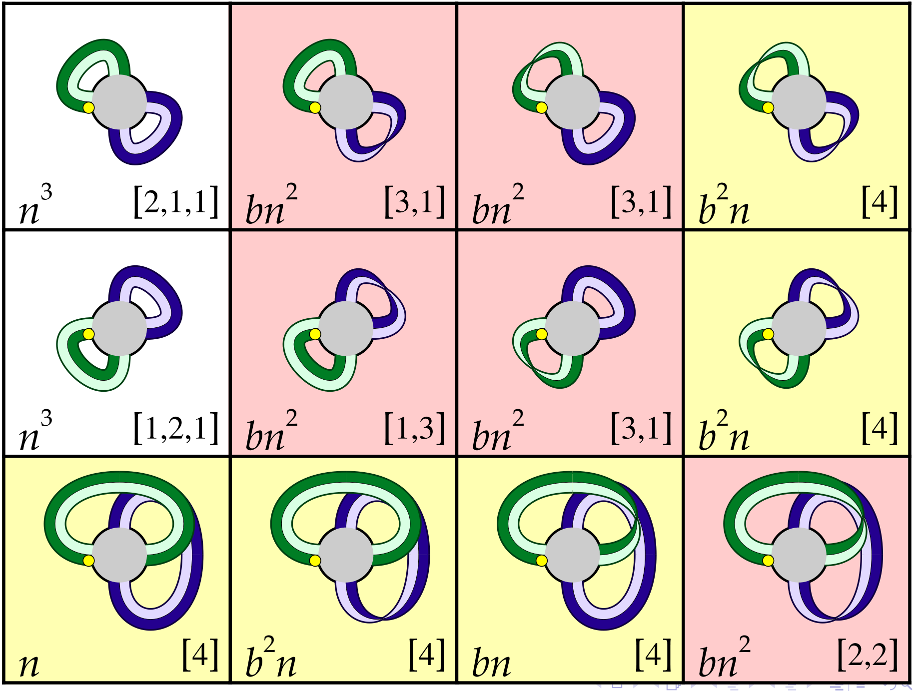


Root Edge Deletion

A rooted map with k edges can be thought of as a sequence of k maps.



Consecutive submaps differ in genus by 0, 1, or 2, and these steps are marked by 1, b, and a to assign a weight to a rooted map.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bn^2 [3,1]
bn^2 [3,1]
b^2n [4]
n^3 [1,2,1]
bn^2 [1,3]
bn^2 [3,1]
b^2n [4]
n [4]
b^2n [4]
bn [4]
bn^2 [2,2]

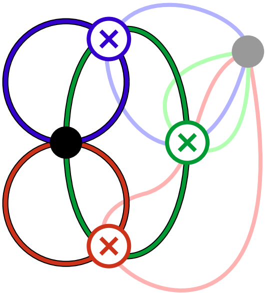
Other Ensembles

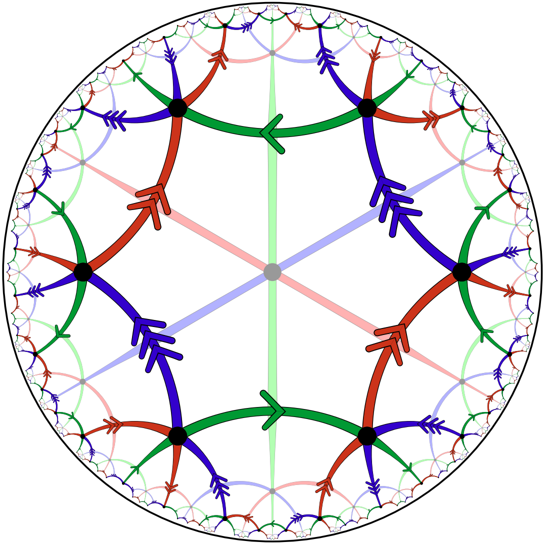
We also get combinatorial interpretations for the moments of:

- The Laguerre Unitary Ensemble (LUE)
- The Laguerre Orthogonal Ensemble (LOE)
- The Complex Ginibre Ensemble.
- Complex Symmetric Matrices.

