β-Gaussian Ensembles and the Non-orientability of Polygonal Glueings

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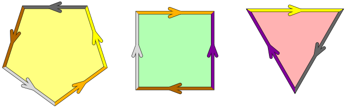
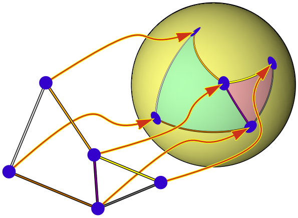




Gaussian Ensembles

For $\beta \in \{1, 2, 4\}$ an element of the β-Gaussian ensemble is constructed as

$$A = G + G^*$$

where G is $n \times n$ with i.i.d. Guassian entries selected from $\{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.

Motivating Question

What is the value of $E(f(A))$, when f is a symmetric function of the eigenvalues of its argument?

Example

$$E(\mathrm{tr}(A^4)) = \begin{cases} 5n + 5n^2 + 2n^3 & \beta = 1 \\ n + 2n^3 & \beta = 2 \\ \frac{5}{4}n - \frac{5}{2}n^2 + \frac{3}{4}n^3 & \beta = 4 \end{cases}$$

General β via Eigenvalue Density

The eigenvalues of A are all real with joint density proportional to

$$\prod_{1\le i<j\le n} |\lambda_i - \lambda_j|^\beta \exp\left(-\frac{\beta}{2}\sum_{i=1}^{n}\frac{\lambda_i^2}{2}\right)$$

Theorem

For every θ, $E(p_\theta(\boldsymbol{\lambda}))_\beta$ is a polynomial in the variables n and $b = \frac{2}{\beta} - 1$.

Example

$$E(p_4(\boldsymbol{\lambda}))_\beta = (1 + b + 3b^2)n + 5bn^2 + 2n^3$$

A Recurrence behind the theorem

Set $\Omega := \mathrm{e}^{-\frac{1}{2(1+b)}p_2(\mathbf{x})}|V(\mathbf{x})|^{\frac{2}{1+b}}$, so that $\langle f \rangle = E(f(\mathbf{x})) = c_{b,n} \int_{\mathbb{R}^n} f\Omega \,\mathrm{d}\mathbf{x}$.
Integrate

$$\frac{\partial}{\partial x_1} x_1^{j+1} p_\theta(\mathbf{x})\Omega = \frac{\partial}{\partial x_1} x_1^{j+1} p_\theta(\mathbf{x}) |V(\mathbf{x})|^{\frac{2}{1+b}} \mathrm{e}^{-\frac{p_2(\mathbf{x})}{2(1+b)}}$$

$$= (j+1)x_1^j p_\theta(\mathbf{x})\Omega + \sum_{i\in\theta} i m_i(\theta) x_1^{i+j} p_{\theta\setminus i}(\mathbf{x})\Omega + \frac{2}{1+b}\sum_{i=2}^{N} \frac{x_1^{j+1} p_\theta(\mathbf{x})}{x_1 - x_i}\Omega - \frac{1}{1+b} x_1^{j+2} p_\theta(\mathbf{x})\Omega$$

to get

An Algebraic recurrence

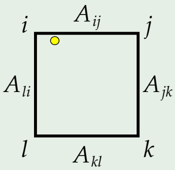
▸ Example

$$\langle p_{j+2} p_\theta \rangle = b(j+1)\langle p_j p_\theta \rangle + \alpha \sum_{i\in\theta} i m_i(\theta) \langle p_{i+j} p_{\theta\setminus i} \rangle + \sum_{l=0}^{j} \langle p_l p_{j-l} p_\theta \rangle .$$

What's being counted?

Example (for $\beta \in \{1, 2, 4\}$)

$$\mathrm{tr}(A^4) = \sum_{i,j,k,l} A_{ij}A_{jk}A_{kl}A_{li}$$



$$\begin{aligned}\mathrm{E}\left(\mathrm{tr}(A^4)\right) &= 1!\binom{n}{1}\mathrm{E}(A_{11}A_{11}A_{11}A_{11}) + 2!\binom{n}{2}\mathrm{E}(4A_{11}A_{11}A_{12}A_{21}) \\ &+ 2!\binom{n}{2}\mathrm{E}(A_{12}A_{21}A_{12}A_{21}) + 3!\binom{n}{3}\mathrm{E}(2A_{12}A_{21}A_{13}A_{31}) \\ &+ 4!\binom{n}{4}\mathrm{E}(A_{12}A_{23}A_{34}A_{41}) + 3!\binom{n}{3}\mathrm{E}(4A_{11}A_{12}A_{23}A_{31}) \\ &+ 2!\binom{n}{2}\mathrm{E}(2A_{11}A_{12}A_{22}A_{21})\end{aligned}$$

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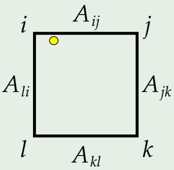

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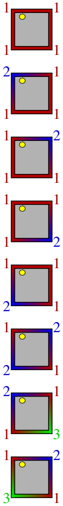


Expectations as Sums

Since the entries of A are independent Guassians,

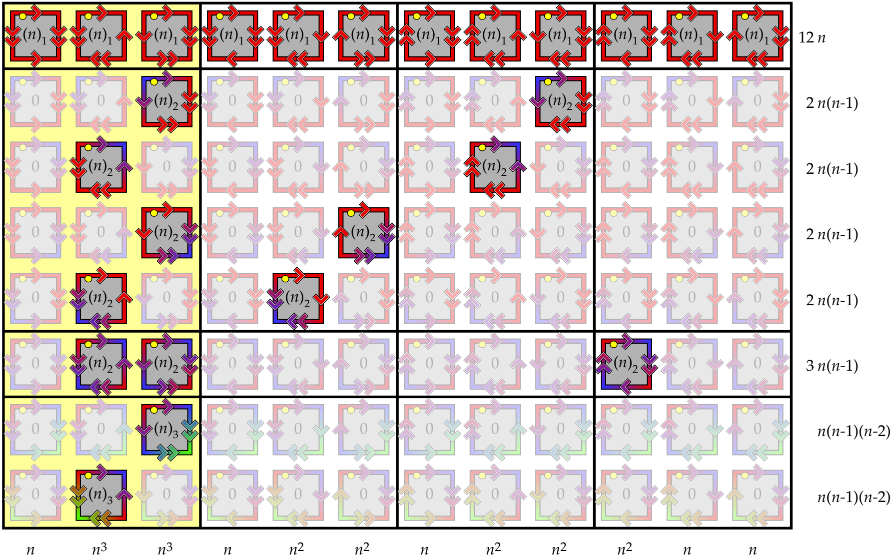
$$E(A_{i_1j_1}A_{i_2j_2}\dots A_{i_kj_k}) = \sum_m \prod_{(u,v)\in m} E(A_uA_v)$$

