Exploring Some Non-Constructive Map Bijections

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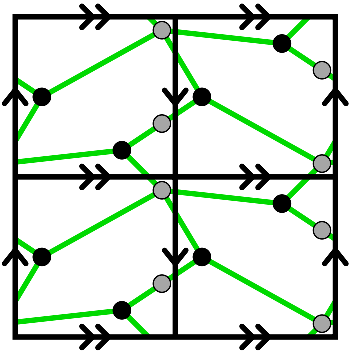


Outline

1. Maps and Hypermaps
   - Maps and their symmetries (Duality and 3 Involutions)
   - Encoding a Map
   - Hypermaps and an $\mathfrak{S}_3$ Symmetry
2. Generating Series
   - Using symmetric Schur functions and zonal polynomials
   - A Jack generalization
3. Partial Solutions and New Mysteries
   - Quantifying non-orientability
   - Root face degree distribution
   - Duality no longer explains the symmetry
   - The Klein Bottle
4. Summary

Outline

Graphs, Surfaces, and Maps

Definition (Surface)

A **surface** is a compact 2-manifold without boundary. (Non-orientable surfaces are permitted.)

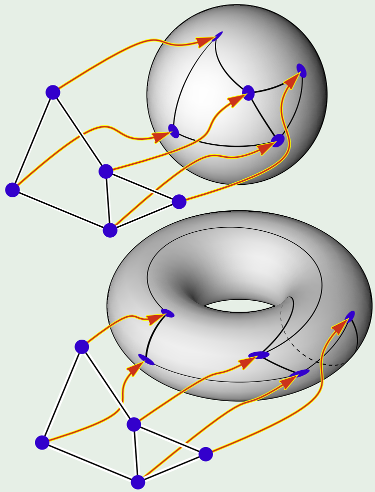
Definition (Graph)

A **graph** is a finite set of *vertices* together with a finite set of *edges*, such that each edge is associated with either one or two vertices. (It may have loops / parallel edges.)

Definition (Map)

A **map** is a 2-cell embedding of a graph in a surface. (It has faces.)

Example



Flags and Rooted Maps

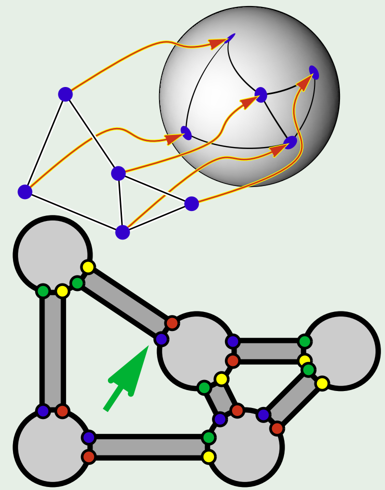
Definition

The neighbourhood of the graph is a **ribbon graph**, and the boundaries of ribbons determine **flags**.

Definition

Automorphisms permute flags, and a **rooted map** is a map together with a distinguished orbit of flags.

Example



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Definition

The neighbourhood of the graph is a **ribbon graph**, and the boundaries of ribbons determine **flags**.

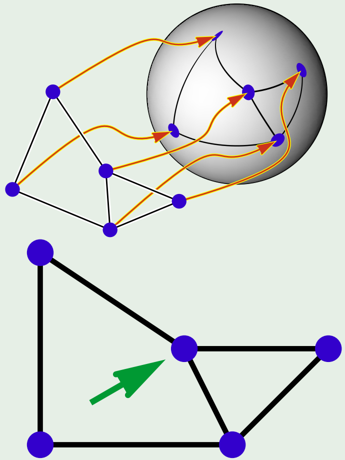
Definition

Automorphisms permute flags, and a **rooted map** is a map together with a distinguished orbit of flags.

Note

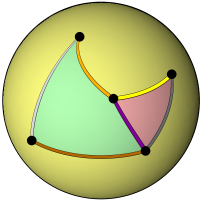
There is a map with no edges. 

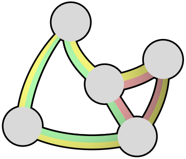
Example

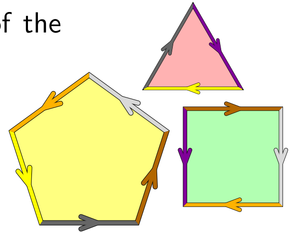


Vertices and Face

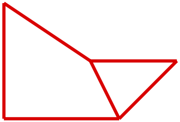
A map can be recovered from a neighbourhood of the graph, or from its faces and surgery instructions.

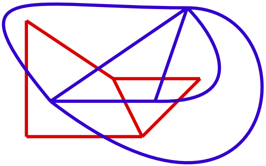


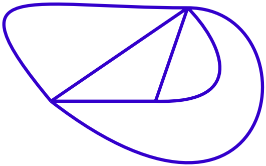




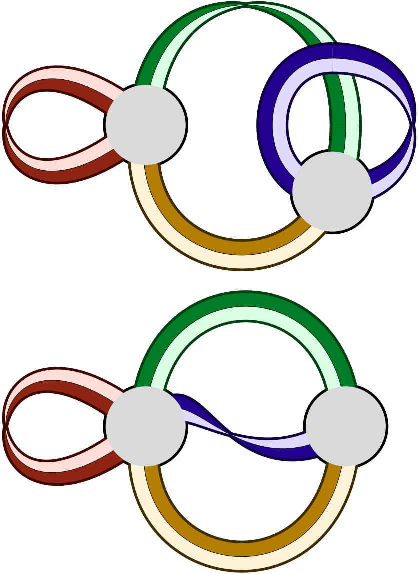
Vertex and face degrees are interchanged by duality.

