Example 6.3 (pp. 101–105)

Model

$$\begin{array}{lcll} X_i|\lambda_i & \sim & \text{Poisson}(\lambda_i) & \text{independently } i = 1, ..., n \\ \lambda_i|\theta & \sim & \text{exponential}(\theta) & \text{also independently} \\ \theta & \sim & \text{gamma}(\alpha, \beta) & \end{array}$$

Goal

$$\pi(\lambda_1, ..., \lambda_n, \theta | x_1, ..., x_n)$$

$$= \frac{f(x_1, ..., x_n | \lambda_1, ..., \lambda_n, \theta) \cdot \pi(\lambda_1, ..., \lambda_n, \theta)}{m(x_1, ..., x_n)}$$

$$= \frac{f(x_1, ..., x_n | \lambda_1, ..., \lambda_n) \cdot \pi(\lambda_1, ..., \lambda_n | \theta) \cdot \pi(\theta)}{m(x_1, ..., x_n)}$$

Technical Bottleneck

$$
\begin{aligned}
& m(x_1, ..., x_n) \\
= & \int f(x_1, ..., x_n | \lambda_1, ..., \lambda_n) \cdot \pi(\lambda_1, ..., \lambda_n | \theta) \cdot \pi(\theta) \; d\theta d\lambda_1 ... d\lambda_n \\
= & \int \left[\prod_{i=1}^{n} \frac{e^{-\lambda_i} \lambda_i^{x_i}}{x_i!} \right] \left[\prod_{i=1}^{n} \theta e^{-\theta \lambda_i} \right] \left[\frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} \right] d\theta d\lambda_1 ... d\lambda_n \\
= & \begin{cases} \text{integrate over } \theta \text{ \underline{first}} \\ \\ \text{integrate over } \lambda_1, ..., \lambda_n \text{ \underline{first}} \end{cases}
\end{aligned}
$$

Integrate Over θ First

$$\int \left[\prod_{i=1}^{n} \frac{e^{-\lambda_i}\lambda_i^{x_i}}{x_i!}\right] \left[\prod_{i=1}^{n} \theta e^{-\theta\lambda_i}\right] \left[\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}\right] d\theta d\lambda_1 ... d\lambda_n$$

$$= \int \left[\prod_{i=1}^{n} \frac{e^{-\lambda_i}\lambda_i^{x_i}}{x_i!}\right] \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} \underbrace{\left[\int \theta^{n+\alpha-1} e^{-(\beta+\sum_{i=1}^{n}\lambda_i)\theta}\, d\theta\right]}_{(\star)} d\lambda_1 ... d\lambda_n$$

$$= \int \left[\prod_{i=1}^{n} \frac{e^{-\lambda_i}\lambda_i^{x_i}}{x_i!}\right] \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(n+\alpha)}{(\beta + \sum_{i=1}^{n}\lambda_i)^{n+\alpha}}\, d\lambda_1 ... d\lambda_n$$

$$= \frac{(\beta^\alpha)\Gamma(n+\alpha)}{\Gamma(\alpha)\prod_{i=1}^{n} x_i!} \underbrace{\int \frac{\left(e^{-\sum_{i=1}^{n}\lambda_i}\right)\left(\prod_{i=1}^{n}\lambda_i^{x_i}\right)}{(\beta+\sum_{i=1}^{n}\lambda_i)^{n+\alpha}}\, d\lambda_1 ... d\lambda_n}_{\text{uh, oh ...}}$$