summed over perfect matchings of the multiset $\{i_1j_1, i_2j_2, \dots, i_kj_k\}$

$$\sum_{p \text{ a painting}} \#\{\text{pairings consistent with } p\}$$

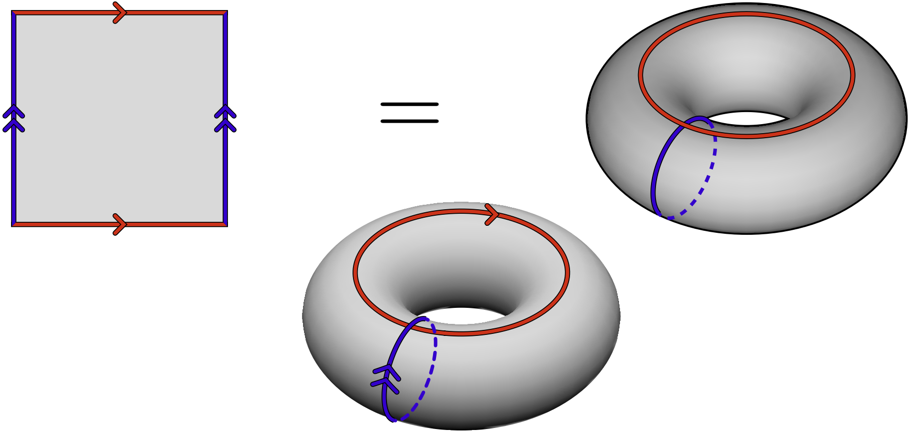
$$= \sum_{m \text{ a matching}} \#\{\text{paintings consistent with } m\}$$

Count the polygon glueings in 2 different ways



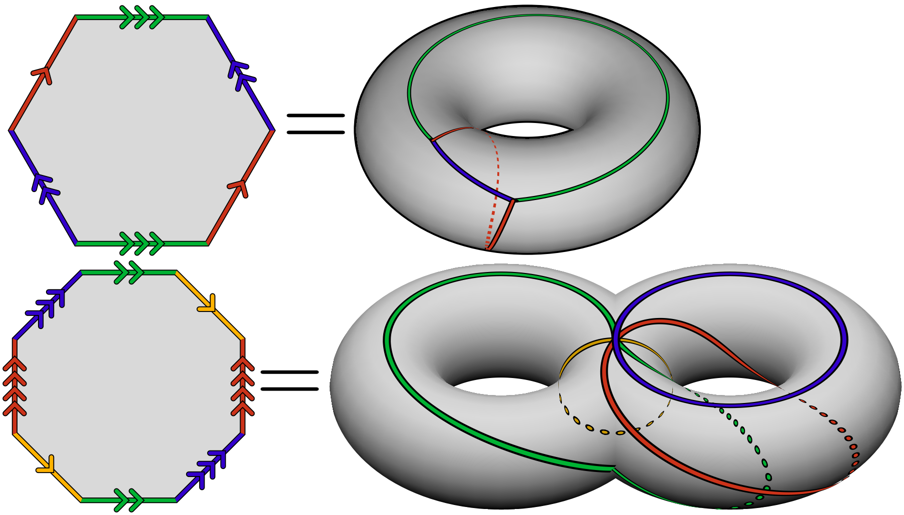
Polygon Glueings = Maps

Identifying the edges of a polygon creates a surface.



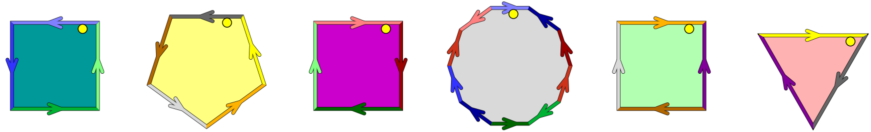
Its boundary is a graph embedded in the surface.

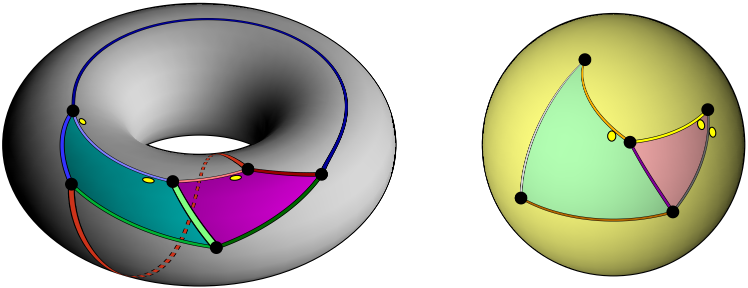
Polygon Glueings = Maps



Polygon Glueings = Maps

Extra polygons give extra faces (and possibles extra components)





Definition

A **surface** is a compact 2-manifold without boundary. (Non-orientable surfaces are permitted.)

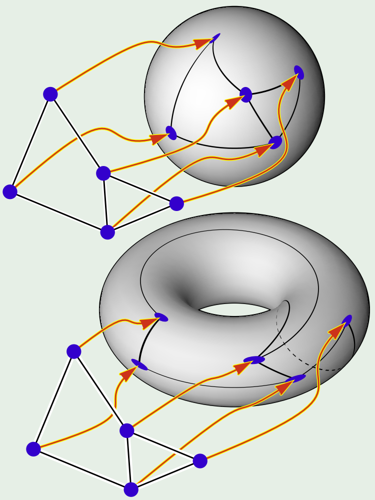
Definition

A **graph** is a finite set of **vertices** together with a finite set of **edges**, such that each edge is associated with either one or two vertices. (It may have loops / parallel edges.)

Definition

A **map** is a 2-cell embedding of a graph in a surface.

Example

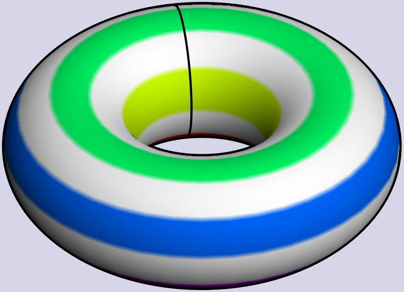


Equivalence of Maps

Two maps are equivalent if the embeddings are homeomorphic.