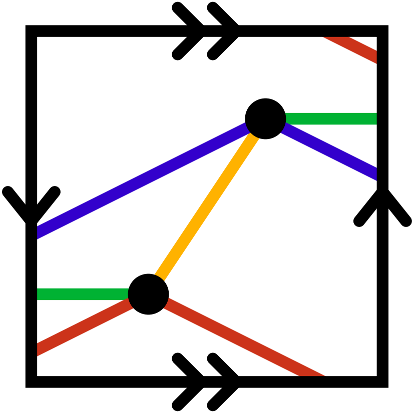






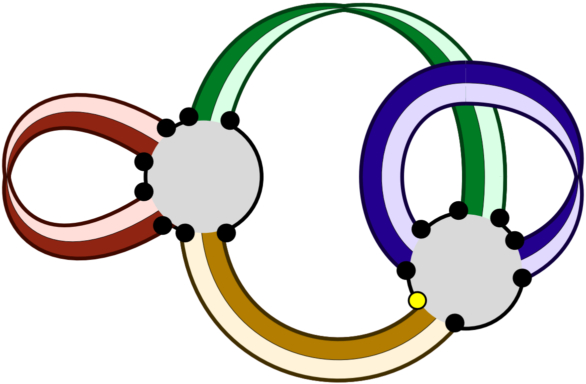
Duality



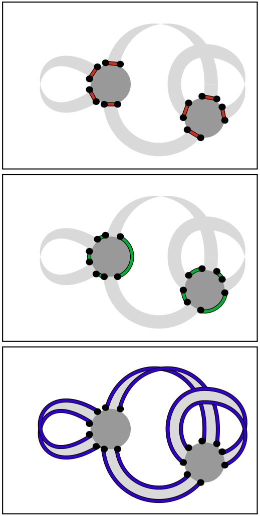


Three Involutions

Three natural involutions reroot a map.

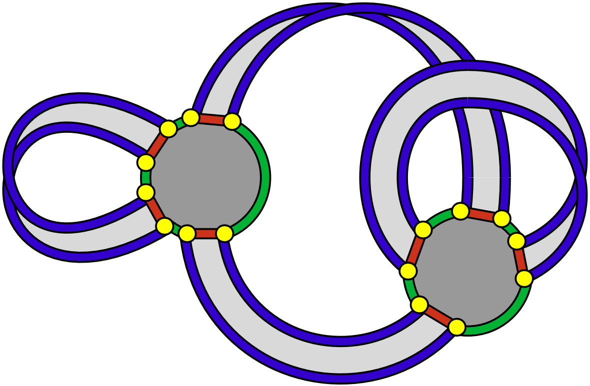


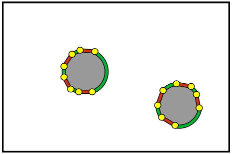
Across Edge | Around Vertex | Along Edge

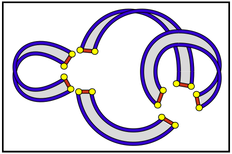


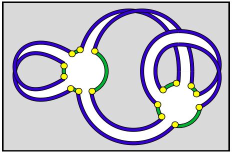
3 Matchings Encode a Map

Each involution gives a perfect matching of flags.









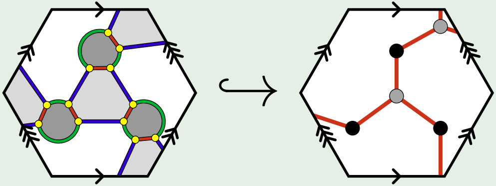
Pairs of matchings recover vertices, edges, and faces.

Hypermaps

Generalizing the combinatorial encoding, an arbitrary triple of perfect matchings determines a **hypermap** when the triple induces a connected graph, with cycles of $M_e \cup M_f$, $M_e \cup M_v$, and $M_v \cup M_f$ determining vertices, hyperfaces, and hyperedges. ▸ Example

Hypermaps both **specialize** and generalize maps.

Example



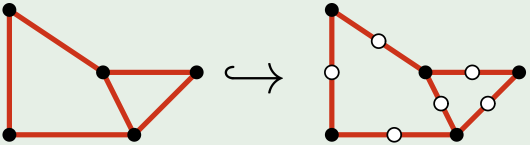
A hypermap can be represented as a bipartite map.

Hypermaps

Generalizing the combinatorial encoding, an arbitrary triple of perfect matchings determines a **hypermap** when the triple induces a connected graph, with cycles of $M_e \cup M_f$, $M_e \cup M_v$, and $M_v \cup M_f$ determining vertices, hyperfaces, and hyperedges. ▸ Example

Hypermaps both specialize and **generalize** maps.

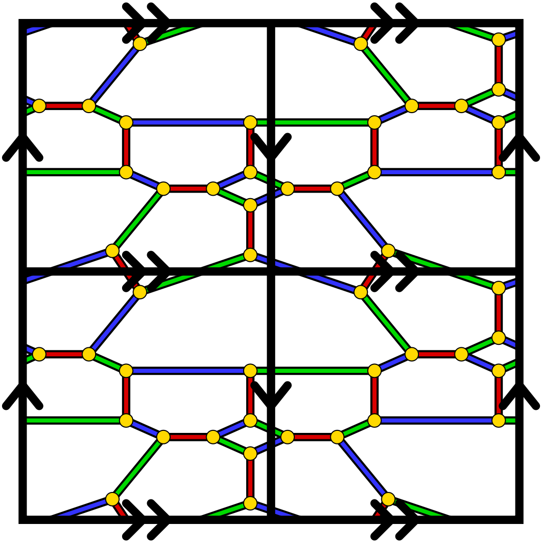
Example

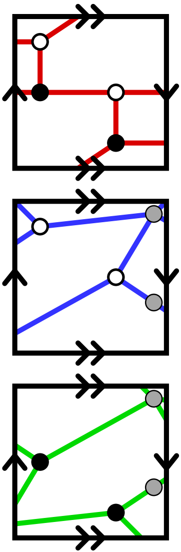


Subdivide edges to get a hypermap from a map.

An $\mathfrak{S}_3$ Action on Hypermaps

Every permutation of the matchings gives a hypermap.





Outline

The Hypermap Series

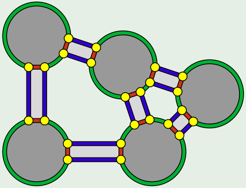
Definition

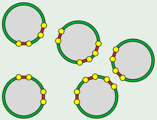
The **hypermap series** for a set $\mathcal{H}$ of hypermaps is the combinatorial sum

$$H(\mathbf{x},\mathbf{y},\mathbf{z}) := \sum_{\mathfrak{h}\in\mathcal{H}} \mathbf{x}^{\nu(\mathfrak{h})}\mathbf{y}^{\phi(\mathfrak{h})}\mathbf{z}^{\epsilon(\mathfrak{h})}$$

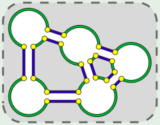
$\nu(\mathfrak{h})$, $\phi(\mathfrak{h})$, and $\epsilon(\mathfrak{h})$ are vertex-, hyperface-, and hyperedge- degrees.

Example

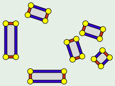




$\nu = [2^3, 3^2]$



$\phi = [3, 4, 5]$



$\epsilon = [2^6]$

contributes $12\,(x_2^3\,x_3^2)\,(y_3\,y_4\,y_5)\,(z_2^6)$.