The End

Thank You



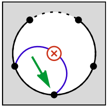


Outline

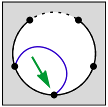
5 Appendix

6 Need Placement

Finding a partial differential equation

Root-edge type	Schematic	Contribution to M
Cross-border		$z\sum_{i\geq 0}(i+1)br_{i+2}\frac{\partial}{\partial r_i}M$
Border		$z\sum_{i\geq 0}\sum_{j=1}^{i+1} r_j y_{i-j+2}\frac{\partial}{\partial r_i}M$
Handle		$z\sum_{i,j\geq 0}(1+b)jr_{i+j+2}\frac{\partial^2}{\partial r_i \partial y_j}M$
Bridge		$z\sum_{i,j\geq 0} r_{i+j+2}\left(\frac{\partial}{\partial r_i}M\right)\left(\frac{\partial}{\partial r_j}M\right)$

Finding a partial differential equation

Root-edge type	Schematic	Contribution to M
Cross-border		$z\sum_{i\geq 0}(i+1)br_{i+2}\frac{\partial}{\partial r_i}M$
Border		$z\sum_{i\geq 0}\sum_{j=1}^{i+1} r_j y_{i-j+2}\frac{\partial}{\partial r_i}M$
Handle		$z\sum_{i,j\geq 0}(1+b)jr_{i+j+2}\frac{\partial^2}{\partial r_i \partial y_j}M$
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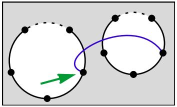
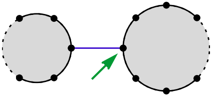
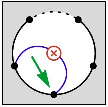
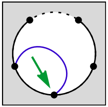
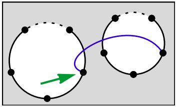
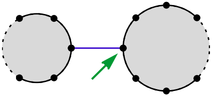
Finding a partial differential equation

Root-edge type	Schematic	Contribution to M
Cross-border		$z\sum_{i\geq 0}(i+1)br_{i+2}\frac{\partial}{\partial r_i}M$
Border		$z\sum_{i\geq 0}\sum_{j=1}^{i+1} r_j y_{i-j+2}\frac{\partial}{\partial r_i}M$
Handle		$z\sum_{i,j\geq 0}(1+b)jr_{i+j+2}\frac{\partial^2}{\partial r_i \partial y_j}M$
Bridge		$z\sum_{i,j\geq 0} r_{i+j+2}\left(\frac{\partial}{\partial r_i}M\right)\left(\frac{\partial}{\partial r_j}M\right)$

Finding a partial differential equation

Root-edge type	Schematic	Contribution to M
Cross-border		$z\sum_{i\geq 0}(i+1)br_{i+2}\frac{\partial}{\partial r_i}M$
Border		$z\sum_{i\geq 0}\sum_{j=1}^{i+1} r_j y_{i-j+2}\frac{\partial}{\partial r_i}M$
Handle		$z\sum_{i,j\geq 0}(1+b)jr_{i+j+2}\frac{\partial^2}{\partial r_i \partial y_j}M$
Bridge		$z\sum_{i,j\geq 0} r_{i+j+2}\left(\frac{\partial}{\partial r_i}M\right)\left(\frac{\partial}{\partial r_j}M\right)$

Finding a partial differential equation

Root-edge type	Schematic	Contribution to M
Cross-border		$z \sum_{i \geq 0} (i+1) b r_{i+2} \frac{\partial}{\partial r_i} M$
Border		$z \sum_{i \geq 0} \sum_{j=1}^{i+1} r_j y_{i-j+2} \frac{\partial}{\partial r_i} M$
Handle		$z \sum_{i,j \geq 0} (1+b) j r_{i+j+2} \frac{\partial^2}{\partial r_i \partial y_j} M$
Bridge		$z \sum_{i,j \geq 0} r_{i+j+2} \left(\frac{\partial}{\partial r_i} M \right) \left(\frac{\partial}{\partial r_j} M \right)$

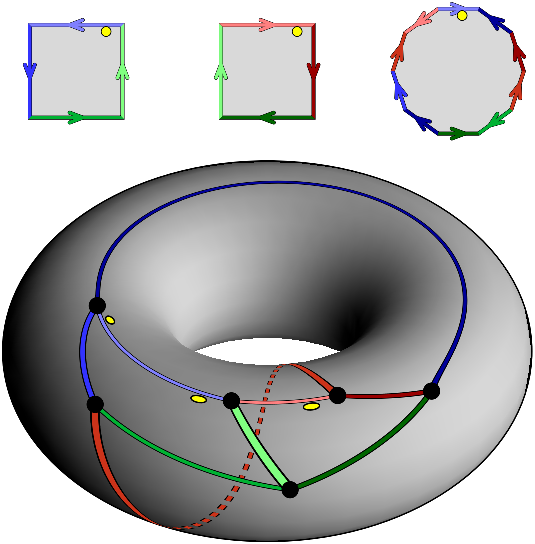
There are $N^{\#f}$ ways to paint an f faced map.

