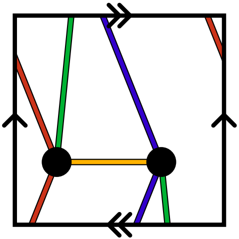
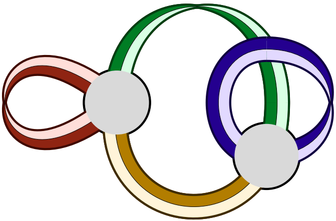
Jack Symmetric Functions and the Non-Orientability of Rooted Maps

Michael La Croix

University of Waterloo

January 4, 2012





Graphs, Surfaces, and Maps

Definition

A **surface** is a compact 2-manifold without boundary.

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A **graph** is a finite set of *vertices* together with a finite set of *edges*, such that each edge is associated with either one or two vertices.

Definition

A **map** is a 2-cell embedding of a graph in a surface.

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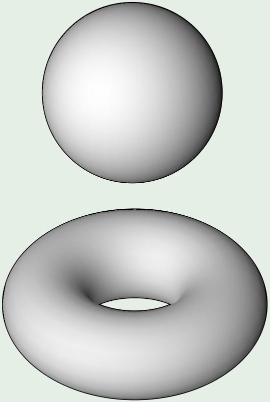

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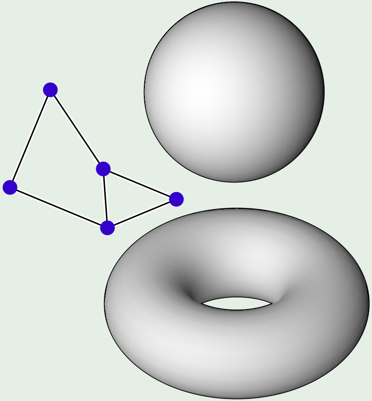

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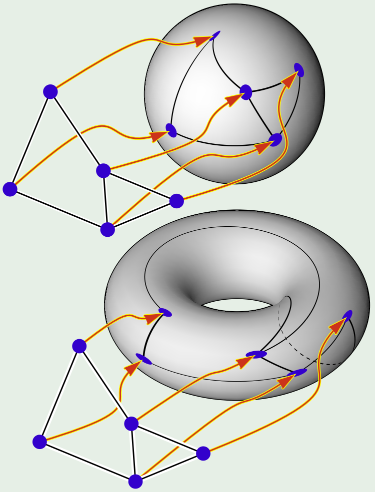

Ribbon Graphs and Flags

Definition

The neighbourhood of the graph determines a **ribbon graph**, and the boundaries of ribbons determine **flags**.

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Automorphisms permute flags, and a **rooted map** is a map together with a distinguished orbit of flags.

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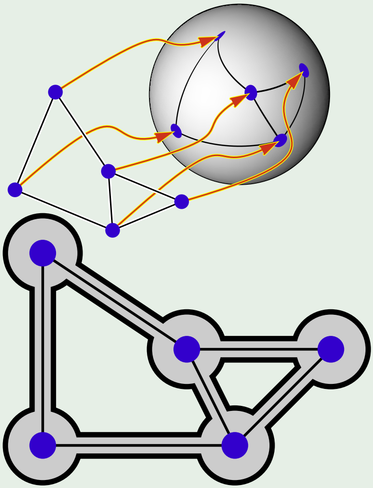

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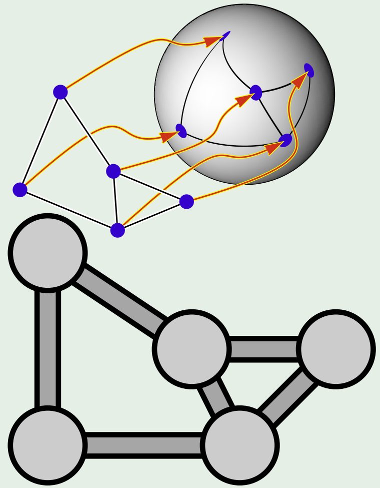

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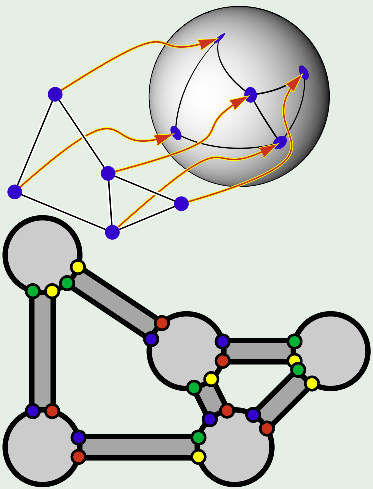

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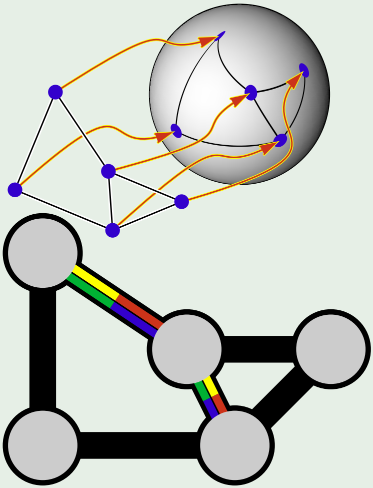