Explicit Generating Series

Some hypermap series can be computed explicitly.

Theorem (Jackson and Visentin - 1990) ▸ Derivation

When $\mathcal{H}$ is the set of orientable hypermaps,

$$H_{\mathcal{O}}\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}); 0\Big) = t\frac{\partial}{\partial t} \ln\left(\sum_{\theta \in \mathscr{P}} t^{|\theta|} H_\theta s_\theta(\mathbf{x}) s_\theta(\mathbf{y}) s_\theta(\mathbf{z})\right)\Bigg|_{t=1}.$$

Theorem (Goulden and Jackson - 1996) ▸ Derivation

When $\mathcal{H}$ is the set of all hypermaps (orientable and non-orientable),

$$H_{\mathcal{A}}\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}); 1\Big) = 2t\frac{\partial}{\partial t} \ln\left(\sum_{\theta \in \mathscr{P}} t^{|\theta|} \frac{1}{H_{2\theta}} Z_\theta(\mathbf{x}) Z_\theta(\mathbf{y}) Z_\theta(\mathbf{z})\right)\Bigg|_{t=1}.$$

A Generalized Series

A common generalization involves Jack symmetric functions, Definition.

b-Conjecture (Goulden and Jackson - 1996)

$$H\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}); b\Big)$$

$$:= (1+b)t\frac{\partial}{\partial t}\ln\left(\sum_{\theta\in\mathscr{P}} t^{|\theta|}\frac{J_\theta^{(1+b)}(\mathbf{x})J_\theta^{(1+b)}(\mathbf{y})J_\theta^{(1+b)}(\mathbf{z})}{\langle J_\theta, J_\theta\rangle_{1+b}\left[p_{1^{|\theta|}}\right]J_\theta}\right)\Bigg|_{t=1}$$

$$= \sum_{n\geq 0}\sum_{\nu,\phi,\epsilon\vdash n} c_{\nu,\phi,\epsilon}(b)p_\nu(\mathbf{x})p_\phi(\mathbf{y})p_\epsilon(\mathbf{z}),$$

enumerates rooted hypermaps with $c_{\nu,\phi,\epsilon}(b) = \sum_{\mathfrak{h}\in\mathcal{H}_{\nu,\phi,\epsilon}} b^{\beta(\mathfrak{h})}$ *for some* β.

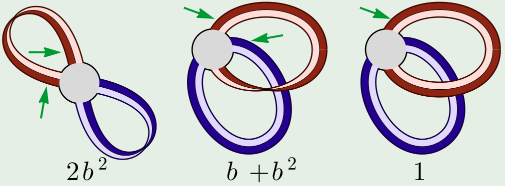
Properties of β

The function $\beta(\mathfrak{h})$ should:

1. be zero for orientable hypermaps,
2. be positive for non-orientable hypermaps,
3. be bounded by cross-cap number,
4. **depend on rooting**,
5. measure departure from orientability.

Example

Rootings of precisely three maps are enumerated by $c_{[4],[4],[2^2]}(b) = 1 + b + 3b^2$.



Properties of β

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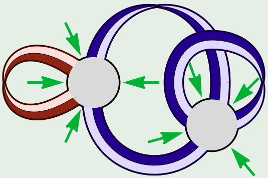
2. be positive for non-orientable hypermaps,
3. be bounded by cross-cap number,
4. depend on rooting,
5. **measure departure from orientability** (probably).

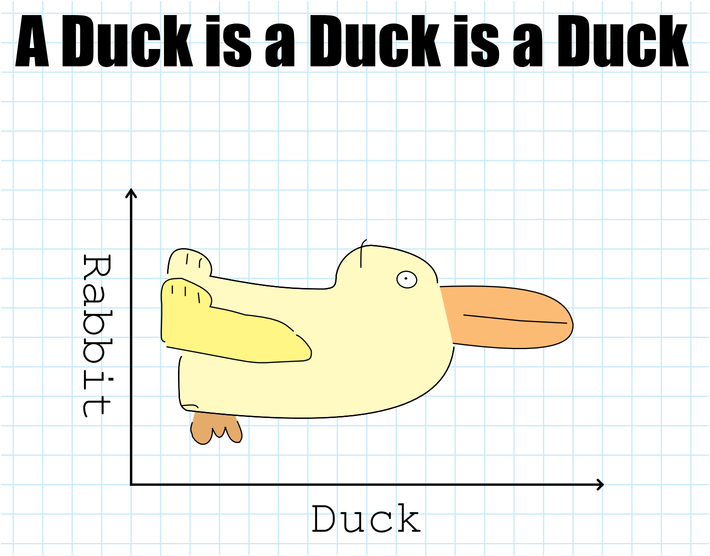
Example

There are precisely eight rooted maps enumerated by $c_{[4,4],[3,5],[2^4]}(b) = 8b^2$.



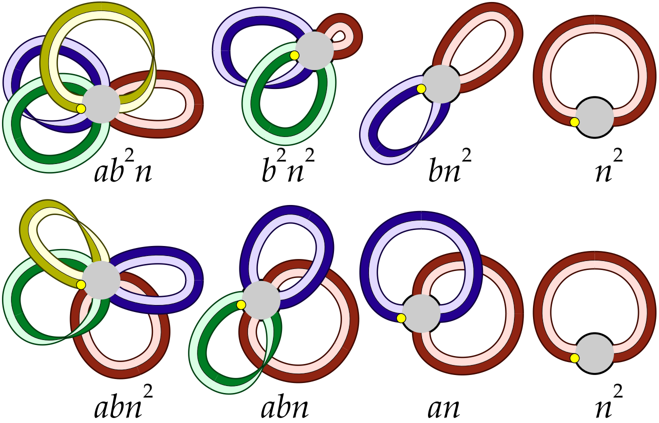
Outline

We can quantify departure from orientability?



Root Edge Deletion

A rooted map with k edges can be thought of as a sequence of k maps.



Consecutive submaps differ in genus by 0, 1, or 2, and these steps are marked by 1, b, and a to assign a weight to a rooted map.

