The Yamabe invariant of Inoue surfaces and their blowups

A Dissertation Presented

by

Michael Albanese

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

August 2019

Stony Brook University

The Graduate School

Michael Albanese

We, the dissertation committee for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation.

Claude LeBrun
Professor of Mathematics

Dennis Sullivan
Distinguished Professor of Mathematics

H. Blaine Lawson, Jr.
Distinguished Professor of Mathematics

Luis Álvarez-Gaumé
Professor of Physics, Director of the Simons Center for Geometry and Physics

This dissertation is accepted by the Graduate School

Eric Wertheimer

Dean of the Graduate School

Abstract of the Dissertation

The Yamabe invariant of Inoue surfaces and their blowups

by

Michael Albanese

Doctor of Philosophy

in

Mathematics

Stony Brook University

2019

The Yamabe invariant of a closed smooth manifold is a real-valued diffeomorphism invariant coming from Riemannian geometry. Using Seiberg-Witten theory, LeBrun showed that the sign of the Yamabe invariant of a Kähler surface is determined by its Kodaira dimension, a complex-geometric invariant of the surface. It is not hard to see that the simplest non-Kähler surfaces, namely Hopf surfaces and their blowups, follow the pattern laid out by LeBrun's theorem. However, we will show that the non-Kähler analogue of LeBrun's theorem does not hold. In particular, we prove that the Yamabe invariants of Inoue surfaces and their blowups are all zero. This is achieved by developing a result which rules out the existence of positive scalar curvature metrics on a larger class of examples.

CONTENTS

ACKNOWLEDGEMENTS

First and foremost, I would like to thank my parents Maria and Jim, my brother Robert, and his wife Karina. Being so far from home has been incredibly difficult, but their continual love and support has made it possible for me to pursue my passion. When I was able to return home, the time spent with my family, including all my aunts, uncles, cousins, and my Nonno, filled me with happiness and served as a reminder that my PhD was not the most important part of who I am. When I could not make it home, my US family, namely Genoveffa, Maria, Anna, and Matthew, always made me feel welcome. I would also like to thank Tiara Miller for her support and encouragement during the final stretch of my PhD.

Mathematically, I am most indebted to my advisor Claude LeBrun. Not only has his help with this thesis been essential, he also helped me to be a better mathematician overall. I am so grateful that he not only put up with my many questions, he encouraged them. He continues to be an excellent model of the type of mathematician I aspire to be, although he is not the only one. Blaine Lawson has effectively been a second advisor to me. His encouragement and willingness to discuss mathematics with me has meant a lot. Similarly, I have had many enlightening discussions with Dennis Sullivan which have shaped the way I think about mathematics. There are many other professors at Stony Brook who have helped me throughout my time here, in particular Ljudmila Kamenova, Mark McLean, John Morgan, Jason Starr, and Aleksey Zinger. I'd also like to thank Demetre Kazaras for his help. I have been incredibly fortunate that he was hired as a postdoc when he was.

One of the reasons I decided to come to the US to do my PhD was the desire to discuss mathematics with a large peer group, and I am glad that came to fruition. I'd like to thank Seyed Ali Aleyasin, Jean-François Arbour, Frederik Benirschke, Jack Burkart, Nathan Chen, Xuemiao Chen, Jae Ho Cho, Charlie Cifarelli, Cameron Crowe, Aleksander Doan, Mohamed El Alami, Paul Frigge, Marlon de Oliveira Gomes, Lisandra Hernandez Vazquez, Jiahao Hu, Yoon-Joo Kim, Matthew Lam, Kirill Lazebnik, Lisa Marquand, James Mathews, Aleksandar Milivojevic, Cristian Minoccheri, Jordan Rainone, Nissim Ranade, Chandrika Sadanand, John Sheridan, Tobias Shin, Lloyd Smith, Yury Sobolev, David Stapleton, Yuhan Sun, Jiasheng Teh, Ying Hong Tham, Alexandra Viktorova, Ben Wu, and Mu Zhao.

