Functions of RVs

Key Question Know joint distribution of $X_1, X_2, ..., X_n$; want distribution of $Y \equiv g(X_1, X_2, ..., X_n)$.

Basic Method

$$F_Y(y) \equiv \mathbb{P}(Y \leq y) = \int_{g(x_1,...,x_n) \leq y} f(x_1, ..., x_n) dx_1 ... dx_n$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

Example 3.6 (pp. 39–40) $X, Y \sim \text{uniform}(0, 1)$, $X \perp\!\!\!\perp Y$, $Z = X + Y \Rightarrow$ distribution of Z?

Exercise 3.6 (p. 43)

$$Z \sim \underset{\substack{\uparrow \\ \phi(z),\Phi(z)}}{\mathrm{N}(0,1)}, \quad X = \sigma Z + \mu \quad \left[\text{recall } \phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\right]$$

$$F(x) = \mathbb{P}(X \le x) = \mathbb{P}\left[Z \overset{\substack{\sigma>0 \\ \downarrow}}{\le} \frac{x-\mu}{\sigma}\right] = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$f(x) = \frac{d}{dx}\Phi\left(\frac{x-\mu}{\sigma}\right) = \phi\left(\frac{x-\mu}{\sigma}\right) \cdot \frac{1}{\sigma} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

Some Special Functions

$X_1, X_2, ..., X_n \overset{iid}{\sim} f_X(x), F_X(x)$. $\mathbb{E}(X_i) = \mu$, $\mathbb{V}\text{ar}(X_i) = \sigma^2$ for all i.

(a) (*sum*) $S_n = X_1 + ... + X_n$. Central Limit Theorem (CLT):

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim \text{N}(0,1) \quad \text{or} \quad \frac{S_n - n\mu}{\sqrt{n}\sigma} \sim \text{N}(0,1) \quad \text{approximately.}$$

(b) (*maximum*) $Y_n = \max\{X_1, X_2, ..., X_n\}$.

$$F(y) = \mathbb{P}(Y_n \leq y) = \mathbb{P}(X_1 \leq y \text{ and } ... \text{ and } X_n \leq y)$$

$$= \prod_{i=1}^{n} \mathbb{P}(X_i \leq y) = [F_X(y)]^n \Rightarrow f(y) = n[F_X(y)]^{n-1} f_X(y)$$

(c) (*minimum*) $Z_n = \min\{X_1, X_2, ..., X_n\}$.

Exercise 3.4a (p. 42) Find the density function for Z_n.

Example

If $X_1, X_2, ..., X_n \sim \text{uniform}(0, \theta)$, then each has density function and CDF,

$$f_X(x) = \frac{1}{\theta} \quad \text{and} \quad F_X(x) = \frac{x}{\theta},$$

respectively on $(0, \theta)$. Thus, $Y_n = \max\{X_1, X_2, ..., X_n\}$ has density

$$f(y) = n[F_X(y)]^{n-1} f_X(y) = n \left[\frac{y}{\theta}\right]^{n-1} \cdot \frac{1}{\theta},$$

and

$$\mathbb{E}(Y_n) = \int_0^\theta y f(y) dy = ... = \frac{n\theta}{n+1} \to \theta$$

as $n \to \infty$. [*<u>Think</u>: Makes sense?*]

Remark Example 3.7 (pp. 41–42) is a slightly simpler version of this example here.

Sum of Independent RVs

- sometimes, $\mathbb{E}(\cdot)$ and $\mathbb{V}\text{ar}(\cdot)$ are enough, $f(\cdot)$ not necessary
- for independent RVs, $X_1, X_2, ..., X_n$, $\exists$ easy formulae:

$$\mathbb{E}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i \mathbb{E}(X_i), \quad \mathbb{V}\text{ar}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i^2 \mathbb{V}\text{ar}(X_i)$$

Warning $\mathbb{V}\text{ar}(\cdot)$ more complicated if RVs are <u>not</u> independent!

