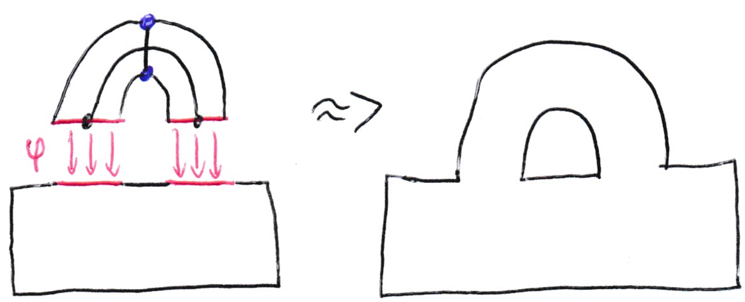
Handles II: Handle Decompositions

Last time: • Defined handles $h^k \cong D^k \times D^{n-k}$ ~ k-cell

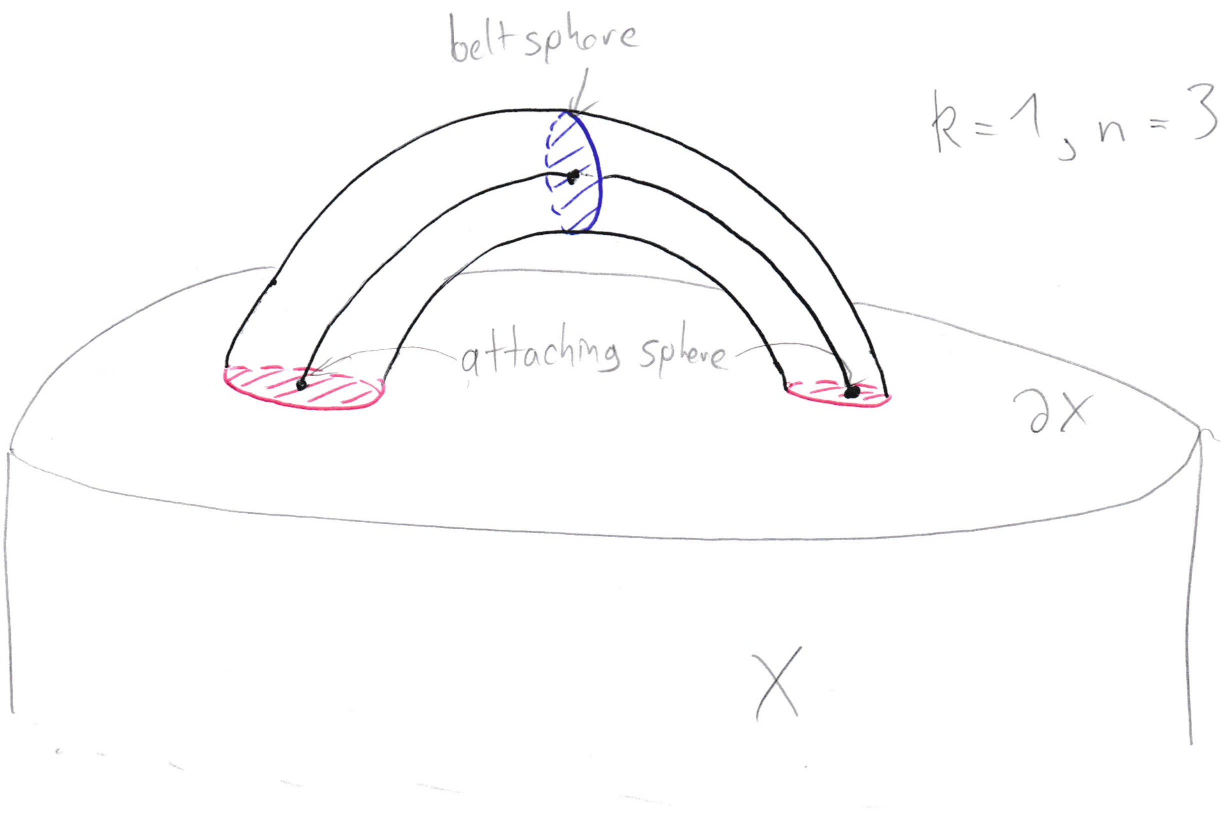
k = index, n = dimension.