I also learnt a lot from users of Mathematics Stack Exchange and MathOverflow. In particular, I'd like to thank Daniele Angella and Mike Miller who I have also had the pleasure to discuss mathematics with in person.

Finally, I'd like to thank Lynne Barnett, Christine Gathman, Donna McWilliams, Lucille Meci, Pat Tonra, and Diane Williams for all their administrative help within the department over the years. Also, thanks to Sharon Williams from the Math, Physics, Astronomy Library.

1. THE YAMABE INVARIANT

Let M be a closed, connected, smooth manifold of dimension $n \geq 2$. For a Riemannian metric g on M, we denote the Ricci curvature by Ric_g, the scalar curvature by s_g, and the Riemannian volume density by $d\mu_g$. Another Riemannian metric $\tilde{g}$ on M is *conformal* to g if $\tilde{g} = ug$ for some positive smooth function u; if u is constant, we say $\tilde{g}$ is *homothetic* to g. If $\tilde{g} = f^*g$ for some diffeomorphism $f : M \to M$, we say $\tilde{g}$ is *isometric* to g. The *conformal class* of g, denoted $[g]$, is the set of Riemannian metrics conformal to g, i.e. $[g] = \{ug \mid u \in C^\infty(M), u > 0\}$.

The Uniformisation Theorem states that every closed, connected surface admits a constant sectional curvature metric. One might hope that this result generalises to higher dimensions. The analogous statement is not true beyond dimension two because an n-dimensional manifold which admits a constant sectional curvature metric has universal cover diffeomorphic to $\mathbb{R}^n$ or S^n, see Theorem 11.12 of [67]. So, for example, the manifold $S^2 \times S^1$ does not admit a constant sectional curvature metric.

If one replaces the constant sectional curvature requirement with constant scalar curvature, which are equivalent requirements in dimension two, then the higher dimensional analogue does hold. In fact, we have a much stronger statement.

Theorem 1.1. *Let M be a closed smooth manifold. Every conformal class contains a constant scalar curvature metric.*

The task of establishing this result was known as the Yamabe problem due to the fact that Yamabe [111] had claimed to prove the statement, but a gap in the argument was later pointed out by Trudinger [108]. Subsequent work by Aubin [8] and Schoen [89] closed the gap.

In dimension two, constant sectional curvature is also equivalent to constant Ricci curvature. More precisely, a Riemannian metric g is said to have constant Ricci curvature if $\mathrm{Ric}_g = \lambda g$ for some constant $\lambda \in \mathbb{R}$; such a metric is more commonly called an *Einstein metric*. In dimension three, a metric is Einstein if and only if it has constant sectional curvature, see Proposition 1.120 of [13]. So, for example, the manifold $S^2 \times S^1$ does not admit an Einstein metric. In dimension four, there are topological obstructions to the existence of Einstein metrics such as the Hitchin-Thorpe inequality [41].

Theorem 1.2. *Let M be a closed orientable smooth four-manifold which admits an Einstein metric. Then $\chi(M) \geq \frac{3}{2}|\sigma(M)|$ with equality if and only if M is finitely covered by a torus or a K3 surface.*

So, for example, the manifolds $S^2 \times \Sigma_g$ do not admit Einstein metrics for $g > 0$. It is worth noting that the Hitchin-Thorpe inequality has been strengthened several times; see [34] page 87, [57], and [58]. Moreover, the Hitchin-Thorpe inequality is itself an improvement of an earlier result by Berger [12] which states that a closed smooth four-manifold M which admits an Einstein metric satisfies $\chi(M) \geq 0$ with equality if and only if M is flat.

In higher dimensions, it is not known if the existence of an Einstein metric has any topological implications, which naturally leads to the following question.

Question 1.3. Does every closed smooth manifold of dimension at least five admit an Einstein metric?