Integrate Over $\lambda_1, ..., \lambda_n$ First

$$\int \left[\prod_{i=1}^{n} \frac{e^{-\lambda_i}\lambda_i^{x_i}}{x_i!}\right] \left[\prod_{i=1}^{n} \theta e^{-\theta\lambda_i}\right] \left[\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}\right] d\theta d\lambda_1 ... d\lambda_n$$

$$= \int \left\{ \prod_{i=1}^{n} \underbrace{\left[\int \frac{e^{-\lambda_i}\lambda_i^{x_i}}{x_i!} \times \theta e^{-\theta\lambda_i} \; d\lambda_i\right]}_{(\star\star)} \right\} \times \left[\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}\right] d\theta$$

$$= \int \left[\frac{\theta}{1+\theta}\right]^n \times \left[\frac{1}{1+\theta}\right]^{n\bar{x}} \times \left[\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}\right] d\theta$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \underbrace{\int \frac{\theta^{n+\alpha-1}e^{-\beta\theta}}{(1+\theta)^{n(\bar{x}+1)}} \; d\theta}_{\text{uh, oh ...}}$$

Monte Carlo

For $\boldsymbol{z} \sim f(\boldsymbol{z})$, $\boldsymbol{z} = (z_1, z_2, ..., z_d) \in \mathbb{R}^d$, want

$$\mathbb{E}[h(\boldsymbol{z})] \equiv \int h(\boldsymbol{z}) f(\boldsymbol{z}) d\boldsymbol{z}, \quad \mathbb{P}[g(\boldsymbol{z}) \in A] \equiv \int_{g(\boldsymbol{z}) \in A} f(\boldsymbol{z}) d\boldsymbol{z}.$$

Can approximate by

$$\int h(\boldsymbol{z}) f(\boldsymbol{z}) d\boldsymbol{z} \approx \frac{1}{n} \sum_{i=1}^{n} h(\boldsymbol{z}_i),$$

$$\int_{g(\boldsymbol{z}) \in A} f(\boldsymbol{z}) d\boldsymbol{z} \approx \frac{1}{n} \sum_{i=1}^{n} I[g(\boldsymbol{z}_i) \in A]$$

if can obtain (somehow) a sample $\{\boldsymbol{z}_1, ..., \boldsymbol{z}_n\} \sim f(\boldsymbol{z})$.

Using the Gibbs Sampler

- want, but don't know how, to sample from a multivariate distribution, $f(z_1, z_2, ..., z_d)$
- can repeatedly sample from the conditional distributions

$$f(z_j|z_1, ..., z_{j-1}, z_{j+1}, ..., z_d), \quad j = 1, 2, ..., d, 1, 2, ..., d, ...$$

until "things stabilize" (called "burn in")

Back to Example 6.3

Want

$$\pi(\lambda_1, ..., \lambda_n, \theta | x_1, ..., x_n).$$

So repeatedly sample from

$$\pi(\lambda_i \quad | \quad \lambda_1, ..., \lambda_{i-1}, \lambda_{i+1}, ..., \lambda_n, \theta, x_1, ..., x_n), \quad i = 1, ..., n;$$

$$\pi(\theta \quad | \quad \lambda_1, ..., \lambda_n, x_1, ..., x_n).$$

Exercise Show that the integral marked earlier by ($\star$) implies

$$\pi(\theta | \lambda_1, ..., \lambda_n, x_1, ..., x_n) \sim \text{gamma}\Big(\alpha + n, \beta + \sum_{i=1}^{n} \lambda_i\Big)$$

and the integral marked earlier by ($\star\star$) implies

$$\pi(\lambda_i | \lambda_1, ..., \lambda_{i-1}, \lambda_{i+1}, ..., \lambda_n, \theta, x_1, ..., x_n) \sim \text{gamma}(x_i + 1, 1 + \theta).$$

Algorithm

initialize $t = 1$ and $\lambda_i^{(0)} = x_i$ for $i = 1, 2, ..., n$

while $(t < t_{\text{max}})$

draw $\theta^{(t)} \sim \text{gamma}\left(\alpha + n, \beta + \sum_{i=1}^{n} \lambda_i^{(t-1)}\right)$

draw $\lambda_i^{(t)} \sim \text{gamma}\left(x_i + 1, 1 + \theta^{(t)}\right)$ for $i = 1, 2, ..., n$

increment $t = t + 1$

end while

return $\left\{(\lambda_1^{(t)}, ..., \lambda_n^{(t)}, \theta^{(t)}) : t > t_{\text{burn.in}}\right\}$