• Attaching handles to the (possibly empty) boundary of a manifold X^n:
choosing an embedding $\varphi: S^{k-1} \times D^{n-k} \hookrightarrow \partial X$ (codim 0)

Form $X \cup_\varphi h^k$:



$k=1, n=3.$



Today

I) Handlebodies and Handle Decompositions.

II) Handle Decompositions Exist.

III) Simplifying Handle Decompositions.

?) Example: $\mathbb{C}P^4$

We'll assume all manifolds are <u>compact</u>.

<u>Recall</u>: <u>Transversality</u>

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

f, g are <u>transverse</u> ($f \pitchfork g$) if
$\forall x \in X \; \forall y \in Y \;\; f(x) = g(y) \Rightarrow f_* T_x X + g_* T_y Y = T_{f(x)} Z$

Suppose f, g are <u>embeddings</u>, and $f \pitchfork g$.

$\Rightarrow \dim(X \cap Y) = \dim X + \dim Y - \dim Z$.

$\therefore$ · If $\dim X + \dim Y = \dim Z$, $\dim(X \cap Y) = 0$.

· If $\dim X + \dim Y < \dim Z$, $X \cap Y = \emptyset$.

§I. Handlebodies and Handle Decompositions

Def

Let Y^{n-1} be closed (and possibly empty).

- Say a <u>relative handlebody built on Y</u> is a manifold of the form

$$(Y\times I) \cup_{\varphi_1} h_1^{k_1} \cup \ldots \cup_{\varphi_m} h_m^{k_m}$$

where we interpret the union by attaching the $(i+1)$-st handle to $(Y\times I) \bigcup_{j=1}^{i} h_j^{k_j} \setminus (Y\times\{0\})$

If $Y=\emptyset$ we just call it a <u>handlebody</u>.

<u>Note:</u> If $Y=\emptyset$, the first handle attachment must be a 0-handle, since the attaching region is $(\partial D^0)\times D^n=\emptyset$.

- Let X^n be a manifold with $\partial X = \partial_+ X \sqcup \overline{\partial_- X}$.
A <u>handle decomposition of X</u> is a diffeomorphism

$$X \cong_{\partial_- X} Z$$

where Z is a relative handlebody built on $\partial_- X$.

II) Handle Decompositions Exist.

Morse Theory

(good reference: Milnor, 1963)

Idea: Non-degenerate critical points of smooth functions tell us about topological changes.

Morse Lemma

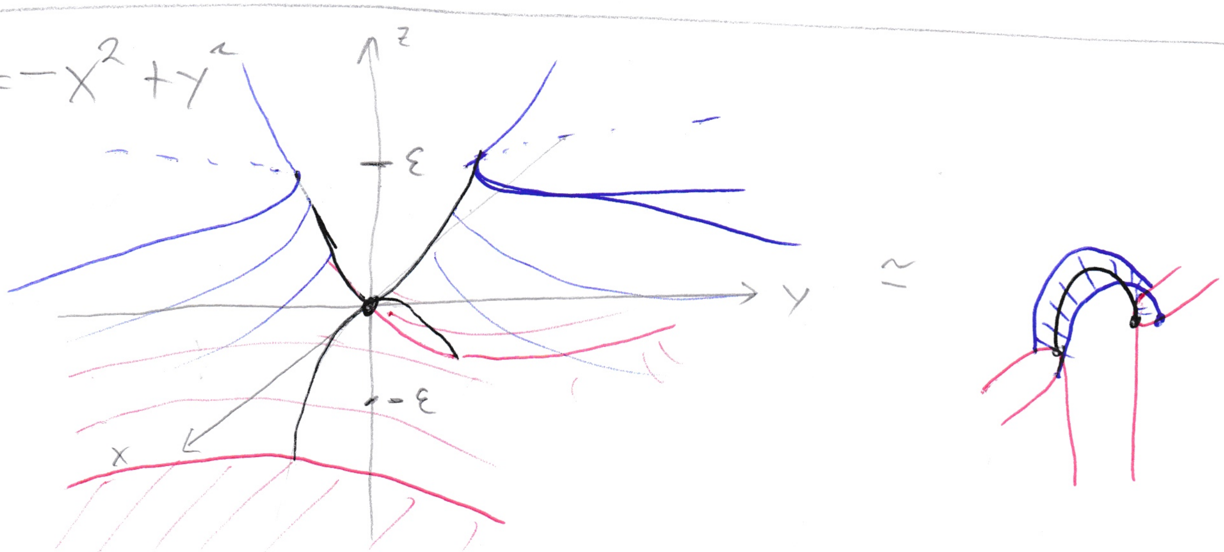
Let $f: X^n \to \mathbb{R}$ be smooth, and let $p \in X$ be a non-degenerate critical point, i.e. has a non-singular Hessian.