In the early twentieth century, Hilbert found a variational characterisation of Ricci-flat metrics in terms of the total scalar curvature. More generally, Einstein metrics can be characterised in terms of the normalised total scalar curvature, now known as the *Einstein-Hilbert functional*. Explicitly, the Einstein-Hilbert functional on the space of

Riemannian metrics on M is given by

$$\mathcal{E}(g) = \frac{\int_M s_g d\mu_g}{\mathrm{Vol}(M,g)^{\frac{n-2}{n}}}.$$

The exponent in the denominator has been chosen so that $\mathcal{E}(cg) = \mathcal{E}(g)$ for all constants $c > 0$, i.e. homothetic metrics have the same value. Moreover, for every diffeomorphism f, we have $\mathcal{E}(f^*g) = \mathcal{E}(g)$, i.e. isometric metrics have the same value; a functional with this property is called a *Riemannian functional.* Hilbert showed that for $n \geq 3$, a Riemannian metric is a critical point of $\mathcal{E}$ if and only if it is an Einstein metric, see Theorem 4.21 of [13]. Note that when $n = 2$, the scalar curvature is twice the sectional curvature which is itself the Gaussian curvature, so we have $\mathcal{E}(g) = \int_M s_g d\mu_g = 4\pi\chi(M)$ by the Gauss-Bonnet Theorem, so every metric is a critical point, even if it isn't Einstein.

One might hope to find global maxima and minima of $\mathcal{E}$ in the search for critical points, but in general, the functional is neither bounded above or below. If one is willing to restrict to a conformal class however, we obtain the following.

Proposition 1.4. *When restricted to a conformal class, the Einstein-Hilbert functional $\mathcal{E}$ is bounded below. More precisely, for any $\tilde{g} \in [g]$, we have*

$$\mathcal{E}(\tilde{g}) \geq -\left(\int_M |s_g|^{\frac{n}{2}} d\mu_g\right)^{\frac{2}{n}}.$$

Equality occurs if and only if g has constant non-positive scalar curvature and $\tilde{g}$ is homothetic to g.

Proof. The conformal Laplacian is $L_g u = 4\frac{n-1}{n-2}\Delta_g u + s_g u$ where $\Delta_g = d^*d + dd^*$ is the Laplace-Beltrami operator (with non-negative spectrum). If $\tilde{g} = u^{\frac{4}{n-2}}g$, then $s_{\tilde{g}} = u^{-\frac{n+2}{n-2}} L_g u$.

The total scalar curvature of $\tilde{g}$ is

$$\begin{aligned}
\int_M s_{\tilde{g}} d\mu_{\tilde{g}} &= \int_M u^{-\frac{n+2}{n-2}} \left(4\frac{n-1}{n-2}\Delta_g u + s_g u\right) u^{\frac{2n}{n-2}} d\mu_g \\
&= \int_M 4\frac{n-1}{n-2} u\Delta_g u + s_g u^2 d\mu_g \\
&= \int_M 4\frac{n-1}{n-2}|du|^2 + s_g u^2 d\mu_g \\
&\geq \int_M s_g u^2 d\mu_g \\
&\geq -\int_M |s_g u^2| d\mu_g \\
&\geq -\left(\int_M |s_g|^{\frac{n}{2}} d\mu_g\right)^{\frac{2}{n}} \left(\int_M u^{\frac{2n}{n-2}} d\mu_g\right)^{\frac{n-2}{n}}
\end{aligned}$$

where the final inequality is Hölder's inequality with $p = \frac{n}{2}$ and $q = \frac{n}{n-2}$. Note that $d\mu_{\tilde{g}} = u^{\frac{2n}{n-2}} d\mu_g$, so

$$\mathrm{Vol}(M, \tilde{g})^{\frac{n-2}{n}} = \left(\int_M u^{\frac{2n}{n-2}} d\mu_g\right)^{\frac{n-2}{n}},$$

and therefore

$$\mathcal{E}(\tilde{g}) = \frac{\int_M s_{\tilde{g}} d\mu_{\tilde{g}}}{\mathrm{Vol}(M, \tilde{g})^{\frac{n-2}{n}}} \geq \frac{-\left(\int_M |s_g|^{\frac{n}{2}} d\mu_g\right)^{\frac{2}{n}} \left(\int_M u^{\frac{2n}{n-2}} d\mu_g\right)^{\frac{n-2}{n}}}{\left(\int_M u^{\frac{2n}{n-2}} d\mu_g\right)^{\frac{n-2}{n}}} = -\left(\int_M |s_g|^{\frac{n}{2}} d\mu_g\right)^{\frac{2}{n}}.$$