Homeomorphisms are more complicated than we might think

Dehn Twists



Y-Homeomorphisms

Not Present for Photo

Equivalence of Maps

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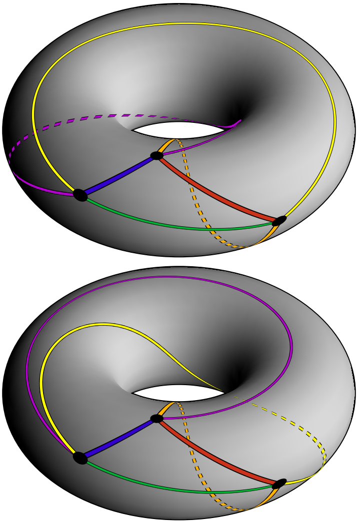
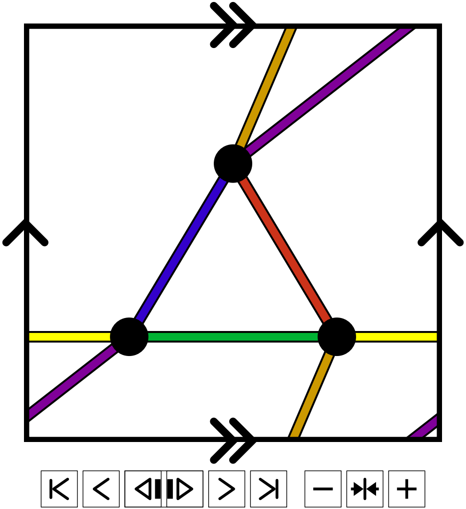




Ribbon Graphs, Flags, and Rooted Maps

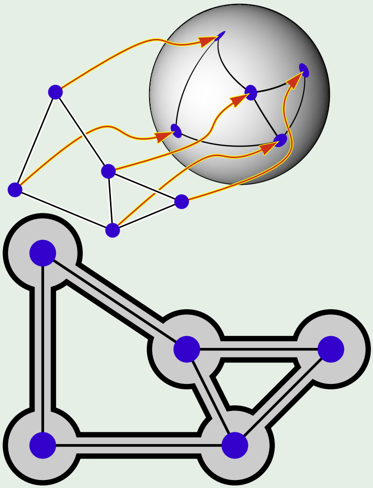
Definition

The neighbourhood of the graph determines a **ribbon graph**, and the boundaries of ribbons determine flags.

Definition

Automorphisms permute flags, and a rooted map is a map together with a distinguished orbit of flags.

Example



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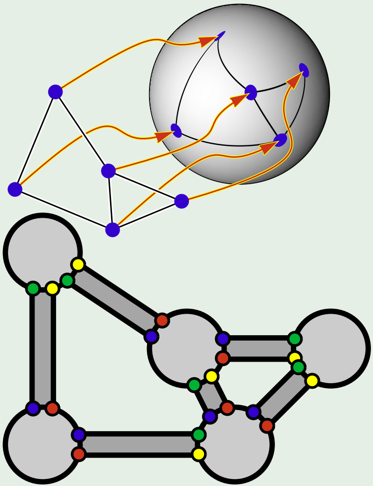

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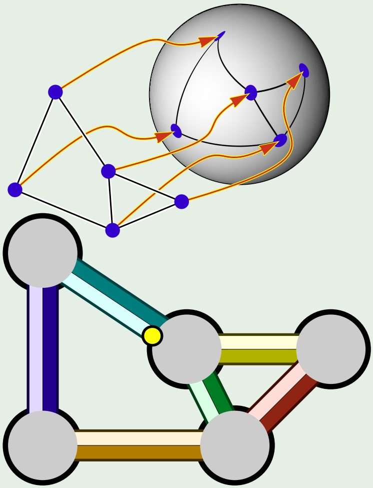
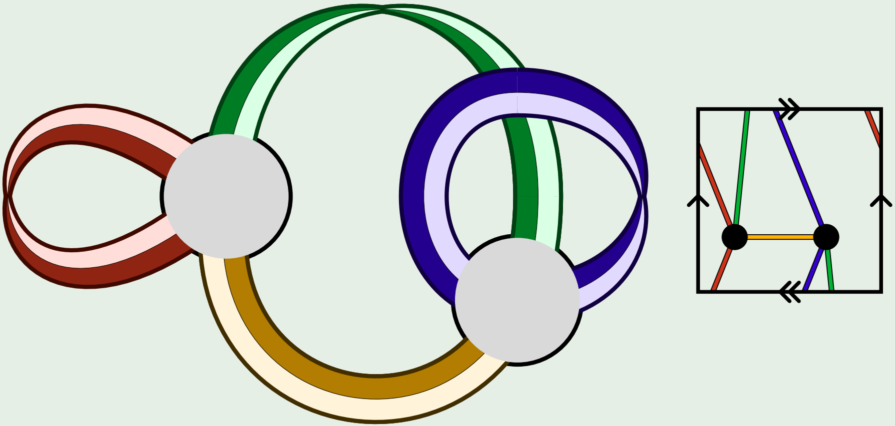


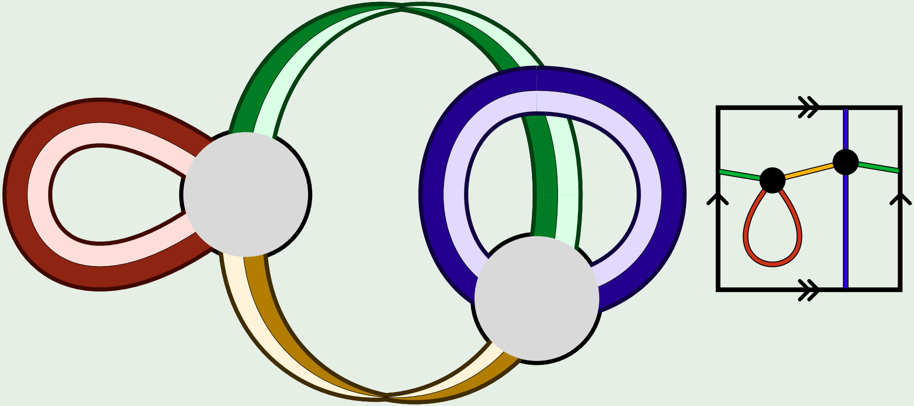
Twisted Ribbons allow Non-Orientable Maps

Example (A map on the Klein Bottle)

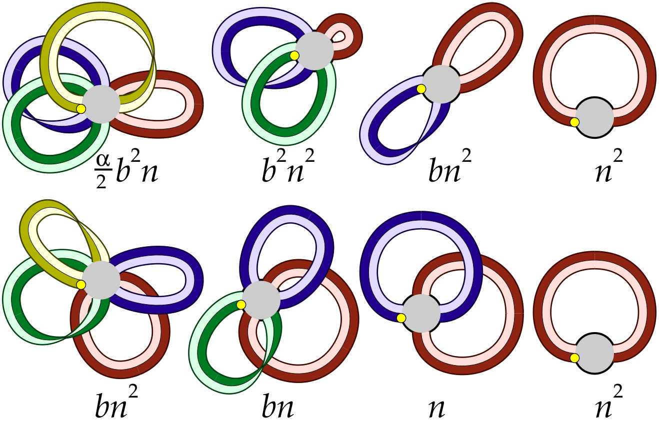


Twisted Ribbons allow Non-Orientable Maps

Example (A map on the Torus)



A rooted map with k edges can be thought of as a sequence of k maps.