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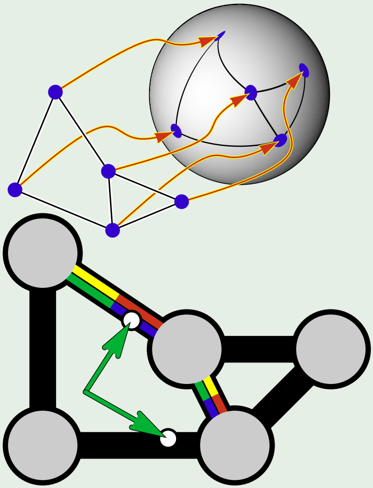

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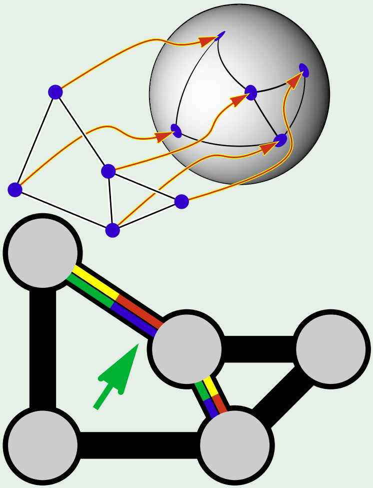

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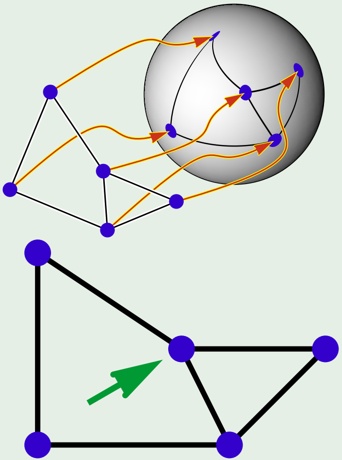
Note

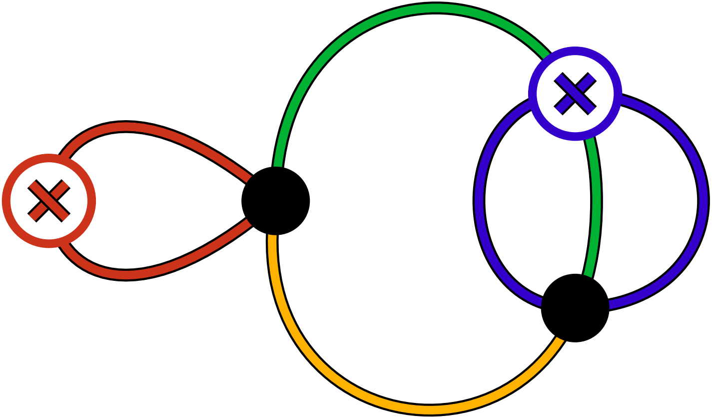
The map with no edges, , has a rooting.

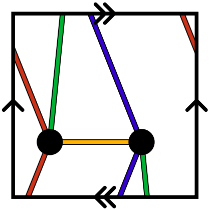
Example



A Combinatorial Encoding of Maps

Equivalence classes can be encoded by perfect matchings of flags.





Start with a ribbon graph.

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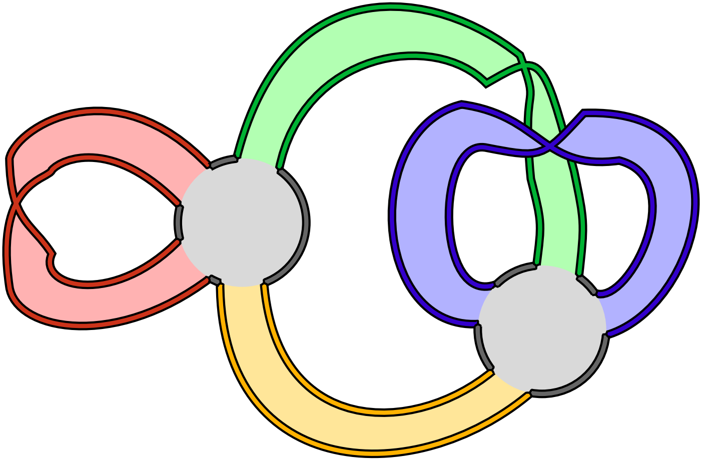