Then there is a chart $\varphi: U \hookrightarrow X^n$ with $\varphi(0) = p$ s.t.

$$\forall (x_1, \dots, x_n) \in U \quad f \circ \varphi(x_1, \dots, x_n) = f(p) + \sum_{i=1}^{k} -x_i^2 + \sum_{i=k+1}^{n} x_i^2$$

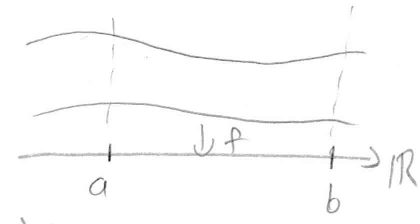
(We call k the <u>index</u> of p).

$z = -x^2 + y^2$



Def $f: X \to \mathbb{R}$ is Morse if all its critical points are non-degenerate.

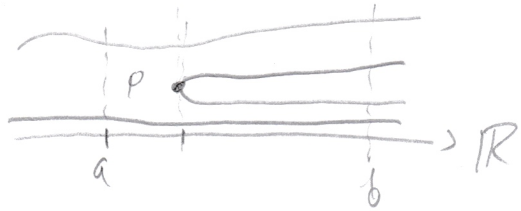
Theorems



Say $f: X \to \mathbb{R}$ is smooth, $a < b \in \mathbb{R}$.

If $f^{-1}[a,b]$ contains no critical points then $X_a \cong X_b$, and X_b deformation retracts onto X_a

Now suppose $f^{-1}[a,b]$ contains a unique critical point $p \in f^{-1}(a,b)$, of index k.



Then $X_b \cong X_a \cup h^k$.

Morse functions are dense in $C(X, \mathbb{R})$

Corollary

Smooth manifolds have handle decompositions.

III) Simplifying Handle Decompositions.

Proposition 4.27

A handle decomposition can be modified so that handles of index k are added before handles of index $k' > k$. Handles of the same index can be attached in any order, or simultaneously. (Let $X_k = (\partial X \times I) \cup \left\{ \begin{smallmatrix} \text{handles with} \\ \text{index} \leq k \end{smallmatrix} \right\}$.)

Pf Suppose we have handles h^k, $h^{k'}$ $k' \geq k$.

Suppose $h^{k'}$ is already attached. Its belt sphere has dimension $n-k'-1$.

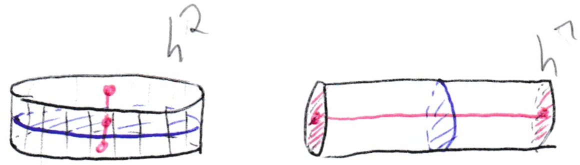
The attaching sphere of h^k has dimension $k-1$.

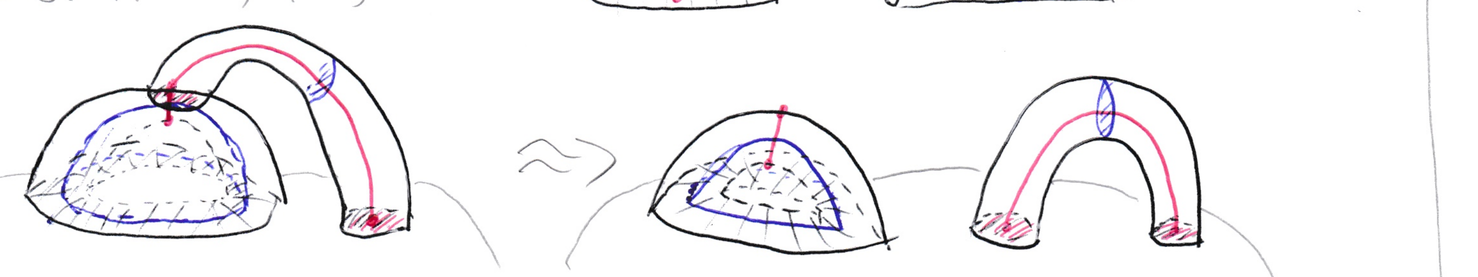
Notice

$$(n-k'-1)+(k-1) = n-(k'-k)-2 \leq n-2 < n-1.$$

$\therefore$ Transversality means we can isotope the attaching map so these spheres are disjoint, and from there we can make the attaching region of h^k disjoint from $h^{k'}$ altogether.