Consecutive submaps differ in genus by 0, 1, or 2, and these steps are marked by 1, b, and $a = \frac{\alpha}{2}$ to assign a weight to a glueing.

Algebraic and Combinatorial Recurrences agree

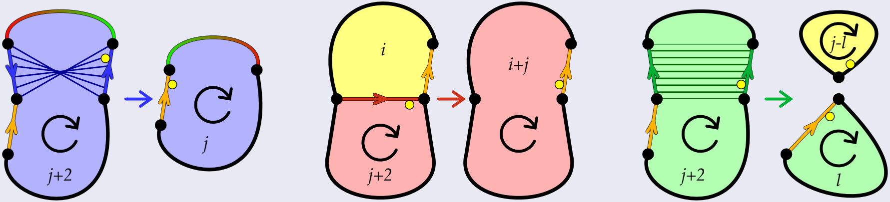
An Algebraic Recurrence

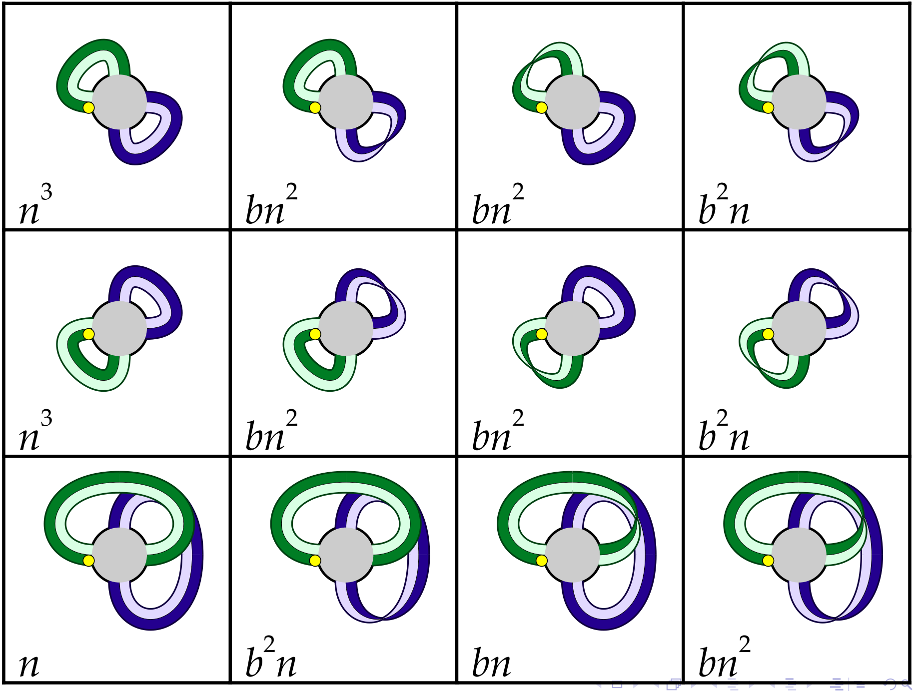
Example

$$\langle p_{j+2}p_\theta \rangle = b(j+1)\,\langle p_j p_\theta \rangle + \alpha \sum_{i\in\theta} i m_i(\theta)\,\langle p_{i+j} p_{\theta\setminus i} \rangle + \sum_{l=0}^{j} \langle p_l p_{j-l} p_\theta \rangle.$$

A Combinatorial Recurrence

It corresponds to a combinatorial recurrence for counting polygon glueings.



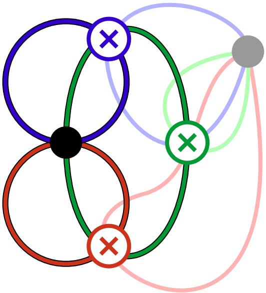

n^3
bn^2
bn^2
b^2n
n^3
bn^2
bn^2
b^2n
n
b^2n
bn
bn^2

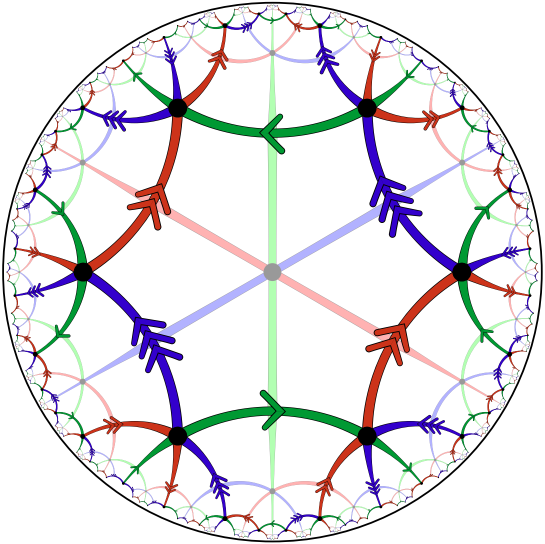
The Future

- The combinatorial interpretation has a two-parameter refinement. Is there a corresponding matrix question?
- At $b = 0$, we obtain glueings in $2^g : 1$ correspondence with orientable glueings. Can this correspondence be made to preserve vertex degrees as well as face degrees?
- A similar recurrence describes moments of the β-Laguerre distribution, with maps replaced by ▸ hypermaps.
- For $\beta \in \{1, 2, 4\}$ we can refine the combinatorial model and compute moments of XA. Is there a model for the β-Ensembles where this interpretation makes sense.
- For $\beta = 1$ and $\beta = 2$, there is a natural duality between vertices and faces. What operation replaces it for b-weighted glueings?
- The connection with ▸ Jack symmetric functions that needs to be explored.