A partial interpretation

$$M = M(x, \mathbf{y}, z, \mathbf{r}; a, b) := \sum_{\mathfrak{m} \in \mathcal{M}} x^{|V(\mathfrak{m})|} \mathbf{y}^{\phi(\mathfrak{m}) \setminus r(\mathfrak{m})} z^{|E(\mathfrak{m})|} r_{\rho(\mathfrak{m})} a^{\tau(\mathfrak{m})} b^{\eta(\mathfrak{m})},$$

Satisfies the PDE ▸ Why?

$$M = r_0 x + b\, z \sum_{i \geq 0} (i+1) r_{i+2} \frac{\partial}{\partial r_i} M + z \sum_{i \geq 0} \sum_{j=1}^{i+1} r_j y_{i-j+2} \frac{\partial}{\partial r_i} M$$
$$+ 2a\, z \sum_{i,j \geq 0} j r_{i+j+2} \frac{\partial^2}{\partial r_i \partial y_j} M + z \sum_{i,j \geq 0} r_{i+j+2} \left(\frac{\partial}{\partial r_i} M \right) \left(\frac{\partial}{\partial r_j} M \right).$$

With $a = \frac{1}{2}(1+b)$ and $x = N$, so does

$$M = (1+b) \sum_{j \geq 0} j r_j \frac{\partial}{\partial y_j} \ln \int_{\mathbb{R}^N} \mathrm{e}^{\sum_{k \geq 1} \frac{p_k(\boldsymbol{\lambda})}{k(1+b)} y_k \sqrt{z^k}} \left| V(\boldsymbol{\lambda}) \right|^{\frac{2}{1+b}} \mathrm{e}^{-\frac{1}{2(1+b)} p_2(\boldsymbol{\lambda})} \, \mathrm{d}\boldsymbol{\lambda}.$$

We can track the degree of the root face

To guess the integral form, we had to replace $2z\dfrac{\partial}{\partial z}$ with $\sum_{j\geq 0} jr_j \dfrac{\partial}{\partial y_j}$.

This means that among all maps with a given set of face degrees, (τ, η) and root-face degree are independently distributed.

For $b = 0$ and $b = 1$ this is a consequence of the re-rooting involutions.

Question

Why does this work for arbitrary b?

An Integral Representation for General b

The integral comes from an evaluation of H, and lets us interpret:

$$d_{k,\phi}(b) = \sum_{\ell(\nu)=k} c_{\nu,\phi,[2^{|\phi|/2}]}(b).$$

Definition

For a function $f\colon \mathbb{R}^n \to \mathbb{R}$, define an expectation operator $\langle \cdot \rangle$ by

$$\langle f \rangle_{1+b} := c_{1+b} \int_{\mathbb{R}^N} |V(\boldsymbol{\lambda})|^{\frac{2}{1+b}} f(\boldsymbol{\lambda}) \mathrm{e}^{-\frac{1}{2(1+b)} p_2(\boldsymbol{\lambda})} \, \mathrm{d}\boldsymbol{\lambda},$$

with c_{1+b} chosen such that $\langle 1 \rangle_{1+b} = 1$.

Theorem (Okounkov - 1997)

If n is a positive integer, $1+b$ is a positive real number, and $\theta \vdash 2n$, then

$$\left\langle J_\theta^{(1+b)}(\boldsymbol{\lambda}) \right\rangle_{1+b} = J_\theta^{(1+b)}(I_N)[p_{[2^n]}] J_\theta^{(1+b)}.$$

Algebraic and Combinatorial Recurrences agree

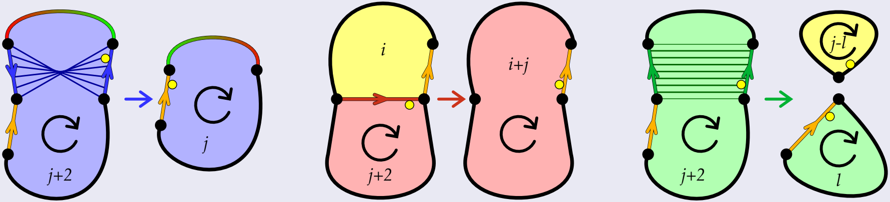
An Algebraic Recurrence

Derivation   Example

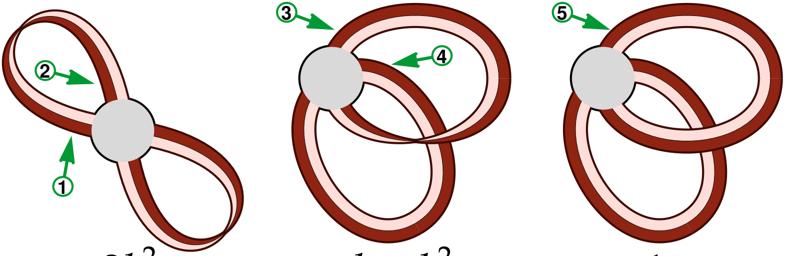
$$\langle p_{j+2} p_\theta \rangle = b(j+1) \langle p_j p_\theta \rangle + \alpha \sum_{i \in \theta} i m_i(\theta) \left\langle p_{i+j} p_{\theta \setminus i} \right\rangle + \sum_{l=0}^{j} \langle p_l p_{j-l} p_\theta \rangle.$$

A Combinatorial Recurrence

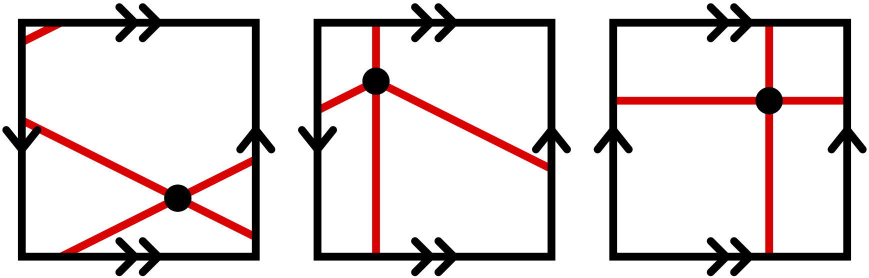
It corresponds to a combinatorial recurrence for counting polygon glueings.