Repaint $N = 1$

Repaint $N = 2$

Repaint $N = 3$

Repaint $N = 4$



Show Neighbourhoods

Show Map

A Recurrence behind the theorem

Set $\Omega := \mathrm{e}^{-\frac{1}{2(1+b)}p_2(\mathbf{x})}|V(\mathbf{x})|^{\frac{2}{1+b}}$, so that $\langle f \rangle = E(f(\mathbf{x})) = c_{b,n}\int_{\mathbb{R}^n} f\Omega\,\mathrm{d}\mathbf{x}$.
Integrate

$$\frac{\partial}{\partial x_1} x_1^{j+1} p_\theta(\mathbf{x})\Omega = \frac{\partial}{\partial x_1} x_1^{j+1} p_\theta(\mathbf{x}) |V(\mathbf{x})|^{\frac{2}{1+b}}\ \mathrm{e}^{-\frac{p_2(\mathbf{x})}{2(1+b)}}$$

$$= (j+1)x_1^j p_\theta(\mathbf{x})\Omega + \sum_{i\in\theta} i m_i(\theta) x_1^{i+j} p_{\theta\setminus i}(\mathbf{x})\Omega + \frac{2}{1+b}\sum_{i=2}^{N} \frac{x_1^{j+1}p_\theta(\mathbf{x})}{x_1 - x_i}\Omega - \frac{1}{1+b} x_1^{j+2} p_\theta(\mathbf{x})\Omega$$

to get

An Algebraic recurrence

Return

$$\langle p_{j+2}p_\theta\rangle = b(j+1)\langle p_j p_\theta\rangle + \alpha \sum_{i\in\theta} i m_i(\theta)\langle p_{i+j} p_{\theta\setminus i}\rangle + \sum_{l=0}^{j}\langle p_l p_{j-l} p_\theta\rangle.$$

Recurrence Example

$$\langle p_{j+2}p_\theta\rangle = b(j+1)\langle p_j p_\theta\rangle + (1+b)\sum_{i\in\theta} i m_i(\theta)\langle p_{i+j}p_{\theta\setminus i}\rangle + \sum_{l=0}^{j}\langle p_l p_{j-l} p_\theta\rangle$$

Example

$$\begin{aligned}
\langle 1\rangle &= 1\\
\langle p_0\rangle &= n\\
\langle p_2\rangle &= b\langle p_0\rangle + \langle p_0p_0\rangle = bn + n^2\\
\langle p_1p_1\rangle &= (1+b)\langle p_0\rangle = (1+b)n\\
\langle p_4\rangle &= 3b\langle p_2\rangle + \langle p_0p_2\rangle + \langle p_1p_1\rangle + \langle p_2p_0\rangle = (1+b+3b^2)n + 5bn^2 + 2n^3\\
\langle p_3p_1\rangle &= 2b\langle p_1p_1\rangle + (1+b)\langle p_2\rangle + \langle p_0p_1p_1\rangle + \langle p_1p_0p_1\rangle = (3b+3b^2)n + (3+3b)n^2\\
\langle p_2p_2\rangle &= b\langle p_0p_2\rangle + 2(1+b)\langle p_2\rangle + \langle p_0p_0p_2\rangle = 2b(1+b)n + (2+2b+b^2)n^2 + 2bn^3 + n^4\\
\langle p_2p_{1,1}\rangle &= b\langle p_0p_{1,1}\rangle + 2(1+b)\langle p_{1,1}\rangle + \langle p_0p_0p_{1,1}\rangle = 2(1+b)^2n + (b+b^2)n^2 + (1+b)n^3\\
\langle p_1p_3\rangle &= 3(1+b)\langle p_2\rangle = (3b+3b^2)n + (3+3b)n^2\\
\langle p_1p_{2,1}\rangle &= 2(1+b)\langle p_{1,1}\rangle + (1+b)\langle p_0p_2\rangle = (2+4b+2b^2)n + (b+b^2)n^2 + (1+b)n^3\\
\langle p_{1,1,1,1}\rangle &= 3(1+b)\langle p_0p_{1,1}\rangle = (1+2b+b^2)n^2
\end{aligned}$$

Return

Jack Symmetric Functions

With respect to the inner product defined by

$$\langle p_\lambda(\mathbf{x}), p_\mu(\mathbf{x}) \rangle_\alpha = \delta_{\lambda,\mu} \frac{|\lambda|!}{|C_\lambda|} \alpha^{\ell(\lambda)},$$

Jack symmetric functions are the unique family satisfying:

(P1) (Orthogonality) If $\lambda \neq \mu$, then $\left\langle J_\lambda^{(\alpha)}, J_\mu^{(\alpha)} \right\rangle_\alpha = 0$.