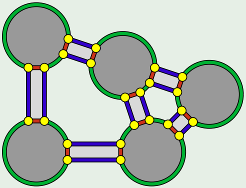
The End

Thank You

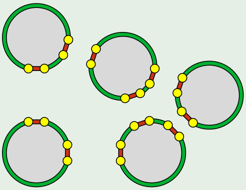




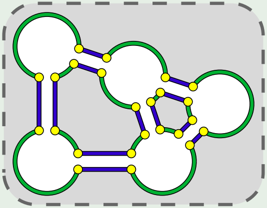
Example



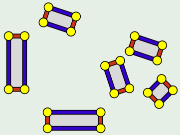
is enumerated by $\left(x_2^3\, x_3^2\right)\,(y_3\, y_4\, y_5)\,\left(z_2^6\right)$.



$\nu = [2^3, 3^2]$



$\phi = [3, 4, 5]$



$\epsilon = [2^6]$

Return

Explicit Formulae

The hypermap series can be computed explicitly when $\mathcal{H}$ consists of orientable hypermaps or all hypermaps.

Theorem (Jackson and Visentin - 1990)

When $\mathcal{H}$ is the set of orientable hypermaps,

$$H_{\mathcal{O}}\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}); 0\Big) = t\frac{\partial}{\partial t} \ln \left(\sum_{\theta \in \mathscr{P}} t^{|\theta|} H_\theta s_\theta(\mathbf{x}) s_\theta(\mathbf{y}) s_\theta(\mathbf{z}) \right)\Bigg|_{t=1}.$$

Theorem (Goulden and Jackson - 1996)

When $\mathcal{H}$ is the set of all hypermaps (orientable and non-orientable),

$$H_{\mathcal{A}}\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}); 1\Big) = 2t\frac{\partial}{\partial t} \ln \left(\sum_{\theta \in \mathscr{P}} t^{|\theta|} \frac{1}{H_{2\theta}} Z_\theta(\mathbf{x}) Z_\theta(\mathbf{y}) Z_\theta(\mathbf{z}) \right)\Bigg|_{t=1}.$$

A Generalized Series

Jack symmetric functions, ▸ Definition, are a one-parameter family, denoted by $\{J_\theta(\alpha)\}_\theta$, that generalizes both Schur functions and zonal polynomials.

b-Conjecture (Goulden and Jackson - 1996)

The generalized series,

$$H\Big(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}); b\Big)$$

$$:= (1+b)t\frac{\partial}{\partial t}\ln\left(\sum_{\theta\in\mathscr{P}} t^{|\theta|}\frac{J_\theta(\mathbf{x};1+b)J_\theta(\mathbf{y};1+b)J_\theta(\mathbf{z};1+b)}{\langle J_\theta, J_\theta\rangle_{1+b}}\right)\Bigg|_{t=1}$$

$$=\sum_{n\geq 0}\sum_{\nu,\phi,\epsilon\vdash n} c_{\nu,\phi,\epsilon}(b)p_\nu(\mathbf{x})p_\phi(\mathbf{y})p_\epsilon(\mathbf{z}),$$

has an combinatorial interpretation involving hypermaps. In particular $c_{\nu,\phi,\epsilon}(b) = \sum_{\mathfrak{h}\in\mathcal{H}_{\nu,\phi,\epsilon}} b^{\beta(\mathfrak{h})}$ *for some invariant* β *of rooted hypermaps.*

For general β, integrate over eigenvalues

Definition

For a function $f\colon \mathbb{R}^n \to \mathbb{R}$, define an expectation operator $\langle\cdot\rangle$ by

$$\langle f\rangle_{1+b} := c_{1+b}\int_{\mathbb{R}^n} |V(\boldsymbol{\lambda})|^{\frac{2}{1+b}} f(\boldsymbol{\lambda}) \mathrm{e}^{-\frac{1}{2(1+b)}p_2(\boldsymbol{\lambda})}\,\mathrm{d}\boldsymbol{\lambda},$$

with c_{1+b} chosen such that $\langle 1\rangle_{1+b} = 1$.

Theorem (Okounkov - 1997)

If n is a positive integer, $1+b$ is a positive real number, and θ is an integer partition of $2n$, then

$$\left\langle J_\theta^{(1+b)}(\boldsymbol{\lambda})\right\rangle_{1+b} = J_\theta^{(1+b)}(I_n)[p_{[2^n]}]J_\theta^{(1+b)}.$$

Jack Symmetric Functions

With respect to the inner product defined by

$$\langle p_\lambda(\mathbf{x}), p_\mu(\mathbf{x}) \rangle_\alpha = \delta_{\lambda,\mu} \frac{|\lambda|!}{|C_\lambda|} \alpha^{\ell(\lambda)},$$

Jack symmetric functions are the unique family satisfying:

(P1) (Orthogonality) If $\lambda \neq \mu$, then $\langle J_\lambda, J_\mu \rangle_\alpha = 0$.

(P2) (Triangularity) $J_\lambda = \sum_{\mu \preccurlyeq \lambda} v_{\lambda\mu}(\alpha) m_\mu$, where $v_{\lambda\mu}(\alpha)$ is a rational function in α, and '$\preccurlyeq$' denotes the natural order on partitions.

(P3) (Normalization) If $|\lambda| = n$, then $v_{\lambda,[1^n]}(\alpha) = n!$.

Return

Jack Symmetric Functions

Jack symmetric functions, are a one-parameter family, denoted by $\{J_\theta(\alpha)\}_\theta$, that generalizes both Schur functions and zonal polynomials.

Proposition (Stanley - 1989)

Jack symmetric functions are related to Schur functions and zonal polynomials by:

$$J_\lambda(1) = H_\lambda s_\lambda, \qquad \langle J_\lambda, J_\lambda \rangle_1 = H_\lambda^2,$$

and

$$J_\lambda(2) = Z_\lambda, \qquad \langle J_\lambda, J_\lambda \rangle_2 = H_{2\lambda},$$

where 2λ is the partition obtained from λ by multiplying each part by two.