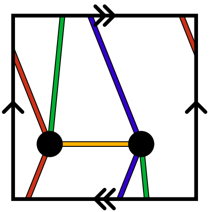

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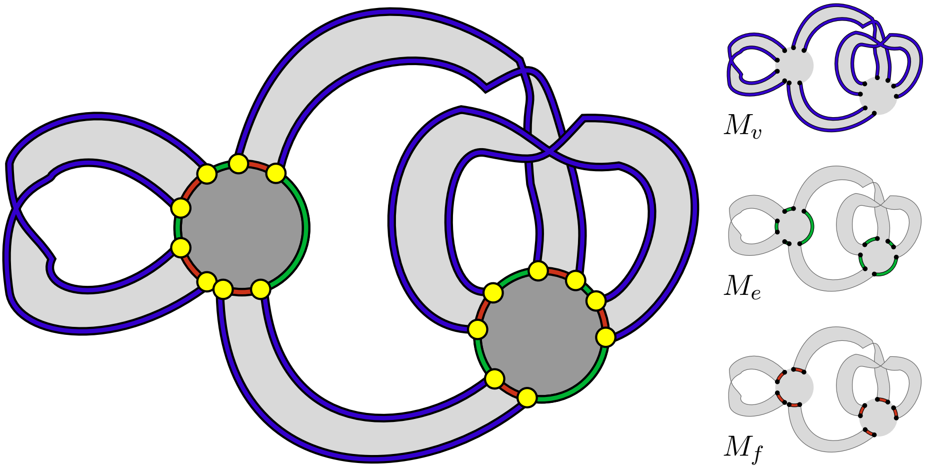

Ribbon boundaries determine 3 perfect matchings of flags.

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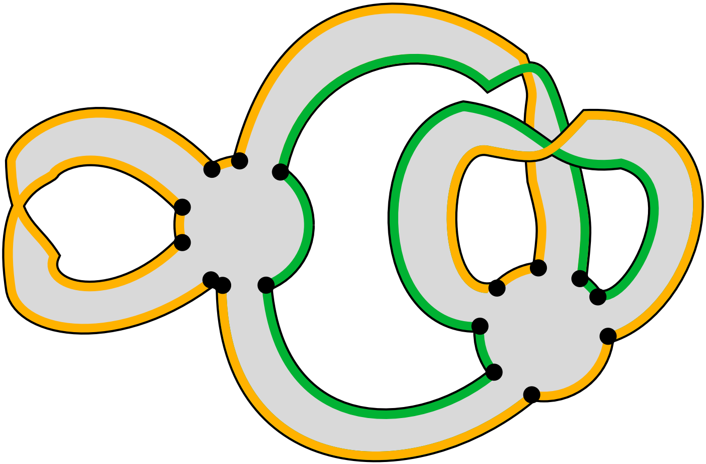
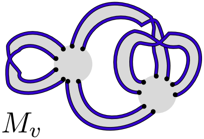
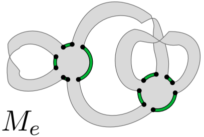




M_v



M_e

Pairs of matchings determine, faces,

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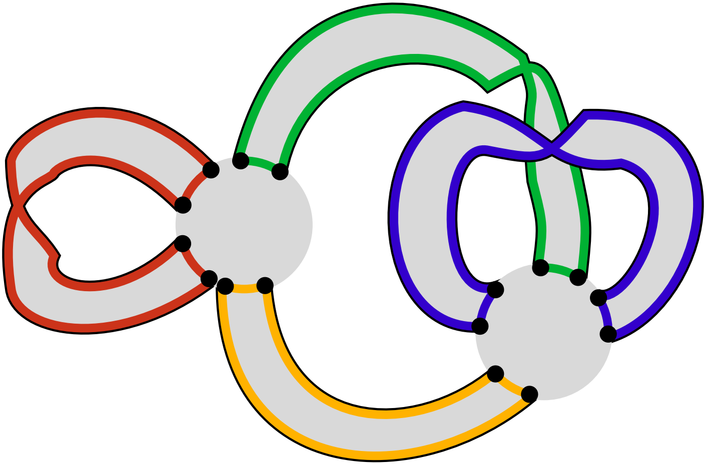

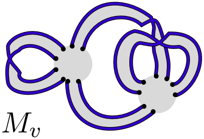

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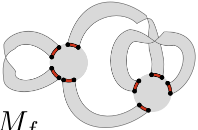

M_f

Pairs of matchings determine, faces, edges,

A Combinatorial Encoding of Maps

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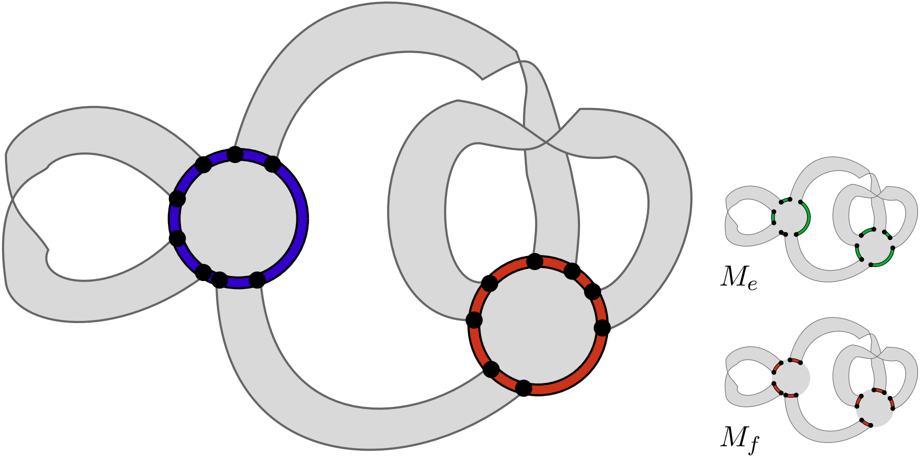

Pairs of matchings determine, faces, edges, and vertices.