The first inequality in the initial computation is an equality if and only if $du = 0$ (i.e. u is constant), while the second inequality is an equality if and only if $s_g \leq 0$. Equality occurs in Hölder's inequality if and only if the integrands are constant multiples of one another, so if u is constant, we see that s_g is constant. □

In view of this proposition, we define the *Yamabe constant* of a conformal class $\mathcal{C}$ to be

$$Y(M, \mathcal{C}) = \inf_{g \in \mathcal{C}} \mathcal{E}(g).$$

Note that the critical points of the Einstein-Hilbert functional restricted to a conformal class need not be critical points for the Einstein-Hilbert functional itself. In fact, for $n \geq 3$, a Riemannian metric is a critical point of the restricted functional if and only if it has constant scalar curvature, see Proposition 4.25 of [8]. The aforementioned Yamabe problem was solved by showing that the above infimum is actually attained, i.e. the Einstein-Hilbert functional has a minimum when restricted to $[g]$. Such metrics are called *Yamabe minimisers.* The following proposition shows that if $Y(M,[g]) \leq 0$ and $n \geq 3$, Yamabe minimisers are effectively unique.

Proposition 1.5. *Let M be a closed, connected manifold with a conformal class $\mathcal{C}$, and suppose $g, \tilde{g} \in \mathcal{C}$ are constant scalar curvature metrics. If $Y(M,\mathcal{C}) \leq 0$, then $\tilde{g}$ is homothetic to g. In particular, there is a unique unit-volume constant scalar curvature metric in $\mathcal{C}$.*

Proof. Suppose that $\tilde{g} = e^{2\varphi}g$. Then

$$s_{\tilde{g}} = e^{-2\varphi}(s_g + 2(n-1)\Delta_g\varphi - (n-1)(n-2)\|\nabla\varphi\|_g^2).$$

where $\Delta_g = d^*d + dd^*$ is the Laplace-Beltrami operator.

Suppose $s_g < 0$. If p is a global maximum of φ, then $(\nabla\varphi)(p) = 0$ and $(\Delta_g\varphi)(p) \geq 0$ so $s_{\tilde{g}} \geq e^{-2\varphi(p)}s_g$. If q is a global minimum of φ, then $(\nabla\varphi)(q) = 0$ and $(\Delta_g\varphi)(q) \leq 0$ so $s_{\tilde{g}} \leq e^{-2\varphi(q)}s_g$. As $e^{-2\varphi(p)}s_g \leq e^{-2\varphi(q)}s_g$ and $s_g < 0$, it follows that $\varphi(p) \leq \varphi(q)$ and hence φ is constant so $\tilde{g}$ is homothetic to g.

If $s_g = 0$, then the above argument shows that $s_{\tilde{g}} = 0$ and hence $\Delta_g\varphi = \frac{1}{2}(n-2)\|\nabla\varphi\|^2 \geq 0$, so φ is subharmonic. By the maximum principle, the function φ is constant, so $\tilde{g}$ is homothetic to g. □

One might suspect that the above proposition follows from the equality case of Proposition 1.4, together with the existence of Yamabe minimisers. This is not quite the case.

If one took g to be a Yamabe minimiser and assumed $\mathcal{E}(\tilde{g}) = \mathcal{E}(g)$, then the conclusion would follow, but a priori, we could have $\mathcal{E}(\tilde{g}) > \mathcal{E}(g)$.