Return

$$\langle p_{j+2} p_\theta \rangle = b(j+1)\langle p_j p_\theta \rangle + (1+b)\sum_{i\in\theta} i m_i(\theta)\langle p_{i+j} p_{\theta\setminus i}\rangle + \sum_{l=0}^{j}\langle p_l p_{j-l} p_\theta\rangle$$

Example

$$\begin{aligned}
\langle 1 \rangle &= 1 \\
\langle p_0 \rangle &= n \\
\langle p_2 \rangle &= b\langle p_0 \rangle + \langle p_0 p_0 \rangle = bn + n^2 \\
\langle p_1 p_1 \rangle &= (1+b)\langle p_0 \rangle = (1+b)n \\
\langle p_4 \rangle &= 3b\langle p_2 \rangle + \langle p_0 p_2 \rangle + \langle p_1 p_1 \rangle + \langle p_2 p_0 \rangle = (1+b+3b^2)n + 5bn^2 + 2n^3 \\
\langle p_3 p_1 \rangle &= 2b\langle p_1 p_1 \rangle + (1+b)\langle p_2 \rangle + \langle p_0 p_1 p_1 \rangle + \langle p_1 p_0 p_1 \rangle = (3b+3b^2)n + (3+3b)n^2 \\
\langle p_2 p_2 \rangle &= b\langle p_0 p_2 \rangle + 2(1+b)\langle p_2 \rangle + \langle p_0 p_0 p_2 \rangle = 2b(1+b)n + (2+2b+b^2)n^2 + 2bn^3 + n^4 \\
\langle p_2 p_{1,1} \rangle &= b\langle p_0 p_{1,1} \rangle + 2(1+b)\langle p_{1,1} \rangle + \langle p_0 p_0 p_{1,1} \rangle = 2(1+b)^2 n + (b+b^2)n^2 + (1+b)n^3 \\
\langle p_1 p_3 \rangle &= 3(1+b)\langle p_2 \rangle = (3b+3b^2)n + (3+3b)n^2 \\
\langle p_1 p_{2,1} \rangle &= 2(1+b)\langle p_{1,1} \rangle + (1+b)\langle p_0 p_2 \rangle = (2+4b+2b^2)n + (b+b^2)n^2 + (1+b)n^3 \\
\langle p_{1,1,1,1} \rangle &= 3(1+b)\langle p_0 p_{1,1} \rangle = (1+2b+b^2)n^2
\end{aligned}$$

◂ Return

$$\langle p_{j+2}p_\theta\rangle = b(j+1)\langle p_j p_\theta\rangle + (1+b)\sum_{i\in\theta} i m_i(\theta)\langle p_{i+j}p_{\theta\setminus i}\rangle + \sum_{l=0}^{j}\langle p_l p_{j-l} p_\theta\rangle$$

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\langle p_4\rangle &= 3b\langle p_2\rangle + \langle p_0p_2\rangle + \langle p_1p_1\rangle + \langle p_2p_0\rangle = (1+b+3b^2)n + 5bn^2 + 2n^3\\
\langle p_3p_1\rangle &= 2b\langle p_1p_1\rangle + (1+b)\langle p_2\rangle + \langle p_0p_1p_1\rangle + \langle p_1p_0p_1\rangle = (3b+3b^2)n + (3+3b)n^2\\
\langle p_2p_2\rangle &= b\langle p_0p_2\rangle + 2(1+b)\langle p_2\rangle + \langle p_0p_0p_2\rangle = 2b(1+b)n + (2+2b+b^2)n^2 + 2bn^3 + n^4\\
\langle p_2p_{1,1}\rangle &= b\langle p_0p_{1,1}\rangle + 2(1+b)\langle p_{1,1}\rangle + \langle p_0p_0p_{1,1}\rangle = 2(1+b)^2n + (b+b^2)n^2 + (1+b)n^3\\
\langle p_1p_3\rangle &= 3(1+b)\langle p_2\rangle = (3b+3b^2)n + (3+3b)n^2\\
\langle p_1p_{2,1}\rangle &= 2(1+b)\langle p_{1,1}\rangle + (1+b)\langle p_0p_2\rangle = (2+4b+2b^2)n + (b+b^2)n^2 + (1+b)n^3\\
\langle p_{1,1,1,1}\rangle &= 3(1+b)\langle p_0p_{1,1}\rangle = (1+2b+b^2)n^2
\end{aligned}$$

Return

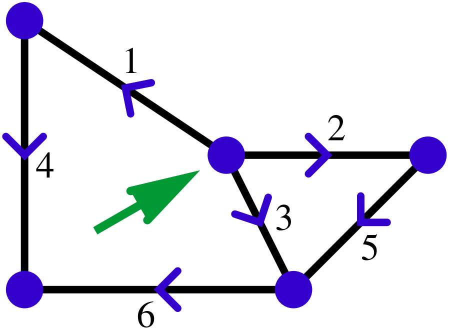
b is ubiquitous

The many lives of b

	$b = 0$		$b = 1$
Hypermaps	Orientable	**?**	Locally Orientable
Symmetric Functions	s_θ	$J_\theta(b)$	Z_θ
Matrix Integrals	GUE	**?**	GOE
Moduli Spaces	over $\mathbb{C}$	**?**	over $\mathbb{R}$
Matching Systems	Bipartite	**?**	All

Encoding Orientable Maps

1. Orient and label the edges.
2. This induces labels on flags.
3. Clockwise circulations at each vertex determine ν.
4. Face circulations are the cycles of $\epsilon\nu$.



$$\epsilon = (1\ 1')(2\ 2')(3\ 3')(4\ 4')(5\ 5')(6\ 6')$$
$$\nu = (1\ 2\ 3)(1'\ 4)(2'\ 5)(3'\ 5'\ 6)(4'\ 6')$$
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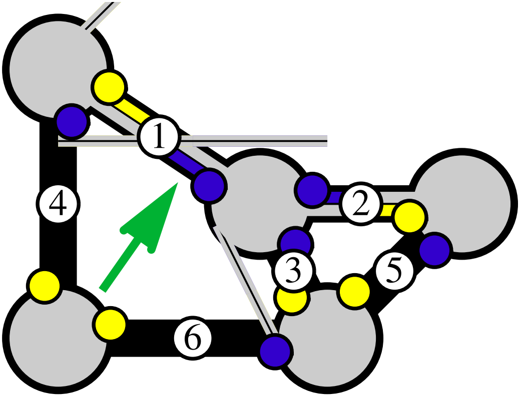

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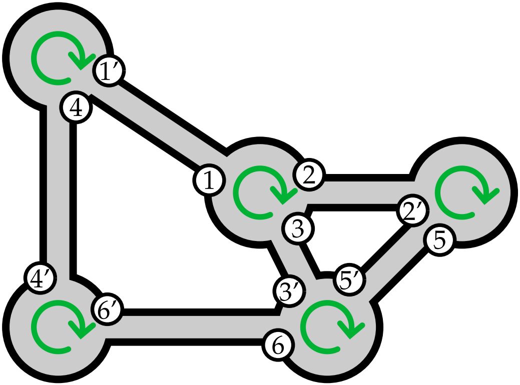