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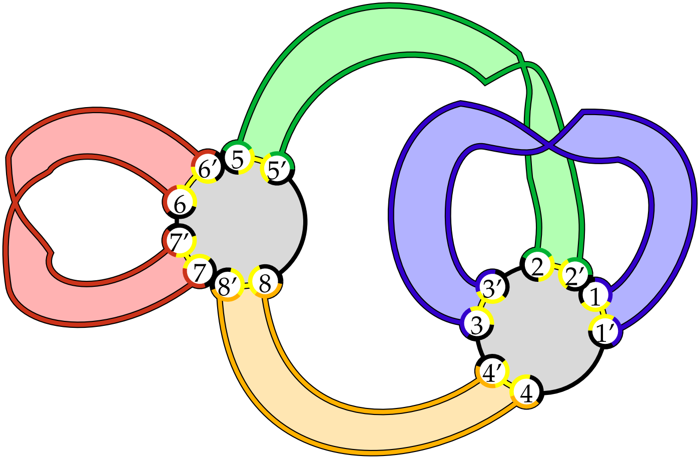
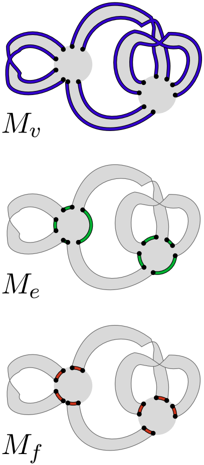




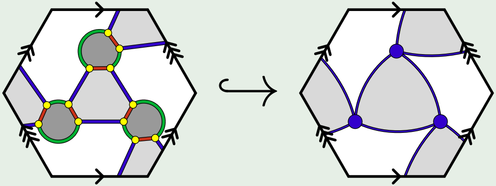
$$M_v = \{\{1,3\},\{1',3'\},\{2,5\},\{2',5'\},\{4,8'\},\{4',8\},\{6,7\},\{6',7'\}\}$$
$$M_e = \{\{1,2'\},\{1',4\},\{2,3'\},\{3,4'\},\{5,6'\},\{5',8\},\{6,7'\},\{7,8'\}\}$$
$$M_f = \{\{1,1'\},\{2,2'\},\{3,3'\},\{4,4'\},\{5,5'\},\{6,6'\},\{7,7'\},\{8,8'\}\}$$

Hypermaps

Generalizing the combinatorial encoding, an arbitrary triple of perfect matchings determines a **hypermap** when the triple induces a connected graph, with cycles of $M_e \cup M_f$, $M_e \cup M_v$, and $M_v \cup M_f$ determining vertices, hyperfaces, and hyperedges. ▸ Example

Hypermaps both **specialize** and generalize maps.

Example



Hypermaps can be represented as face-bipartite maps.

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Hypermaps both specialize and **generalize** maps.

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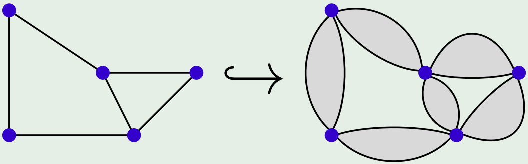

Maps can be represented as hypermaps with $\epsilon = [2^n]$.

The Hypermap Series

Definition

The **hypermap series** for a set $\mathcal{H}$ of hypermaps is the combinatorial sum

$$H(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \sum_{\mathfrak{h} \in \mathcal{H}} \mathbf{x}^{\nu(\mathfrak{h})} \mathbf{y}^{\phi(\mathfrak{h})} \mathbf{z}^{\epsilon(\mathfrak{h})}$$

where $\nu(\mathfrak{h})$, $\phi(\mathfrak{h})$, and $\epsilon(\mathfrak{h})$ are the vertex-, hyperface-, and hyperedge-degree partitions of $\mathfrak{h}$. ▸ Example

Example

Rootings of contribute $12\,(x_2^3\, x_3^2)\,(y_3\, y_4\, y_5)\, z_2^6$ to the sum.

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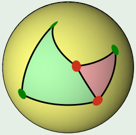
Rootings of  contribute $12\,(x_2^3\, x_3^2)\,(y_3\, y_4\, y_5)\, z_2^6$ to the sum.