(P2) (Triangularity) $J_\lambda^{(\alpha)} = \sum_{\mu \preccurlyeq \lambda} v_{\lambda\mu}(\alpha) m_\mu$, where $v_{\lambda\mu}(\alpha)$ is a rational function in α, and '$\preccurlyeq$' denotes the natural order on partitions.

(P3) (Normalization) If $|\lambda| = n$, then $v_{\lambda,[1^n]}(\alpha) = n!$.

Return

Jack Symmetric Functions

Jack symmetric functions, are a one-parameter family, denoted by $\{J_\theta^{(\alpha)}\}_\theta$, that generalizes both Schur functions and zonal polynomials.

Proposition (Stanley - 1989)

Jack symmetric functions are related to Schur functions and zonal polynomials by:

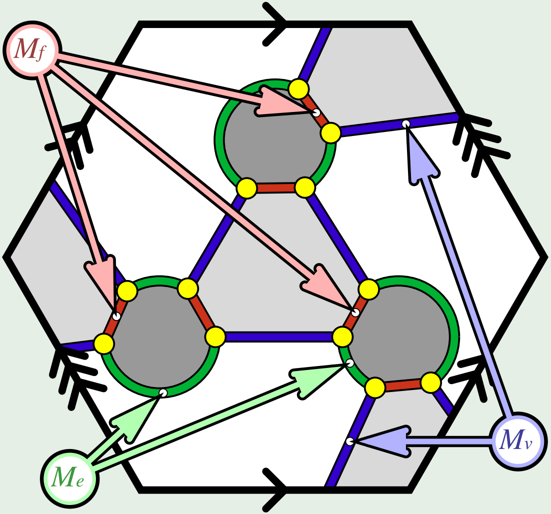
$$J_\lambda^{(1)} = H_\lambda s_\lambda, \qquad \left\langle J_\lambda^{(1)}, J_\lambda^{(1)} \right\rangle_1 = H_\lambda^2,$$

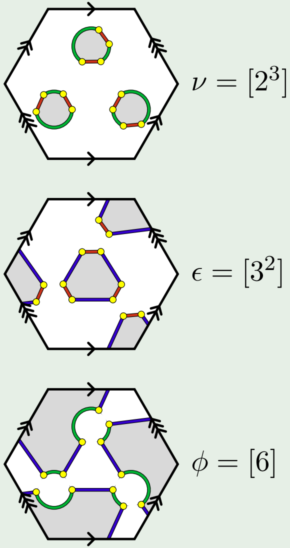
$$J_\lambda^{(2)} = Z_\lambda, \qquad \textit{and} \qquad \left\langle J_\lambda^{(2)}, J_\lambda^{(2)} \right\rangle_2 = H_{2\lambda},$$

where 2λ is the partition obtained from λ by multiplying each part by two.

Return

Example

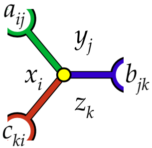




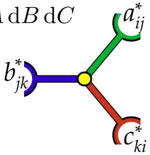
Return

Oriented Derivation (Matrices over $\mathbb{C}$)

With $X = \operatorname{diag}(x_1, x_2, \ldots, x_M)$, $Y = \operatorname{diag}(y_1, y_2, \ldots, y_N)$, $Z = \operatorname{diag}(z_1, z_2, \ldots, z_O)$