E.g. $n=3$, $k=1$, $k'=2$.

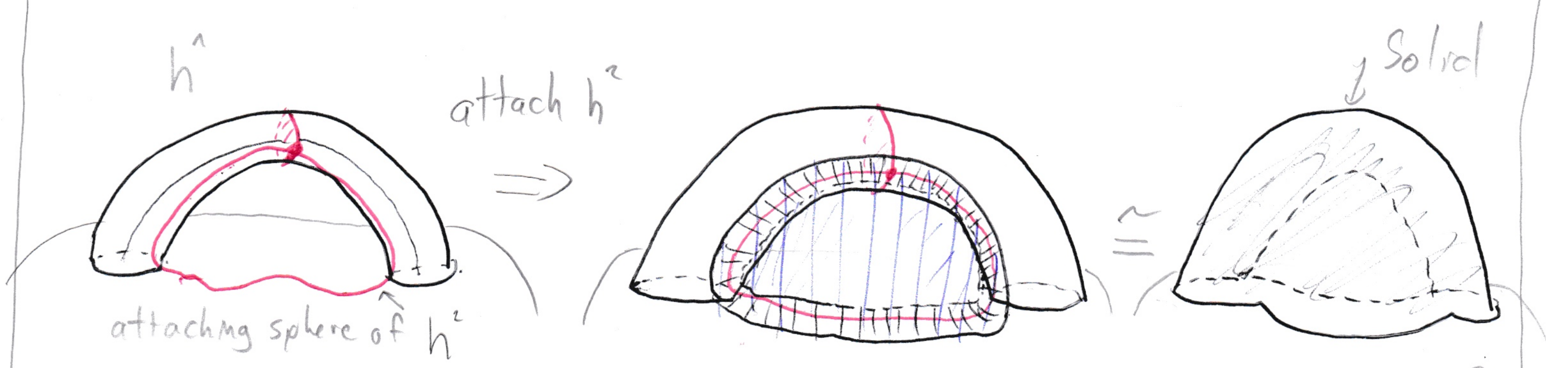




Handle Cancellation

Let h^{k-1}, h^k be $(k-1)$- and k-handles respectively, $1 \le k \le n$.
Suppose we attach h^{k-1} and then h^k to ∂Y so that the attaching sphere of h^k intersects the belt sphere of h^{k-1} transversely at 1 point.
Then $Y \cup h^{k-1} \cup h^k \cong Y$.

Pf

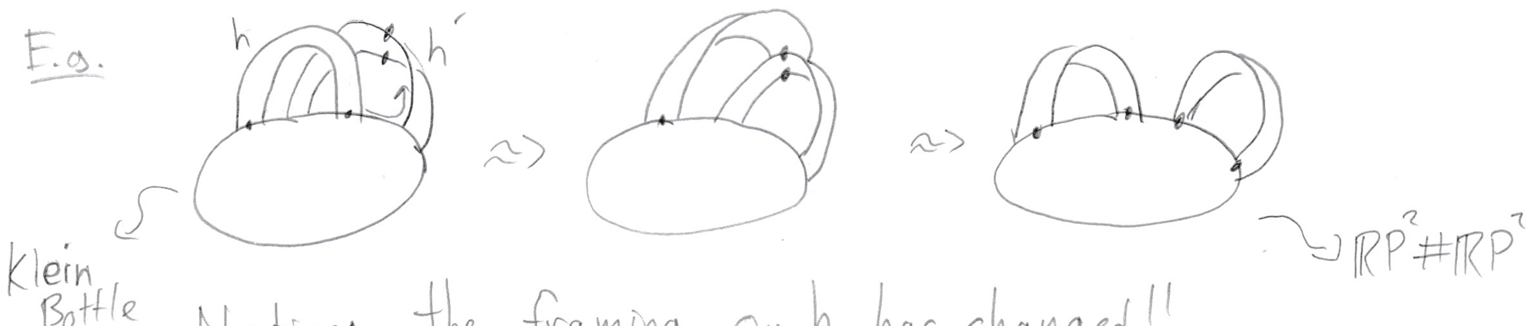


□

Def^n: Handle Slide

Say h, h' are two handles, disjointly attached to ∂Y.
Suppose they both have index k, $0<k<n$.
We handle slide h by isotoping its attaching sphere to pass through the belt sphere of h'.