Explicit Formulae

The hypermap series can be computed explicitly when $\mathcal{H}$ consists of orientable hypermaps or all hypermaps. ▸ sketch

Theorem (Jackson and Visentin - 1990)

When $\mathcal{H}$ is the set of orientable hypermaps, ▸ encoding details

$$H_{\mathcal{O}}\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}); 0\Big) = t\frac{\partial}{\partial t}\ln\left(\sum_{\theta\in\mathscr{P}} H_\theta s_\theta(\mathbf{x})s_\theta(\mathbf{y})s_\theta(\mathbf{z})\right)\Bigg|_{t=0}.$$

Theorem (Goulden and Jackson - 1996)

When $\mathcal{H}$ is the set of all hypermaps (orientable and non-orientable),

$$H_{\mathcal{A}}\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}); 1\Big) = 2t\frac{\partial}{\partial t}\ln\left(\sum_{\theta\in\mathscr{P}} \frac{1}{H_{2\theta}} Z_\theta(\mathbf{x})Z_\theta(\mathbf{y})Z_\theta(\mathbf{z})\right)\Bigg|_{t=0}.$$

Jack Symmetric Functions

Jack symmetric functions, ▸ Definition, are a one-parameter family, denoted by $\{J_\theta(\alpha)\}_\theta$, that generalizes both Schur functions and zonal polynomials.

Proposition (Stanley - 1989)

Jack symmetric functions are related to Schur functions and zonal polynomials by:

$$J_\lambda(1) = H_\lambda s_\lambda, \qquad \langle J_\lambda, J_\lambda \rangle_1 = H_\lambda^2,$$

and

$$J_\lambda(2) = Z_\lambda, \qquad \langle J_\lambda, J_\lambda \rangle_2 = H_{2\lambda},$$

where 2λ is the partition obtained from λ by multiplying each part by two.

A Generalized Series

b-Conjecture (Goulden and Jackson - 1996)

The generalized series,

$$H\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}); b\Big)$$
$$:= (1+b)t\frac{\partial}{\partial t}\ln\left(\sum_{\theta\in\mathscr{P}}\frac{J_\theta(\mathbf{x};1+b)J_\theta(\mathbf{y};1+b)J_\theta(\mathbf{z};1+b)}{\langle J_\theta, J_\theta\rangle_{1+b}}\right)\Bigg|_{t=0}$$
$$= \sum_{n\geq 0}\sum_{\nu,\phi,\epsilon\vdash n} c_{\nu,\phi,\epsilon}(b)p_\nu(\mathbf{x})p_\phi(\mathbf{y})p_\epsilon(\mathbf{z}),$$

has an combinatorial interpretation involving hypermaps. In particular $c_{\nu,\phi,\epsilon}(b) = \sum_{\mathfrak{h}\in\mathcal{H}_{\nu,\phi,\epsilon}} b^{\beta(\mathfrak{h})}$ *for some invariant* β *of rooted hypermaps.*

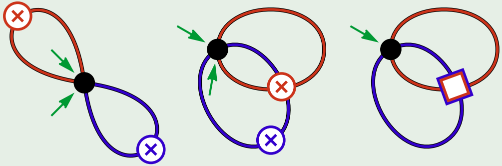
A b-Invariant

The b-Conjecture assumes that $c_{\nu,\phi,\epsilon}(b)$ is a polynomial, and numerical evidence suggests that its degree is the genus of the hypermaps it enumerates. A b-invariant must:

1. be **zero** for orientable hypermaps,
2. be positive for non-orientable hypermaps, and
3. depend on rooting.

Example

Rootings of precisely three maps are enumerated by $c_{[4],[4],[2^2]}(b) = 1 + b + 3b^2$.



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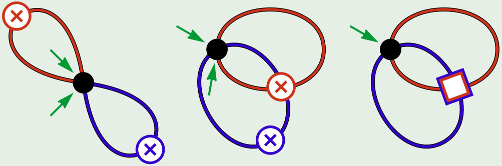

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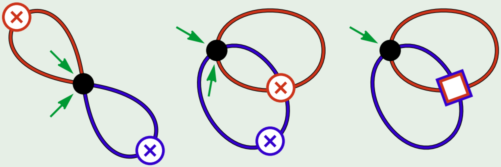

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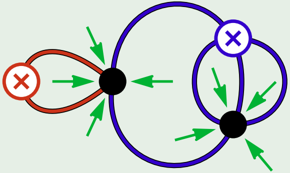

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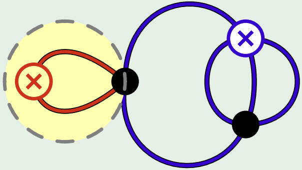
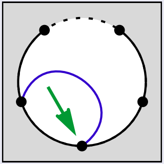
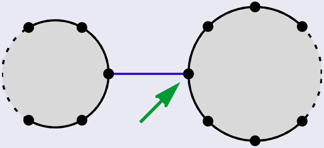


There are four possible types of root edges in a map.