Duality no longer explains the symmetry

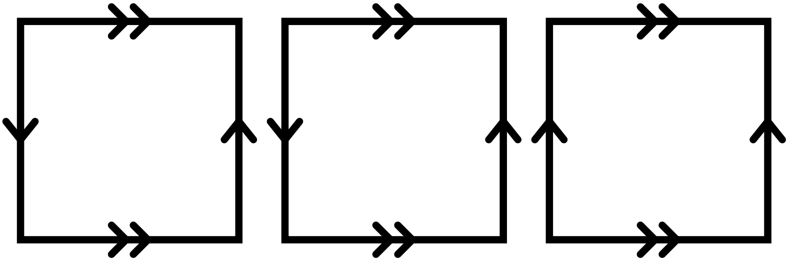


$2b^2$ $\quad b + b^2$ $\quad 1$

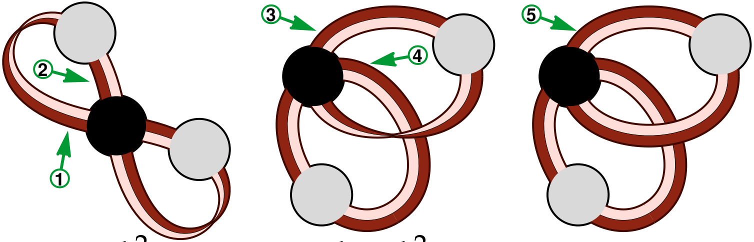


PLAY Start End Double

Triality doesn't Help Either



Reset | Red | Blue | Green | Dual Red | Dual Blue | Dual Green | Zoom



$2b^2$ $\qquad$ $b + b^2$ $\qquad$ 1

The Special Case of the Klein Bottle

If $c_{\nu,\phi,\epsilon}(b)$ enumerates hypermaps on the Klein bottle and torus, then

$$c_{\nu,\phi,\epsilon}(b) = r(1+b) + sb^2$$

This gives an implicit bijection between maps on the torus and a subset of maps on the Klein bottle.

Question

We implicitly have a bijection that preserves number of vertices, and face degrees. Can we preserve vertex degrees as well?

Outline

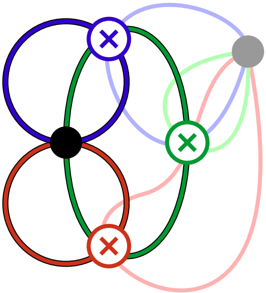
Summary and Points to Ponder

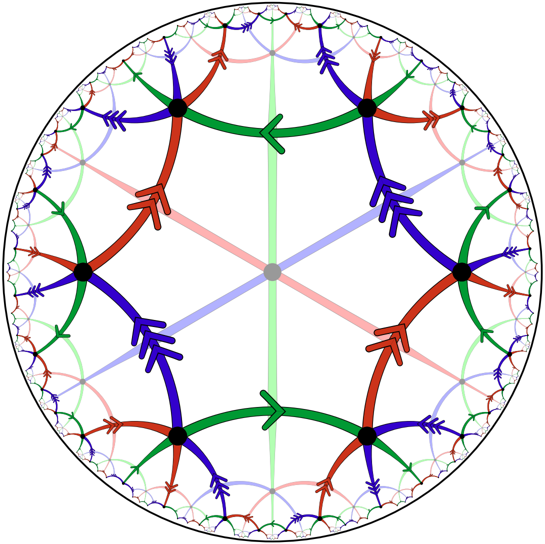
At least for some questions, we can simultaneously enumerate oriented and non-oriented (hyper)maps. This involves refining maps according to a quantification of non-orientability. In the process, we break several symmetries of the original problems, but the solutions still exhibit these symmetries.

- Why is root-face degree independent of non-orientability?
- Is the degree of the root-vertex also independent of non-orientability?
- How can we explain the symmetry between the different variables?
- In particular, is there a natural involution that can replace duality?

The End

Thank You





Jack Symmetric Functions

With respect to the inner product defined by

$$\langle p_\lambda(\mathbf{x}), p_\mu(\mathbf{x})\rangle_\alpha = \delta_{\lambda,\mu}\frac{|\lambda|!}{|C_\lambda|}\alpha^{\ell(\lambda)},$$

Jack symmetric functions are the unique family satisfying:

(P1) (Orthogonality) If $\lambda \neq \mu$, then $\left\langle J_\lambda^{(\alpha)}, J_\mu^{(\alpha)}\right\rangle_\alpha = 0$.

(P2) (Triangularity) $J_\lambda^{(\alpha)} = \sum_{\mu \preccurlyeq \lambda} v_{\lambda\mu}(\alpha) m_\mu$, where $v_{\lambda\mu}(\alpha)$ is a rational function in α, and '$\preccurlyeq$' denotes the natural order on partitions.

(P3) (Normalization) If $|\lambda| = n$, then $v_{\lambda,[1^n]}(\alpha) = n!$.

Return

Jack Symmetric Functions

Jack symmetric functions, are a one-parameter family, denoted by $\{J_\theta^{(\alpha)}\}_\theta$, that generalizes both Schur functions and zonal polynomials.

Proposition (Stanley - 1989)

Jack symmetric functions are related to Schur functions and zonal polynomials by:

$$J_\lambda^{(1)} = H_\lambda s_\lambda, \qquad \left\langle J_\lambda^{(1)}, J_\lambda^{(1)} \right\rangle_1 = H_\lambda^2,$$

and

$$J_\lambda^{(2)} = Z_\lambda, \qquad \left\langle J_\lambda^{(2)}, J_\lambda^{(2)} \right\rangle_2 = H_{2\lambda},$$

where 2λ is the partition obtained from λ by multiplying each part by two.

◂ Return

Jack Polynomials

	$p_{[1^4]}$	$p_{[2,1^2]}$	$p_{[2^2]}$	$p_{[3,1]}$	$p_{[4]}$
$J^{(1+b)}_{[1^4]}$	1	-6	3	8	-6
$J^{(1+b)}_{[2,1^2]}$	1	$b-2$	$-b-1$	$-2b$	$2b+2$
$J^{(1+b)}_{[2^2]}$	1	$2b$	b^2+3b+3	$-4b-4$	$-b^2-b$
$J^{(1+b)}_{[3,1]}$	1	$3b+2$	$-b-1$	$2b^2+2b$	$-2b^2-4b-2$
$J^{(1+b)}_{[4]}$	1	$6b+6$	$3b^2+6b+3$	$8b^2+16b+8$	$6b^3+18b^2+18b+6$

θ	$\langle J_\theta, J_\theta \rangle_{1+b}$
$[1^4]$	$24b^4+240b^3+840b^2+1200b+576$
$[2,1^2]$	$4b^5+40b^4+148b^3+256b^2+208b+64$
$[2^2]$	$8b^6+84b^5+356b^4+780b^3+932b^2+576b+144$
$[3,1]$	$12b^6+100b^5+340b^4+604b^3+592b^2+304b+64$
$[4]$	$144b^7+1272b^6+4752b^5+9744b^4+11856b^3+8568b^2+3408b+576$