E.g.



Notice: the framing on h has changed!!

Theorem (Cerf)

This is a sufficient set of moves.

That is, given two handle decompositions of $(X, \partial_- X)$, there is a sequence of

- handle slides
- handle pair creation/cancellation
- isotopies within levels

taking one decomposition to the other.

Continuing to simplify:

Proposition

If X^n is compact and connected, then

- If $\partial_- X = \emptyset$ X has a handle decomp with 1 0-handle
- If $\partial_- X \neq \emptyset$ " " " 0 0-handles.

Similarly

- If $\partial_+ X = \emptyset$, we can assume 1 n-handle
- If $\partial_+ X \neq \emptyset$ " " 0 n-handles

Proof: If $\partial_- X = \emptyset$ there must be at least one 0-handle.

If there are more than one, they must be connected by a 1-handle, since h^k $k>1$ has a connected attaching sphere.

But we can cancel one of the 0-handles with the 1-handle, and proceed by induction.

What about the "similarly" part?

Dual Handle Decompositions

Given a relative handlebody built on $\partial_- X$, we can dualize each handle $h^k \cong D^k \times D^{n-k} = D^k \times D^{n-k} \cong h_{n-k}$ and attach them in reverse order to $\overline{\partial_+ X}$.

The belt sphere of h^k becomes the attaching sphere of h_{n-k} and vice versa.

In terms of Morse functions, if our handle decomposition is induced by $f : X \to [0,1]$ then the dual is induced by $\tilde{f}(x) = 1 - f(x)$.

Now we can finish the proof of the proposition by applying the argument to the dual decomposition.

$\boxed{\text{Example } \mathbb{C}P^2}$

Standard charts $\psi_i : \mathbb{C}^2 \longrightarrow \mathbb{C}P^2$ position i

$(z_1, z_2) \longmapsto [z_1 : \ldots : 1 : \ldots : z_2]$

Fix $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$, and let $B_i = \psi_i(D \times D)$.

Idea: B_0, B_1, B_2 are the 0-, 2-, and 4-handle of our decomposition.

For $p \in \mathbb{C}P^2$, write $p = [z_0 : z_1 : z_2]$ where $\max_i |z_i| = 1$.

Then 1) $p \in B_i$ iff $|z_i| = 1$

2) $p \in \operatorname{int} B_i$ iff $|z_j| < 1$ for $j \neq i$.

i.e. They intersect along points where multples coordinates have norm 1.

If $p \in B_0 \cap B_1$, we can write

$p = [1 : w_1 : w_2] = [z_1 : 1 : z_2]$

$\Rightarrow w_1 = z_1^{-1}$ and $w_2 = z_1^{-1} z_2$

B_1 intersects B_0 at $\psi_1(\partial D \times D)$ where $\partial D \times D \cong S^1 \times D^2 \subset \partial D^3$

So we can view B_1 as a 2-handle attached to B_0 via

$\varphi : S^1 \times D^2 \hookrightarrow \partial B_0$

$(z_1, z_2) \longmapsto [1 : z_1^{-1} : z_1^{-1} z_2]$

Viewing $S^1 \times D^2$ as a disk bundle, φ preserves the fibres.

If we traverse the S^1 factor via $z_1 = e^{2\pi i t}$, then our disk coordinate is sent to $e^{-2\pi i t} z_2$.

$\rightsquigarrow$ this rotates the fibres once realizing a generator of $\pi_1(O(2)) \cong \mathbb{Z}$.

$\rightsquigarrow$ $\therefore$ $B_0 \cup B_1$ is a disc bundle over S^2 with Euler class $+1$.

The boundary is then the circle-bundle over S^2 with Euler class $+1$.

But attaching $B_2 \cong D^4 \times D^0$ results in a closed manifold, so we must have $\partial(B_0 \cup B_1) \cong S^3$.

$\rightsquigarrow$ Have a bundle $S^1 \hookrightarrow S^3 \twoheadrightarrow S^2$, the Hopf fibration.