Borders



Bridges



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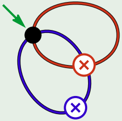
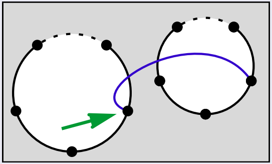
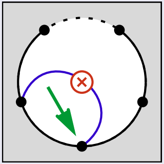


A handle

Handles



Cross-Borders



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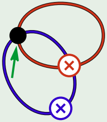
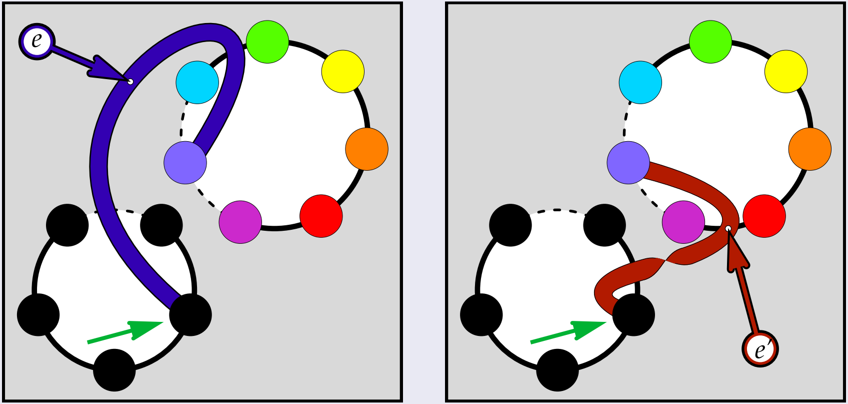


A cross-border

A root-edge classification

Handles occur in pairs



Untwisted

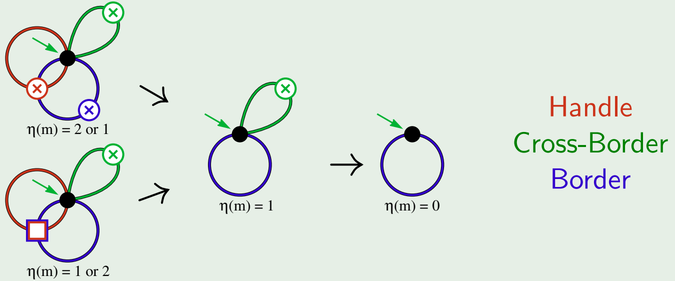
Twisted

A family of invariants

The invariant η

- Iteratively deleting the root edge assigns a type to each edge in a map.
- An invariant, η, is given by
$$\eta(\mathfrak{m}) := (\#\text{ of cross-borders}) + (\#\text{ of twisted handles})\,.$$
- Different handle twisting determines a different invariant.

Example



Main result (marginal b-invariants exist)

Theorem (La Croix)

If ϕ partitions $2n$ and η is a member of the family of invariants then,

$$d_{v,\phi}(b) := \sum_{\ell(\nu)=v} c_{\nu,\phi,[2^n]}(b) = \sum_{\mathfrak{m}\in\mathcal{M}_{v,\phi}} b^{\eta(\mathfrak{m})}.$$

Proof (sketch).

- A ▸ generating series for maps with respect η satisfies a ▸ PDE with a unique solution.
- The corresponding specialization of H has an analytic presentation. ▸ Details
- An algebraic refinement to distinguish between root and non-root faces in the generating series satisfies the same PDE. □

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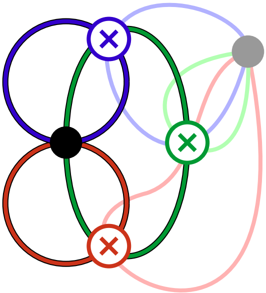
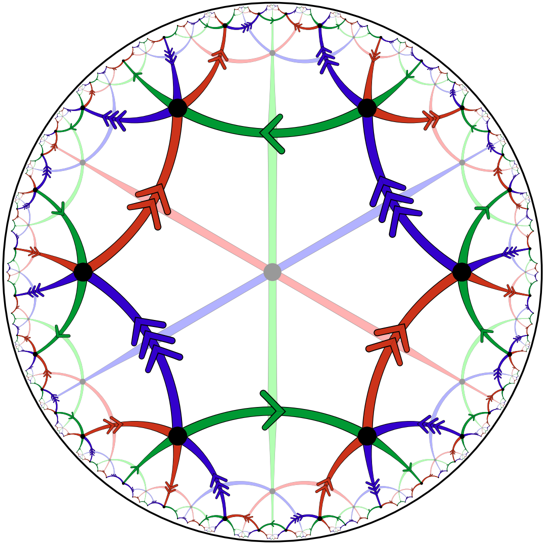
Implications of the proof

- $d_{v,\phi}(b)$ is of the form $\displaystyle\sum_{0\leq i\leq g/2} h_{v,\phi,i} b^{g-2i}(1+b)^i$.
- The degree of $d_{v,\phi}(b)$ is the genus of the maps it enumerates.
- The top coefficient, $h_{v,\phi,0}$, enumerates **unhandled** maps.
- η and root-face degree are independent among maps with given ϕ.