A natural question to ask is whether the above proposition also holds for conformal classes $\mathcal{C}$ for which $Y(M,\mathcal{C}) > 0$. As the following example demonstrates, it doesn't in general.

Example 1.6. Note that $Y(S^2, [g_{\text{round}}]) = 4\pi > 0$ where g_{round} denotes the round metric. If $f : S^2 \to S^2$ is a conformal diffeomorphism, then $f^*g = ug$ for some smooth positive function u, and f^*g has constant scalar curvature as $s_{f^*g} = s_g \circ f = s_g$. If u were constant, we would have $s_{f^*g} = s_{ug} = u^{-1}s_g$. Therefore, if f^*g is homothetic to g, we must have $u = 1$, i.e. f is an isometry of (S^2, g). So for any conformal diffeomorphism which is not an isometry, f^*g is a constant scalar curvature metric which is not homothetic to g. To see that such maps f exist, first note that an orientation-preserving conformal diffeomorphism is precisely a biholomorphism of $\mathbb{CP}^1$, so the conformal diffeomorphism group is an extension of $PGL(2,\mathbb{C})$ by $\mathbb{Z}_2$ while the isometry group of g_{round} is $O(3)$. As the isometry group is three-dimensional and the conformal diffeomorphism group is six-dimensional, there are many conformal diffeomorphisms which are not isometries.

The conformal classes $\mathcal{C}$ for which $Y(M,\mathcal{C}) > 0$ have the following useful characterisation.

Proposition 1.7. *The Yamabe constant of a conformal class $\mathcal{C}$ is positive if and only if $\mathcal{C}$ contains a positive scalar curvature metric.*

This proposition can be established without resorting to the existence of Yamabe minimisers, but for simplicity, we will do exactly that.

Proof. One direction is fairly straightforward. Let $\tilde{g}$ be a unit-volume Yamabe minimiser of $\mathcal{C}$, so that $Y(M,\mathcal{C}) = \mathcal{E}(\tilde{g}) = \int_M s_{\tilde{g}} d\mu_{\tilde{g}}$. As $\tilde{g}$ has constant scalar curvature, we see that $\int_M s_{\tilde{g}} d\mu_{\tilde{g}} = s_{\tilde{g}} \operatorname{Vol}(M,\tilde{g}) = s_{\tilde{g}}$ and hence $s_{\tilde{g}} = Y(M,\mathcal{C}) > 0$, so $\tilde{g}$ is a positive scalar curvature metric.

For the opposite direction, suppose $\mathcal{C}$ contains a positive scalar curvature metric g. Let $\tilde{g}$ be a Yamabe minimiser of $\mathcal{C}$, then $\tilde{g} = e^{2\varphi}g$ for some function φ and $s_{\tilde{g}} = e^{-2\varphi}(s_g + 2(n-1)\Delta_g\varphi - (n-1)(n-2)\|\nabla\varphi\|^2)$. If p is a local maximum of φ, then $(\nabla\varphi)(p) = 0$ and $(\Delta_g\varphi)(p) \geq 0$ so $s_{\tilde{g}}(p) = e^{-2\varphi(p)}(s_g(p) + 2(n-1)(\Delta_g u)(p)) \geq e^{-2\varphi(p)}s_g(p) > 0$. As $s_{\tilde{g}}$ is constant, we see that $s_{\tilde{g}} = s_{\tilde{g}}(p) > 0$, so $\tilde{g}$ is a positive scalar curvature metric. $\square$

Now we see that Example 1.6 generalises to higher dimensions. More precisely, as g_{round}, the round metric on S^n, has positive scalar curvature, we have $Y(S^n, [g_{\text{round}}]) > 0$ by Proposition 1.7. The conformal diffeomorphism group of $(S^n, [g_{\text{round}}])$ is $O(n+1,1)/\{\pm I\}$ which has dimension $\binom{n+2}{2}$, while the isometry group of g_{round} is $O(n+1)$ which has dimension $\binom{n+1}{2}$. As $\binom{n+1}{2} < \binom{n+2}{2}$, there are many conformal diffeomorphisms which are not isometries of g_{round}. For such a conformal diffeomorphism f, the metric f^*g_{round} has constant scalar curvature but is not homothetic to g_{round}.