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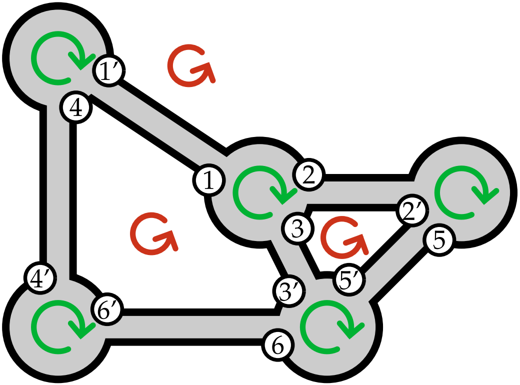

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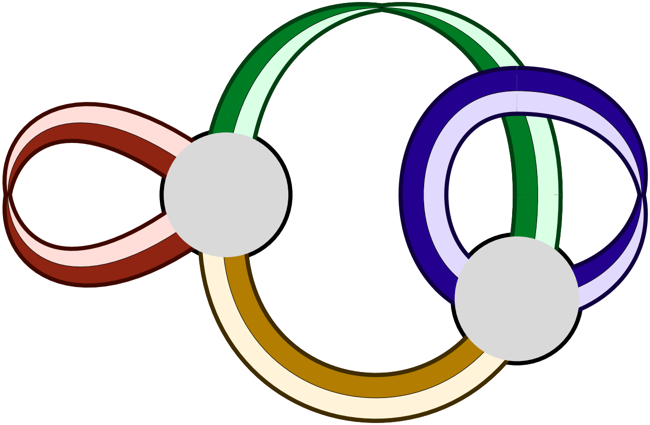
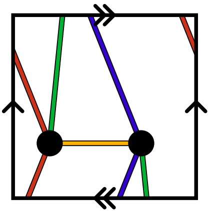
Equivalence classes can be encoded by perfect matchings of flags.

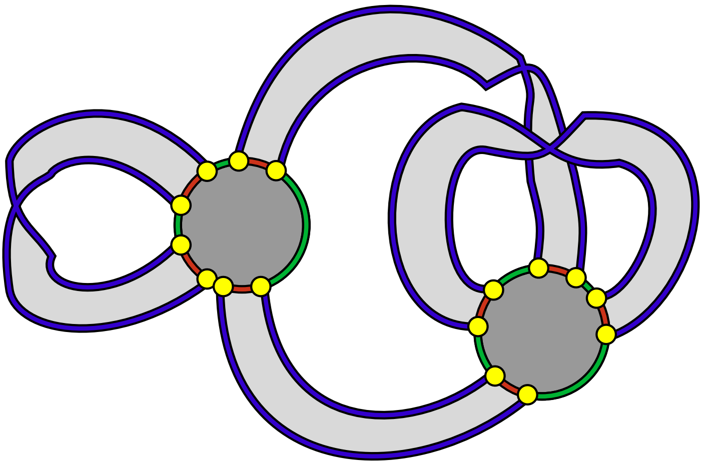


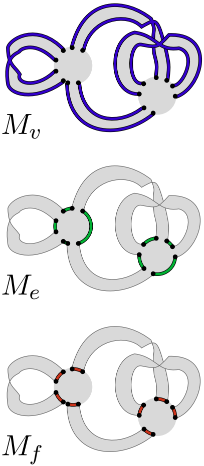


Start with a ribbon graph.

Encoding all Maps

Equivalence classes can be encoded by perfect matchings of flags.

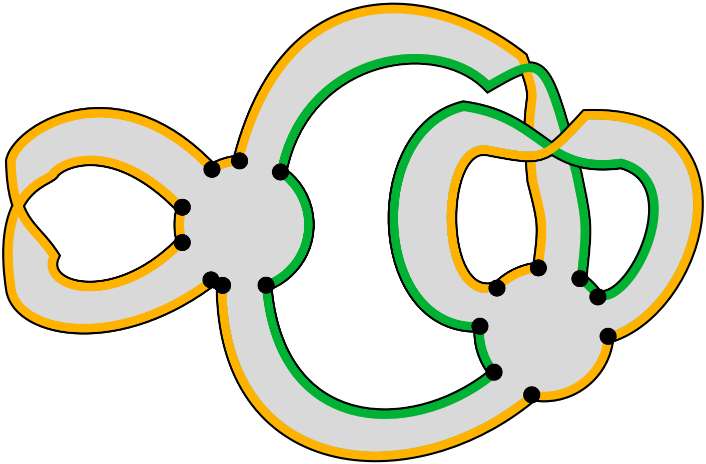


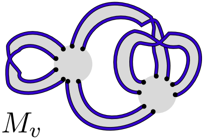


Ribbon boundaries determine 3 perfect matchings of flags.

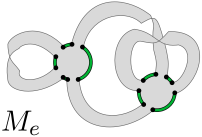
Encoding all Maps

Equivalence classes can be encoded by perfect matchings of flags.





M_v

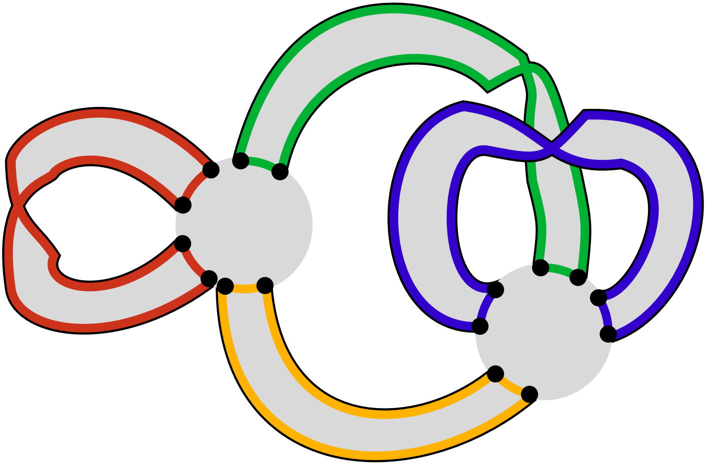


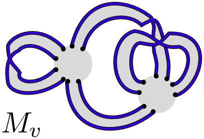
M_e

Pairs of matchings determine, faces,

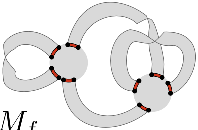
Encoding all Maps

Equivalence classes can be encoded by perfect matchings of flags.