Example Using Zonal Polynomials

$$Z_{[1^4]} = 1p_{[1^4]} - 6p_{[2,1^2]} + 3p_{[2,2]} + 8p_{[3,1]} - 6p_{[4]}$$
$$Z_{[2,1^2]} = 1p_{[1^4]} - p_{[2,1^2]} - 2p_{[2,2]} - 2p_{[3,1]} + 4p_{[4]}$$
$$Z_{[2^2]} = 1p_{[1^4]} + 2p_{[2,1^2]} + 7p_{[2,2]} - 8p_{[3,1]} - 2p_{[4]}$$
$$Z_{[3,1]} = 1p_{[1^4]} + 5p_{[2,1^2]} - 2p_{[2,2]} + 4p_{[3,1]} - 8p_{[4]}$$
$$Z_{[4]} = 1p_{[1^4]} + 12p_{[2,1^2]} + 12p_{[2,2]} + 32p_{[3,1]} + 48p_{[4]}$$

θ	$[1^4]$	$[2,1^2]$	$[2^2]$	$[3,1]$	$[4]$
$\langle p_\theta, p_\theta \rangle_2$	$4! \cdot 2^4 = 384$	$2! \cdot 2 \cdot 2^3 = 32$	$2! \cdot 2^2 \cdot 2^2 = 32$	$3 \cdot 2^2 = 12$	$4 \cdot 2 = 8$
$\langle Z_\theta, Z_\theta \rangle_2$	2880	720	2880	2016	40320

Example

$$\left\langle Z_{[4]}, Z_{[4]} \right\rangle = 1^2 \cdot 384 + 12^2 \cdot 32 + 12^2 \cdot 32 + 32^2 \cdot 12 + 48^2 \cdot 8 = 40320$$

Return

Example Using Zonal Polynomials

$$
\begin{aligned}
Z_{[1^4]} &= 1p_{[1^4]} - 6p_{[2,1^2]} + 3p_{[2,2]} + 8p_{[3,1]} - 6p_{[4]} \\
Z_{[2,1^2]} &= 1p_{[1^4]} - p_{[2,1^2]} - 2p_{[2,2]} - 2p_{[3,1]} + 4p_{[4]} \\
Z_{[2^2]} &= 1p_{[1^4]} + 2p_{[2,1^2]} + 7p_{[2,2]} - 8p_{[3,1]} - 2p_{[4]} \\
Z_{[3,1]} &= 1p_{[1^4]} + 5p_{[2,1^2]} - 2p_{[2,2]} + 4p_{[3,1]} - 8p_{[4]} \\
Z_{[4]} &= 1p_{[1^4]} + 12p_{[2,1^2]} + 12p_{[2,2]} + 32p_{[3,1]} + 48p_{[4]}
\end{aligned}
$$

θ	$[1^4]$	$[2,1^2]$	$[2^2]$	$[3,1]$	$[4]$
$\langle p_\theta, p_\theta \rangle_2$	$4! \cdot 2^4 = 384$	$2! \cdot 2 \cdot 2^3 = 32$	$2! \cdot 2^2 \cdot 2^2 = 32$	$3 \cdot 2^2 = 12$	$4 \cdot 2 = 8$
$\langle Z_\theta, Z_\theta \rangle_2$	2880	720	2880	2016	40320

Example

$$p_{[4]} = -6\frac{8}{2880}Z_{[1^4]} + 4\frac{8}{720}Z_{[2,1^2]} - 2\frac{8}{2880}Z_{[2^2]} - 8\frac{8}{2016}Z_{[3,1]} + 48\frac{8}{40320}Z_{[4]}$$

Return

Example Using Zonal Polynomials

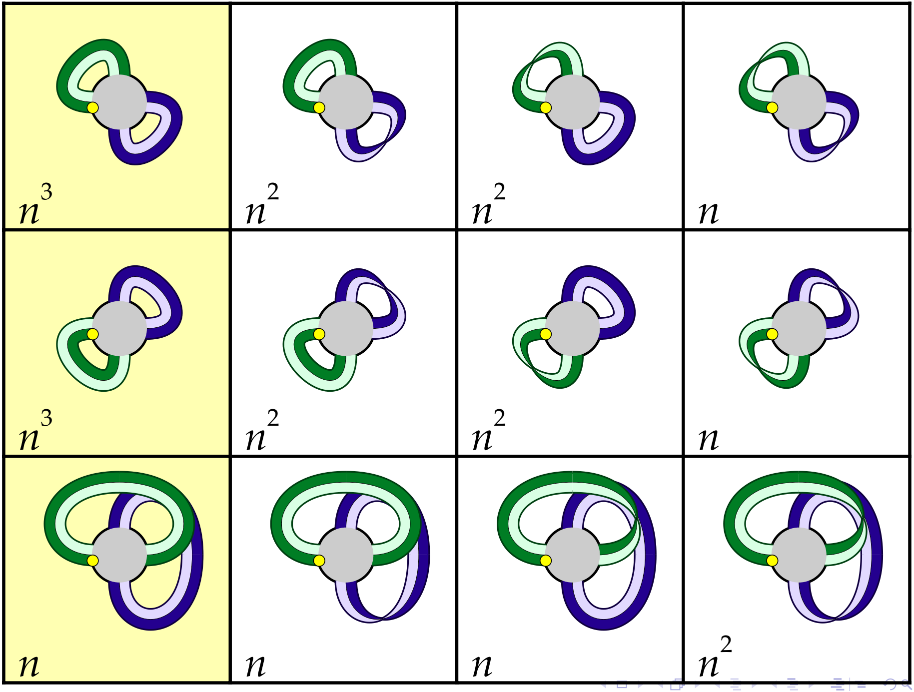
$$
\begin{aligned}
Z_{[1^4]} &= 1p_{[1^4]} - 6p_{[2,1^2]} + 3p_{[2,2]} + 8p_{[3,1]} - 6p_{[4]} \\
Z_{[2,1^2]} &= 1p_{[1^4]} - p_{[2,1^2]} - 2p_{[2,2]} - 2p_{[3,1]} + 4p_{[4]} \\
Z_{[2^2]} &= 1p_{[1^4]} + 2p_{[2,1^2]} + 7p_{[2,2]} - 8p_{[3,1]} - 2p_{[4]} \\
Z_{[3,1]} &= 1p_{[1^4]} + 5p_{[2,1^2]} - 2p_{[2,2]} + 4p_{[3,1]} - 8p_{[4]} \\
Z_{[4]} &= 1p_{[1^4]} + 12p_{[2,1^2]} + 12p_{[2,2]} + 32p_{[3,1]} + 48p_{[4]}
\end{aligned}
$$

Example

$$p_{[4]} = -6\frac{8}{2880}Z_{[1^4]} + 4\frac{8}{720}Z_{[2,1^2]} - 2\frac{8}{2880}Z_{[2^2]} - 8\frac{8}{2016}Z_{[3,1]} + 48\frac{8}{40320}Z_{[4]}$$