Note that f^*g_{round} is not only a constant scalar curvature metric, it is also Einstein (in fact, constant sectional curvature). So f^*g_{round} and g_{round} are conformal Einstein metrics which are not homothetic if f is a conformal diffeomorphism which is not an isometry. It follows from the next result that this is the only case where such metrics can exist.

Theorem 1.8. (Obata [79]) *Let M be a compact, connected n-dimensional manifold with an Einstein metric g, and let $\tilde{g} \in [g]$ be a constant scalar curvature metric. If $(M, [g])$ is not conformally diffeomorphic to $(S^n, [g_{round}])$, then $\tilde{g}$ is homothetic to g. In particular, there is a unique unit-volume constant scalar curvature metric in $[g]$.*

If $Y(M, [g]) \leq 0$, the above statement follows from Proposition 1.5. For $Y(M, [g]) > 0$, see Proposition 6.2 of [79].

In the same paper, Obata shows that the constant scalar curvature metrics on S^n

constructed above are the only such metrics conformal to g_{round}, see Proposition 6.1 of [79]. That is, every constant scalar curvature metric $g \in [g_{\text{round}}]$ is of the form f^*g_{round} for some diffeomorphism $f : S^n \to S^n$, i.e. g is isometric to g_{round}. In particular, although there is not a unique unit-volume constant scalar curvature metric in $[g_{\text{round}}]$, they all have the same value of scalar curvature and are therefore Yamabe minimisers.

Both Proposition 1.5 and Theorem 1.8 allow one to compute the Yamabe constant of certain conformal classes. More precisely, if g is a constant non-positive scalar curvature metric or an Einstein metric, then g is a Yamabe minimiser so $Y(M,[g]) = \mathcal{E}(g)$. The following lemma gives an upper bound on the possible value of the Yamabe constant of every conformal class.

Lemma 1.9. (Aubin [8]) *Let M be a closed smooth n-dimensional manifold. Then for any conformal class $[g]$, the Yamabe constant of $\mathcal{C}$ satisfies $Y(M,\mathcal{C}) \leq Y(S^n,[g_{round}])$ with equality if and only if $(M,\mathcal{C})$ is conformally diffeomorphic to $(S^n,[g_{round}])$.*

For many conformal classes, namely those in the purview of Proposition 1.5 and Theorem 1.8, there is a unique unit-volume Yamabe minimiser. Anderson has shown that this is also true for a generic conformal class, see Theorem 1.1 of [5]. Now suppose $\mathcal{C}$ is a conformal class with $Y(M,\mathcal{C}) > 0$. Even if $\mathcal{C}$ has a unique unit-volume Yamabe minimiser g, it need not have a unique unit-volume constant scalar curvature metric. That is, there could be a unit-volume constant scalar curvature metric $\tilde{g}$ with $s_{\tilde{g}} \neq s_g$ which is equivalent to $\mathcal{E}(\tilde{g}) > \mathcal{E}(g)$. This descrepancy makes the computation of $Y(M,\mathcal{C})$ more difficult in the positive case.

Example 1.10. Let M be a closed m-dimensional smooth manifold which admits a constant positive scalar curvature metric[1] g_M and let N be a closed n-dimensional smooth manifold which admits a scalar-flat metric, i.e. a metric with constant zero

[1] This is equivalent to requiring M to admit a positive scalar curvature metric. To see this, note that if g'_M has positive scalar curvature, then $Y(M,[g'_M]) > 0$ by Proposition 1.7, so if g_M is a Yamabe minimiser for $[g'_M]$, then g_M is a constant positive scalar curvature metric.