Exercise For i.i.d. RVs, $X_1, X_2, ..., X_n$, with $\mathbb{E}(X_i) = \mu$ and $\mathbb{V}\text{ar}(X_i) = \sigma^2$, show that $\mathbb{E}(\bar{X}_n) = \mu$ and $\mathbb{V}\text{ar}(\bar{X}_n) = \sigma^2/n$.
[*Remark: The only big conclusion from the CLT is normality.*]

Sum of Non-Independent RVs

Example For a, b non-random,

$$\begin{aligned}\mathbb{V}\mathrm{ar}(aX+bY) &= \mathbb{E}[(aX+bY)^2]-[\mathbb{E}(aX+bY)]^2 \\ &= \{a^2\,\mathbb{E}(X^2)\ +b^2\,\mathbb{E}(Y^2)\ +2ab\mathbb{E}(XY)\qquad\}- \\ &\quad\ \{a^2[\mathbb{E}(X)]^2+b^2[\mathbb{E}(Y)]^2+2ab\mathbb{E}(X)\mathbb{E}(Y)\} \\ &= a^2\mathbb{V}\mathrm{ar}(X)+b^2\mathbb{V}\mathrm{ar}(Y)+2ab\mathbb{C}\mathrm{ov}(X,Y).\end{aligned}$$

That is, $\exists$ additional $\mathbb{C}\mathrm{ov}(\cdot,\cdot)$ terms.

Remark The textbook, *Essential Statistics*, does not really cover the concept of covariance.

General $\mathbb{V}\mathrm{ar}$ and $\mathbb{C}\mathrm{ov}$ Formulae for Sums

$$\mathbb{V}\mathrm{ar}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i^2 \mathbb{V}\mathrm{ar}(X_i) + \sum_{i \neq j} a_i a_j \mathbb{C}\mathrm{ov}(X_i, X_j)$$

$$\mathbb{C}\mathrm{ov}\left[\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right] = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \mathbb{C}\mathrm{ov}(X_i, Y_j)$$

Analogy Like doing "perfect squares" $(a+b+c)^2$ and "multiplications" $(a+b)(c+d)$.

Covariance

Definition

$$
\begin{aligned}
\mathbb{C}\mathrm{ov}(X, Y) &= \mathbb{E}\{[X - \mathbb{E}(X)][Y - \mathbb{E}(Y)]\} && (1) \\
&\vdots \\
&= \mathbb{E}[XY - Y\mathbb{E}(X) - X\mathbb{E}(Y) + \mathbb{E}(X)\mathbb{E}(Y)] \\
&= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) - \mathbb{E}(Y)\mathbb{E}(X) + \mathbb{E}(X)\mathbb{E}(Y) \\
&\vdots \\
\therefore \quad \text{also} &= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) && (2)
\end{aligned}
$$

Remark Easy to see from definition (1) that $\mathbb{C}\mathrm{ov}(X, X) = \mathbb{E}\{[X - \mathbb{E}(X)]^2\} = \mathbb{V}\mathrm{ar}(X)$, and from convenient computational formula (2) that $\mathbb{C}\mathrm{ov}(X, X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$.

Independence ⇒ Zero Covariance

Definition X and Y are independent if $f(x,y) = f_X(x) f_Y(y)$.

Implication Then,

$$\mathbb{E}(XY) = \int\int xy f(x,y) dx dy = \int\int xy f(x) f(y) dx dy = \ldots$$

$$\ldots = \int y f(y) \underbrace{\left[\int x f(x) dx\right]}_{\mathbb{E}(X)} dy = \mathbb{E}(X)\mathbb{E}(Y)$$

so $\mathbb{C}\mathrm{ov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 0.$

Zero Covariance $\not\Rightarrow$ Independence

Counter Example

$f(x, y)$

$x\backslash y$	-1	0	$+1$
-1	0	0.1	0
0	0.1	0.6	0.1
$+1$	0	0.1	0

Exercise Show that $\mathbb{C}\mathrm{ov}(X, Y) = 0$ but that X and Y are <u>not</u> independent.

Puzzle What kind of dependence is captured by the covariance?

Correlation

Definition

$$\mathbb{C}\text{orr}(X, Y) = \frac{\mathbb{C}\text{ov}(X, Y)}{\sqrt{\mathbb{V}\text{ar}(X)\mathbb{V}\text{ar}(Y)}}$$

Theorem $-1 \leq \mathbb{C}\text{orr}(X, Y) \leq +1$, with equality if and only if X and Y are almost surely linear functions of each other.

Proof Omitted. Main idea is to note that $\mathbb{V}\text{ar}(Y - tX)$, a quadratic function of t, is $\geq 0 \ \forall \ t \in \mathbb{R}$. What does this imply? What does it mean if there exists a t for which this quadratic function is $= 0$, and when does such a t exist?

For the Curious Only

Question What does it mean to say Y is almost surely a linear function of X?

Answer Let

$$A = \{\omega \in S : \quad Y(\omega) \neq tX(\omega) + c\}.$$

It means the set A has probability zero, $\mathbb{P}(A) = 0$.

Remark Here, the notion that random variables are functions mapping from S to $\mathbb{R}$ becomes important.