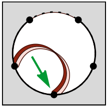
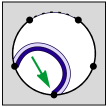
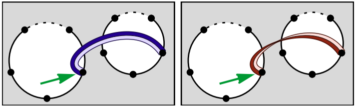
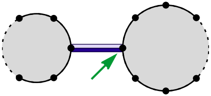
$$
\begin{aligned}
\mathrm{E}(p_{[4]}(A)) = &-6\tfrac{8}{2880}(3)\left(1y_1^4 - 6y_2y_1^2 + 3y_2^2 + 8y_3y_1 - 6y_4\right) + 4\tfrac{8}{720}(-2)\left(1y_1^4 - 1y_2y_1^2 - 2y_2^2 - 2y_3y_1 + 4y_4\right) \\
&- 2\tfrac{8}{2880}(7)\left(1y_1^4 + 2y_2y_1^2 + 7y_2^2 - 8y_3y_1 - 2y_4\right) - 8\tfrac{8}{2016}(-2)\left(1y_1^4 + 5y_2y_1^2 - 2y_2^2 + 4y_3y_1 - 8y_4\right) \\
&+ 48\tfrac{8}{40320}(12)\left(1y_1^4 + 12y_2y_1^2 + 12y_2^2 + 32y_3y_1 + 48y_4\right) \\
= &2y_2y_1^2 + y_2^2 + 4y_3y_1 + 5y_4
\end{aligned}
$$

Return



n^3
n^2
n^2
n
n^3
n^2
n^2
n
n
n
n
n^2

Explaining the partial differential equation

Root-edge type	Schematic	Contribution to M
Cross-border		$z\sum_{i\geq 0}(i+1)br_{i+2}\frac{\partial}{\partial r_i}M$
Border		$z\sum_{i\geq 0}\sum_{j=1}^{i+1} r_j y_{i-j+2}\frac{\partial}{\partial r_i}M$
Handle		$z\sum_{i,j\geq 0}(1+b)jr_{i+j+2}\frac{\partial^2}{\partial r_i \partial y_j}M$
Bridge		$z\sum_{i,j\geq 0} r_{i+j+2}\left(\frac{\partial}{\partial r_i}M\right)\left(\frac{\partial}{\partial r_j}M\right)$

Return

A Recurrence behind the theorem

Set $\Omega := \mathrm{e}^{-\frac{1}{2(1+b)}p_2(\mathbf{x})} |V(\mathbf{x})|^{\frac{2}{1+b}}$, so that $\langle f \rangle = c_{b,n} \int_{\mathbb{R}^n} f \Omega \, \mathrm{d}\mathbf{x}$.

$$\frac{\partial}{\partial x_1} x_1^{j+1} p_\theta(\mathbf{x}) \Omega = \frac{\partial}{\partial x_1} x_1^{j+1} p_\theta(\mathbf{x}) |V(\mathbf{x})|^{\frac{2}{1+b}} \mathrm{e}^{-\frac{p_2(\mathbf{x})}{2(1+b)}}$$

$$= (j+1) x_1^j p_\theta(\mathbf{x}) \Omega + \sum_{i \in \theta} i m_i(\theta) x_1^{i+j} p_{\theta \setminus i}(\mathbf{x}) \Omega + \frac{2}{1+b} \sum_{i=2}^{N} \frac{x_1^{j+1} p_\theta(\mathbf{x})}{x_1 - x_i} \Omega - \frac{1}{1+b} x_1^{j+2} p_\theta(\mathbf{x}) \Omega$$

integrate to get

An Algebraic recurrence

Back Example

$$\langle p_{j+2} p_\theta \rangle = b(j+1) \langle p_j p_\theta \rangle + \alpha \sum_{i \in \theta} i m_i(\theta) \left\langle p_{i+j} p_{\theta \setminus i} \right\rangle + \sum_{l=0}^{j} \langle p_l p_{j-l} p_\theta \rangle.$$

Example of Recurrence

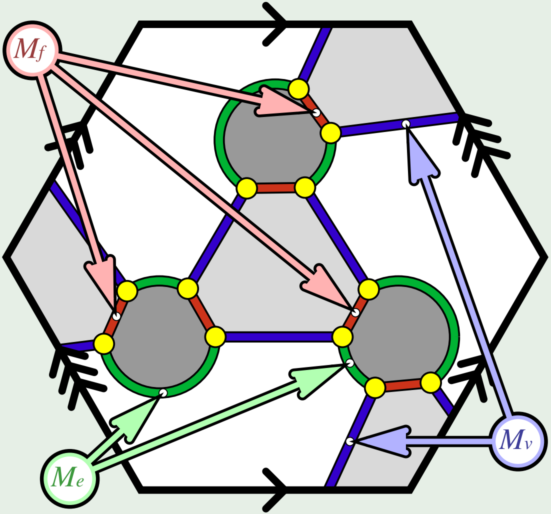
▸ Derivation

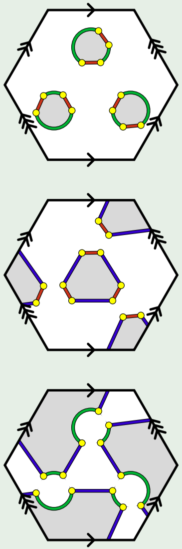
$$\langle p_{j+2}p_\theta\rangle = b(j+1)\langle p_j p_\theta\rangle + (1+b)\sum_{i\in\theta} i m_i(\theta)\left\langle p_{i+j}p_{\theta\setminus i}\right\rangle + \sum_{l=0}^{j}\langle p_l p_{j-l} p_\theta\rangle$$