scalar curvature. Consider the family of metrics $h_t = g_M + tg_N$ on $M \times N$. Note that $s_{h_t} = s_{g_M} + s_{tg_N} = s_{g_M} + t^{-1}s_{g_N} = s_{g_M}$ and $\mathrm{Vol}(M \times N, h_t) = \mathrm{Vol}(M, g_M)\,\mathrm{Vol}(N, tg_N) = t^{n/2}\,\mathrm{Vol}(N, g_N) = t^{n/2}$. So for every $t > 0$, the metric g_t has constant scalar curvature and satisfies $\mathcal{E}(g_t) = s_{g_1}t^{-n/2}$. For small enough t, we see that $\mathcal{E}(h_t) \geq Y(S^{n+m}, [g_{\mathrm{round}}])$ so by Lemma 1.9, the constant scalar curvature metrics h_t cannot be Yamabe minimisers for their conformal class. In particular, we must have $\mathcal{E}(h_t) > \mathcal{E}(h)$ where h is a Yamabe minimiser for $[h_t]$.

For conformal classes where unit-volume constant scalar curvature metrics are not unique, one could ask how many possible scalar curvature values can occur. It follows from section 2 of [90] that for any N, there are conformal classes on $S^n \times S^1$ for which the number of values is at least N; also see [87] for a more general statement. For $n \geq 25$, there exist conformal classes on S^n for which the number of such values is infinite, see [19] and [20].

Thanks to Lemma 1.9, the following quantity is finite:

$$Y(M) = \sup_{[g]} Y(M, [g]).$$

This is known as the *Yamabe invariant* of M. It was defined independently by Kobayashi [55], who denoted it by μ, and Schoen [90], who denoted it by σ. The following result is immediate by Proposition 1.7.

Proposition 1.11. *The Yamabe invariant of M is positive if and only if M admits a positive scalar curvature metric.*

The Yamabe invariant is a diffeomorphism invariant, but it is not a homeomorphism invariant, see Example 3.27 and Example 4.2. So it can be used to distinguish smooth structures on a topological manifold. It can be viewed as a refinement of the $\mathbb{Z}_2$-valued invariant which detects whether or not a manifold admits a positive scalar curvature metric. In particular, it can be used to distinguish smooth structures when neither one admits positive scalar curvature metrics, see Example 4.2.

Given the min-max definition of the Yamabe invariant, one might hope that if M admits a Yamabe minimiser g realising the Yamabe invariant, that is $Y(M,[g]) = Y(M)$, then g is a saddle point of the Einstein-Hilbert functional and hence an Einstein metric; note that Einstein metrics are always saddle points, see page 9 of [109]. If $Y(M) \leq 0$, this is true, see Lemma 1.9 of [55]. If $Y(M) > 0$, it is unknown in general. One case where it is known to be true is when $\frac{s_g}{n-1}$ is not an eigenvalue of the Laplace-Beltrami operator Δ_g, see Proposition 4.47 of [13]. By Theorem 1.8, if g is an Einstein metric and $(M,[g])$ is not conformally diffeomorphic to $(S^n,[g_{\mathrm{round}}])$, then any constant scalar curvature metric is homothetic to g. In the converse direction, if $Y(M,[g]) = Y(M)$ and every constant scalar curvature metric in $[g]$ is homothetic to g, then g is Einstein, see Lemma 1.4 of [71].

1.1 Values of the Yamabe Invariant

Although the Yamabe invariant is a useful diffeomorphism invariant, it is notoriously difficult to compute, especially if the manifold admits positive scalar curvature metrics due to the potential existence of constant scalar curvature metrics which are not Yamabe minimisers as demonstrated in Example 1.10.