M_v

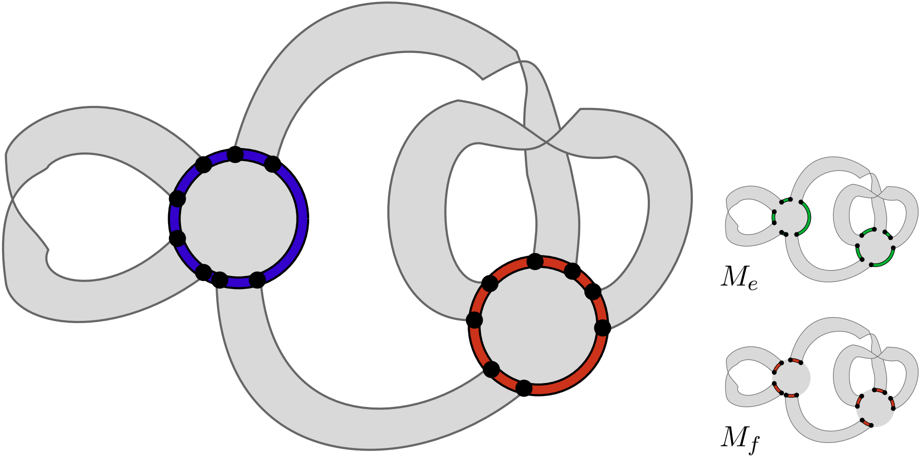


M_f

Pairs of matchings determine, faces, edges,

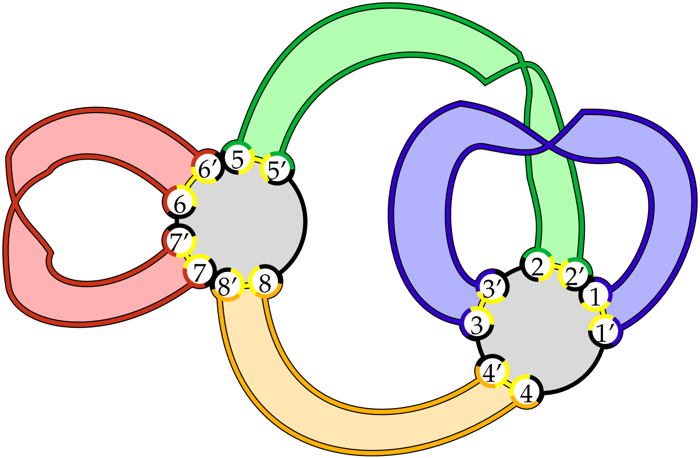
Encoding all Maps

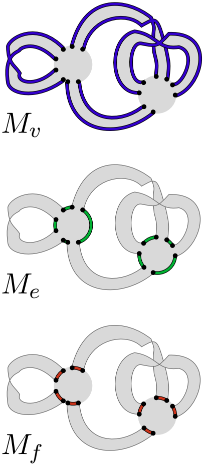
Equivalence classes can be encoded by perfect matchings of flags.



Pairs of matchings determine, faces, edges, and vertices.

Encoding all Maps





$$M_v = \{\{1,3\},\{1',3'\},\{2,5\},\{2',5'\},\{4,8'\},\{4',8\},\{6,7\},\{6',7'\}\}$$

$$M_e = \{\{1,2'\},\{1',4\},\{2,3'\},\{3,4'\},\{5,6'\},\{5',8\},\{6,7'\},\{7,8'\}\}$$

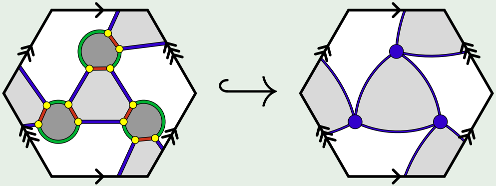
$$M_f = \{\{1,1'\},\{2,2'\},\{3,3'\},\{4,4'\},\{5,5'\},\{6,6'\},\{7,7'\},\{8,8'\}\}$$

Hypermaps

Generalizing the combinatorial encoding, an arbitrary triple of perfect matchings determines a **hypermap** when the triple induces a connected graph, with cycles of $M_e \cup M_f$, $M_e \cup M_v$, and $M_v \cup M_f$ determining vertices, hyperfaces, and hyperedges. ▸ Example

Hypermaps both **specialize** and generalize maps.

Example



Hypermaps can be represented as face-bipartite maps.

◂ Return

Hypermaps

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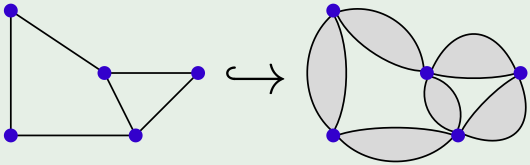
Hypermaps both specialize and **generalize** maps.

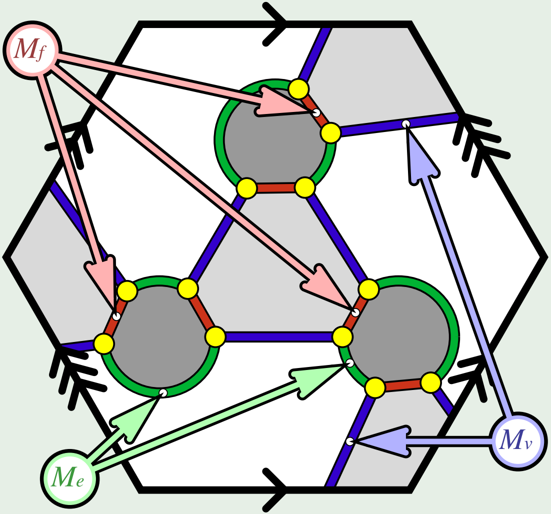
Example

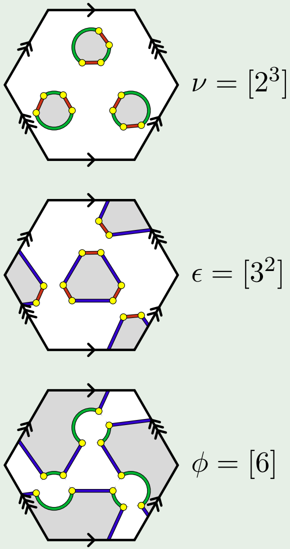


Maps can be represented as hypermaps with $\epsilon = [2^n]$.

Return

Example





Return