Example

$$\langle 1\rangle = 1$$
$$\langle p_0\rangle = n$$
$$\langle p_2\rangle = b\langle p_0\rangle + \langle p_0 p_0\rangle = bn + n^2$$
$$\langle p_1 p_1\rangle = (1+b)\langle p_0\rangle = (1+b)n$$
$$\langle p_4\rangle = 3b\langle p_2\rangle + \langle p_0 p_2\rangle + \langle p_1 p_1\rangle + \langle p_2 p_0\rangle = (1+b+3b^2)n + 5bn^2 + 2n^3$$
$$\langle p_3 p_1\rangle = 2b\langle p_1 p_1\rangle + (1+b)\langle p_2\rangle + \langle p_0 p_1 p_1\rangle + \langle p_1 p_0 p_1\rangle = (3b+3b^2)n + (3+3b)n^2$$
$$\langle p_2 p_2\rangle = b\langle p_0 p_2\rangle + 2(1+b)\langle p_2\rangle + \langle p_0 p_0 p_2\rangle = 2b(1+b)n + (2+2b+b^2)n^2 + 2bn^3 + n^4$$
$$\langle p_2 p_{1,1}\rangle = b\langle p_0 p_{1,1}\rangle + 2(1+b)\langle p_{1,1}\rangle + \langle p_0 p_0 p_{1,1}\rangle = 2(1+b)^2 n + (b+b^2)n^2 + (1+b)n^3$$
$$\langle p_1 p_3\rangle = 3(1+b)\langle p_2\rangle = (3b+3b^2)n + (3+3b)n^2$$
$$\langle p_1 p_{2,1}\rangle = 2(1+b)\langle p_{1,1}\rangle + (1+b)\langle p_0 p_2\rangle = (2+4b+2b^2)n + (b+b^2)n^2 + (1+b)n^3$$
$$\langle p_{1,1,1,1}\rangle = 3(1+b)\langle p_0 p_{1,1}\rangle = (1+2b+b^2)n^2$$

Example

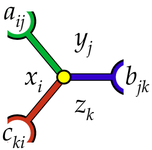




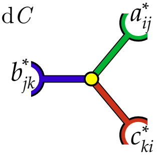
Return

Oriented Derivation

With $X = \operatorname{diag}(x_1, x_2, \dots, x_M)$, $Y = \operatorname{diag}(y_1, y_2, \dots, y_N)$, $Z = \operatorname{diag}(z_1, z_2, \dots, z_O)$



$$
\begin{aligned}
&H_{\mathcal{A}}\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}), t^2; 0\Big) \\
&= t\frac{\partial}{\partial t}\ln \int \mathrm{e}^{\operatorname{tr}(XAYBZC)t - \operatorname{tr}(C^*B^*A^*)t}\mathrm{e}^{-\operatorname{tr}(AA^*+BB^*+CC^*)}\,\mathrm{d}A\,\mathrm{d}B\,\mathrm{d}C \\
&= t\frac{\partial}{\partial t}\ln \int \sum_{\theta,\phi\in\mathscr{P}} \frac{s_\theta(XAYBZC)\overline{s_\phi(ABC)}}{([p_{1|\theta|}]s_\theta)^{-1}([p_{1|\phi|}]s_\phi)^{-1}}\, t^{|\theta|+|\phi|}\mathrm{e}^{-\operatorname{tr}(AA^*+BB^*+CC^*)}\,\mathrm{d}A\,\mathrm{d}B\,\mathrm{d}C \\
&= t\frac{\partial}{\partial t}\ln \int \sum_{\theta\in\mathscr{P}} \frac{s_\theta(\mathbf{x})s_\theta(\mathbf{y})s_\theta(\mathbf{z})s_\theta(ABC)\overline{s_\theta(ABC)}}{s_\theta(I_M)s_\theta(I_N)s_\theta(I_O)([p_{1|\theta|}]s_\theta)^{-2}}\, t^{2|\theta|}\mathrm{e}^{-\operatorname{tr}(AA^*+BB^*+CC^*)}\,\mathrm{d}A\,\mathrm{d}B\,\mathrm{d}C \\
&= t\frac{\partial}{\partial t}\ln \sum_{\theta\in\mathscr{P}} \frac{s_\theta(\mathbf{x})s_\theta(\mathbf{y})s_\theta(\mathbf{z})}{[p_{1|\theta|}]s_\theta}\, t^{2|\theta|}
\end{aligned}
$$

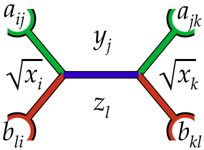


Since $\displaystyle\int_{\mathbb{R}^{M\times N}} s_\theta(XAYA^{\mathrm{T}})\mathrm{e}^{-\operatorname{tr}(AA^{\mathrm{T}})}\,\mathrm{d}A = \frac{s_\theta(X)s_\theta(Y)}{[p_{1|\theta|}]s_\theta}$

Return

Non-Oriented Derivation

With $\sqrt{X} = \operatorname{diag}(\sqrt{x_1}, \sqrt{x_2}, \ldots, \sqrt{x_M})$, $Y = \operatorname{diag}(y_1, y_2, \ldots, y_N)$, $Z = \operatorname{diag}(z_1, z_2, \ldots, z_O)$



$$
\begin{aligned}
&H_{\mathcal{A}}\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}), t; 1\Big) \\
&= 2t\frac{\partial}{\partial t}\ln \int e^{\frac{t}{2}\operatorname{tr}(\sqrt{X}AYA^{\mathrm{T}}\sqrt{X}BZB^{\mathrm{T}})}\, e^{-\frac{1}{2}\operatorname{tr}(AA^{\mathrm{T}}+BB^{\mathrm{T}})} \mathrm{d}A\,\mathrm{d}B \\
&= 2t\frac{\partial}{\partial t}\ln \int \sum_{\theta\in\mathscr{P}} \frac{Z_\theta(\sqrt{X}AYA^{\mathrm{T}}\sqrt{X}BZB^{\mathrm{T}})}{\langle Z_\theta, Z_\theta\rangle_2}\, t^{|\theta|}\, e^{-\frac{1}{2}\operatorname{tr}(AA^{\mathrm{T}}+BB^{\mathrm{T}})} \mathrm{d}A\,\mathrm{d}B \\
&= 2t\frac{\partial}{\partial t}\ln \int \sum_{\theta\in\mathscr{P}} \frac{Z_\theta(XAYA^{\mathrm{T}})Z_\theta(BZB^{\mathrm{T}})}{\langle Z_\theta, Z_\theta\rangle_2\, Z_\theta(I_M)}\, t^{|\theta|}\, e^{-\frac{1}{2}\operatorname{tr}(AA^{\mathrm{T}}+BB^{\mathrm{T}})} \mathrm{d}A\,\mathrm{d}B \\
&= 2t\frac{\partial}{\partial t}\ln \sum_{\theta\in\mathscr{P}} \frac{Z_\theta(\mathbf{x})Z_\theta(\mathbf{y})Z_\theta(\mathbf{z})}{\langle Z_\theta, Z_\theta\rangle_2}\, t^{|\theta|}
\end{aligned}
$$

Since $\displaystyle\int_{\mathbb{R}^{M\times N}} Z_\theta(XAYA^{\mathrm{T}})e^{-\frac{1}{2}\operatorname{tr}(AA^{\mathrm{T}})}\,\mathrm{d}A = Z_\theta(X)Z_\theta(Y)$

Return