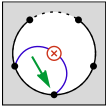
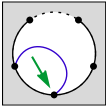
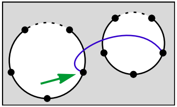
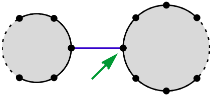
The End

Thank You



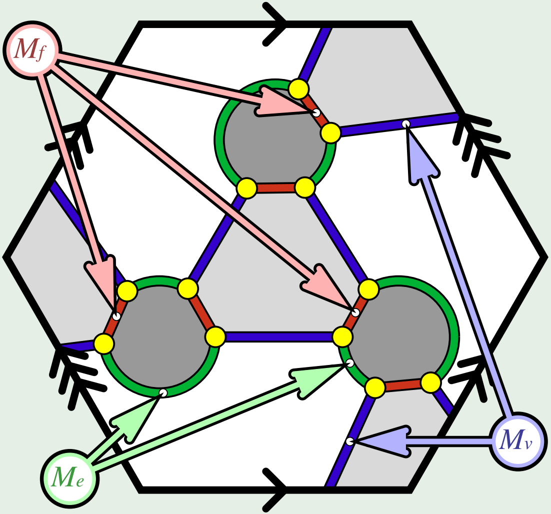


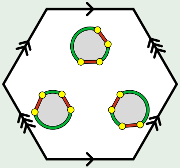
Finding a partial differential equation

Root-edge type	Schematic	Contribution to M
Cross-border		$z\sum_{i\geq 0}(i+1)br_{i+2}\frac{\partial}{\partial r_i}M$
Border		$z\sum_{i\geq 0}\sum_{j=1}^{i+1} r_j y_{i-j+2}\frac{\partial}{\partial r_i}M$
Handle		$z\sum_{i,j\geq 0}(1+b)jr_{i+j+2}\frac{\partial^2}{\partial r_i \partial y_j}M$
Bridge		$z\sum_{i,j\geq 0} r_{i+j+2}\left(\frac{\partial}{\partial r_i}M\right)\left(\frac{\partial}{\partial r_j}M\right)$

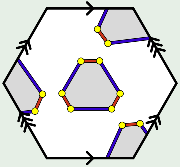
Return

Example

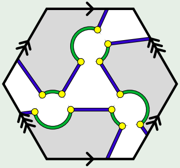




$\nu = [2^3]$



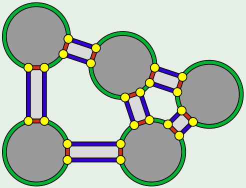
$\epsilon = [3^2]$



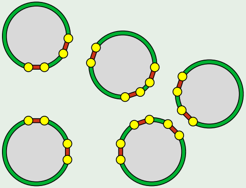
$\phi = [6]$

Return

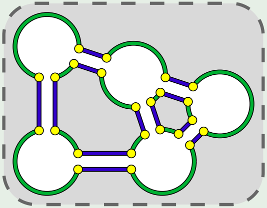
Example



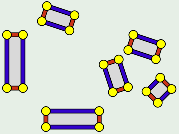
is enumerated by $\left(x_2^3\, x_3^2\right)\left(y_3\, y_4\, y_5\right)\left(z_2^6\right)$.



$\nu = [2^3, 3^2]$



$\phi = [3, 4, 5]$



$\epsilon = [2^6]$

Return

Jack Symmetric Functions

With respect to the inner product defined by

$$\langle p_\lambda(\mathbf{x}), p_\mu(\mathbf{x})\rangle_\alpha = \delta_{\lambda,\mu}\frac{|\lambda|!}{|C_\lambda|}\alpha^{\ell(\lambda)},$$

Jack symmetric functions are the unique family satisfying:

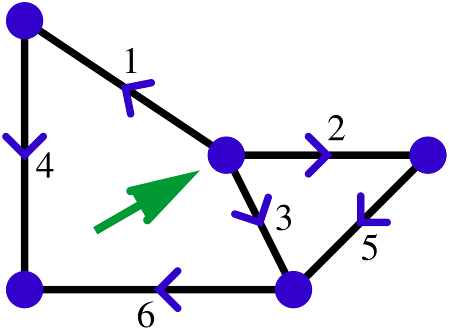
(P1) (Orthogonality) If $\lambda \neq \mu$, then $\langle J_\lambda, J_\mu\rangle_\alpha = 0$.

(P2) (Triangularity) $J_\lambda = \sum_{\mu \preccurlyeq \lambda} v_{\lambda\mu}(\alpha) m_\mu$, where $v_{\lambda\mu}(\alpha)$ is a rational function in α, and '$\preccurlyeq$' denotes the natural order on partitions.

(P3) (Normalization) If $|\lambda| = n$, then $v_{\lambda,[1^n]}(\alpha) = n!$.

Return

1. Orient and label the edges.
2. This induces labels on flags.
3. Clockwise circulations at each vertex determine ν.
4. Face circulations are the cycles of $\epsilon\nu$.



$$\epsilon = (1\ 1')(2\ 2')(3\ 3')(4\ 4')(5\ 5')(6\ 6')$$
$$\nu = (1\ 2\ 3)(1'\ 4)(2'\ 5)(3'\ 5'\ 6)(4'\ 6')$$
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Encoding Orientable Maps

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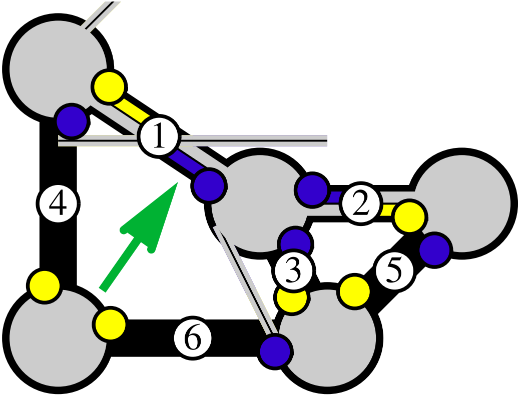