$$
\begin{aligned}
&H_{\mathcal{A}}\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}), t^2; 0\Big) \\
&= t\frac{\partial}{\partial t} \ln \int \mathrm{e}^{[\operatorname{tr}(XAYBZC)+\operatorname{tr}(C^*B^*A^*)]t} \mathrm{e}^{-\operatorname{tr}(AA^*+BB^*+CC^*)} \mathrm{d}A\,\mathrm{d}B\,\mathrm{d}C \\
&= t\frac{\partial}{\partial t} \ln \int \sum_{\theta,\phi \in \mathscr{P}} \frac{s_\theta(XAYBZC) s_\phi(C^*B^*A^*)}{([p_{1^{|\theta|}}]s_\theta)^{-1}([p_{1^{|\phi|}}]s_\phi)^{-1}} t^{|\theta|+|\phi|} \mathrm{e}^{-\operatorname{tr}(AA^*+BB^*+CC^*)} \mathrm{d}A\,\mathrm{d}B\,\mathrm{d}C \\
&= t\frac{\partial}{\partial t} \ln \int \sum_{\theta \in \mathscr{P}} \frac{s_\theta(X)s_\theta(Y)s_\theta(Z)s_\theta(ABC)s_\theta(C^*B^*A^*)}{s_\theta(I_M)s_\theta(I_N)s_\theta(I_O)([p_{1^{|\theta|}}]s_\theta)^{-2}} t^{2|\theta|} \mathrm{e}^{-\operatorname{tr}(AA^*+BB^*+CC^*)} \mathrm{d}A\,\mathrm{d}B\,\mathrm{d}C \\
&= t\frac{\partial}{\partial t} \ln \sum_{\theta \in \mathscr{P}} \frac{s_\theta(X)s_\theta(Y)s_\theta(Z)}{[p_{1^{|\theta|}}]s_\theta} t^{2|\theta|}
\end{aligned}
$$

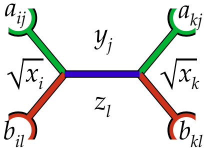


Since $\displaystyle\int_{\mathbb{C}^{M\times N}} s_\theta(XAYA^*)\mathrm{e}^{-\operatorname{tr}(AA^*)}\,\mathrm{d}A = \frac{s_\theta(X)s_\theta(Y)}{[p_{1^{|\theta|}}]s_\theta}$

Return

Non-Oriented Derivation (Matrices over $\mathbb{R}$)

With $\sqrt{X} = \operatorname{diag}(\sqrt{x_1}, \sqrt{x_2}, \dots, \sqrt{x_M})$, $Y = \operatorname{diag}(y_1, y_2, \dots, y_N)$, $Z = \operatorname{diag}(z_1, z_2, \dots, z_O)$



$$
\begin{aligned}
&H_{\mathcal{A}}\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}), t; 1\Big) \\
&\quad = 2t\frac{\partial}{\partial t} \ln \int \mathrm{e}^{\frac{t}{2}\operatorname{tr}(\sqrt{X}AYA^{\mathrm{T}}\sqrt{X}BZB^{\mathrm{T}})}\, \mathrm{e}^{-\frac{1}{2}\operatorname{tr}(AA^{\mathrm{T}}+BB^{\mathrm{T}})} \mathrm{d}A\,\mathrm{d}B \\
&\quad = 2t\frac{\partial}{\partial t} \ln \int \sum_{\theta\in\mathscr{P}} \frac{Z_\theta(\sqrt{X}AYA^{\mathrm{T}}\sqrt{X}BZB^{\mathrm{T}})}{\langle Z_\theta, Z_\theta\rangle_2}\, t^{|\theta|}\, \mathrm{e}^{-\frac{1}{2}\operatorname{tr}(AA^{\mathrm{T}}+BB^{\mathrm{T}})} \mathrm{d}A\,\mathrm{d}B \\
&\quad = 2t\frac{\partial}{\partial t} \ln \int \sum_{\theta\in\mathscr{P}} \frac{Z_\theta(Y)Z_\theta(XBZB^{\mathrm{T}})}{\langle Z_\theta, Z_\theta\rangle_2}\, t^{|\theta|}\, \mathrm{e}^{-\frac{1}{2}\operatorname{tr}(BB^{\mathrm{T}})} \mathrm{d}B \\
&\quad = 2t\frac{\partial}{\partial t} \ln \sum_{\theta\in\mathscr{P}} \frac{Z_\theta(X)Z_\theta(Y)Z_\theta(Z)}{\langle Z_\theta, Z_\theta\rangle_2}\, t^{|\theta|}
\end{aligned}
$$

Since $\displaystyle\int_{\mathbb{R}^{M\times N}} Z_\theta(XAYA^{\mathrm{T}})\mathrm{e}^{-\frac{1}{2}\operatorname{tr}(AA^{\mathrm{T}})}\, \mathrm{d}A = Z_\theta(X)Z_\theta(Y)$

Return

Outline

Polynomiality and half-integers - combinatorial vs bidiagonal vs anti-GUE

How do we deal with additional β

Both our combinatorial and analytic descriptions, when specialized, satisfy the same partial differential equation

An $\mathfrak{S}_3$ Action on Hypermaps

Every permutation of the matchings gives a hypermap.