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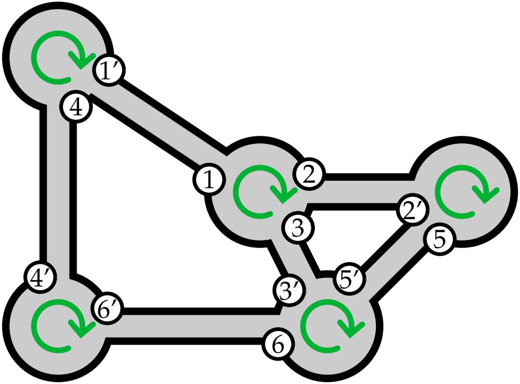

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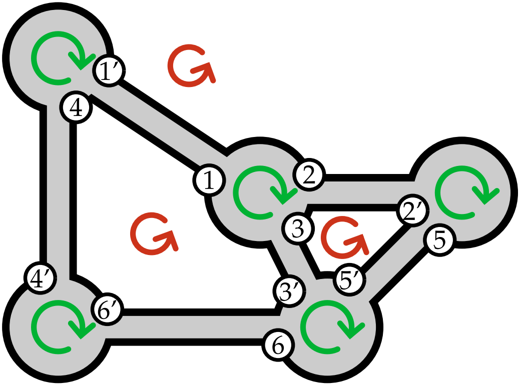

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The Map Series

An enumerative problem associated with maps is to determine the number of rooted maps with specified vertex- and face- degree partitions.

Definition

The **map series** for a set $\mathcal{M}$ of rooted maps is the combinatorial sum

$$M = M(x, \mathbf{y}, z, \mathbf{r}; b) := \sum_{\mathfrak{m}\in\mathcal{M}} x^{|V(\mathfrak{m})|}\mathbf{y}^{\phi(\mathfrak{m})\setminus\rho(\mathfrak{m})}z^{|E(\mathfrak{m})|}r_{\rho(\mathfrak{m})}b^{\eta(\mathfrak{m})},$$

where the sum is taken over all rooted maps, including the map with no edges, $V(\mathfrak{m})$ is the vertex set of $\mathfrak{m}$, $\phi(\mathfrak{m})$ is the face-degree partition of $\mathfrak{m}$, $\rho(\mathfrak{m})$ is the degree of the root face of $\mathfrak{m}$, and $E(\mathfrak{m})$ is the edge set of $\mathfrak{m}$.

Return

How to enumerate maps with symmetric functions

- Instead of counting rooted maps, we can count labelled hypermaps. This adds easily computable multiplicities.
- Labelled counting problems are turned into problems involving counting factorizations.
- These can be answered via character theory.
- Appropriate characters appear as coefficients of symmetric functions.
- The logarithms restrict to connected maps, and the differential operators remove the decoration.

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A Specialization

Definition

For a function $f \colon \mathbb{R}^N \to \mathbb{R}$, define an expectation operator $\langle \cdot \rangle$ by

$$\langle f \rangle_{1+b} := c_{1+b} \int_{\mathbb{R}^N} |V(\boldsymbol{\lambda})|^{\frac{2}{1+b}} f(\boldsymbol{\lambda}) \mathrm{e}^{-\frac{1}{2(1+b)} p_2(\boldsymbol{\lambda})} \, \mathrm{d}\boldsymbol{\lambda},$$

with c_{1+b} chosen such that $\langle 1 \rangle_{1+b} = 1$.

Theorem (Okounkov - 1997)

If N is a positive integer, $1+b$ is a positive real number, and θ is an integer partition of $2n$, then

$$\langle J_\theta(\boldsymbol{\lambda}, 1+b) \rangle_{1+b} = J_\theta(\mathbf{1}_N, 1+b)[p_{[2^n]}] J_\theta,$$

where $\mathbf{1}_N = (1, \ldots, 1, 0, 0, \ldots)$ consists of N leading 1's followed by 0's.

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with c_{1+b} chosen such that $\langle 1 \rangle_{1+b} = 1$.

$$M(N, \mathbf{y}, z, \mathbf{r}; b) = r_0 N + (1+b) \sum_{j \geq 1} r_j \frac{\partial}{\partial y_j} \ln \left\langle \mathrm{e}^{\frac{1}{1+b} \sum_{k \geq 1} \frac{1}{k} y_k p_k(\boldsymbol{\lambda}) \sqrt{z}^k} \right\rangle_{1+b}$$

◂ Return

b is ubiquitous

The many lives of b

	$b=0$		$b=1$
Hypermaps	Orientable	**?**	Locally Orientable
Symmetric Functions	s_θ	$J_\theta(b)$	Z_θ
Matrix Integrals	Hermitian	**?**	Real Symmetric
Moduli Spaces	over $\mathbb{C}$	**?**	over $\mathbb{R}$
Matching Systems	Bipartite	**?**	All