It follows from Obata's results that an Einstein metric g is a Yamabe minimiser, so $Y(M,[g]) = \mathcal{E}(g)$. In particular, if g is a unit-volume Einstein metric with $\mathrm{Ric}_g = \lambda g$, then $\mathcal{E}(g) = \lambda$. This allows us to compute the Yamabe invariant of S^n:

$$Y(S^n) = Y(S^n,[g_{\mathrm{round}}]) = \mathcal{E}(g_{\mathrm{round}}) = n(n-1)\,\mathrm{Vol}(S^n, g_{\mathrm{round}})^{\frac{2}{n}}.$$

It follows from Lemma 1.9 that $Y(M) \leq Y(S^n)$ but unlike the case for the Yamabe constant of a conformal class, equality can occur without M and S^n being diffeomorphic. For example, Schoen proved $Y(S^{n-1} \times S^1) = Y(S^n)$, see pages 132-135 of [90]. More generally, Kobayashi showed that $Y(M) = Y(S^n)$ for any mapping torus of S^{n-1}, see Theorem 2 (b) of [55].

In a fixed dimension, the value of the Yamabe invariant is bounded above by the Yamabe invariant of the corresponding sphere. However, if we consider manifolds of varying dimension, there is no upper bound for the Yamabe invariant.

Proposition 1.12. *The values of the Yamabe invariant are not bounded above. In particular* $\lim_{k\to\infty} Y(S^{2k}) = \infty$.

Proof. We have

$$Y(S^n) = n(n-1)\,\mathrm{Vol}(S^n, g_{\mathrm{round}})^{\frac{2}{n}} = n(n-1)\left(\frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}\right)^{\frac{2}{n}}.$$

So for $n = 2k$ we have

$$Y(S^{2k}) = 2k(2k-1)\left(\frac{2\pi^k\sqrt{\pi}}{\Gamma(k+\frac{1}{2})}\right)^{\frac{1}{k}}.$$

By 6.1.8 and 6.1.12 of [2], we have

$$\Gamma\left(k+\frac{1}{2}\right) = \frac{(2k-1)!!}{2^k}\sqrt{\pi}.$$

As $(2k-1)!! < (2k)!! < (2k)^k < 2(2k)^k$, we see that

$$\Gamma\left(k+\frac{1}{2}\right) < \frac{2(2k)^k}{2^k}\sqrt{\pi} = 2k^k\sqrt{\pi}.$$

Therefore

$$Y(S^{2k}) > 2k(2k-1)\left(\frac{2\pi^k\sqrt{\pi}}{2k^k\sqrt{\pi}}\right)^{\frac{1}{k}} = 2k(2k-1)\left(\frac{\pi^k}{k^k}\right)^{\frac{1}{k}} = 2k(2k-1)\frac{\pi}{k} = (4k-2)\pi$$

and hence $\lim_{k\to\infty} Y(S^{2k}) = \infty$. □

For a surface of genus g we have $Y(\Sigma_g) = 4\pi\chi(\Sigma_g) = 8\pi(1-g)$, so there is also no lower bound on the Yamabe invariant.

One might wonder whether every real number arises as the Yamabe invariant of some closed smooth manifold – this is far from true. The following theorem implies that only countably many real numbers arise.

Theorem 1.13. *There are countably many closed smooth manifolds up to diffeomorphism.*

Every closed smooth manifold admits a compatible piecewise linear structure, this is due to Cairns [22] and Whitehead [110]. Moreover, a PL manifold only admits finitely many possible smoothings, see page 221 of [93]. As closed PL manifolds are in particular finite simplicial complexes, there are only countably many. Combining these facts, Theorem 1.13 follows. Alternatively, the result also follows from Cheeger's Finiteness Theorem, see [25] and [86].

An approach to produce more values of the Yamabe invariant is to take the connected sum of two manifolds M_1 and M_2 whose Yamabe invariants are known, with the hope that $Y(M_1\#M_2)$ can be computed in terms of $Y(M_1)$ and $Y(M_2)$. In general, there is no formula relating $Y(M_1\#M_2)$ with $Y(M_1)$ and $Y(M_2)$, however, there is a lower bound for $Y(M_1\#M_2)$.

Theorem 1.14. (Kobayashi [55]) *If M_1 and M_2 are compact connected manifolds of* *dimension $n \geq 3$, then*

$$Y(M_1 \# M_2) \geq \begin{cases} -(|Y(M_1)|^{n/2} + |Y(M_2)|^{n/2})^{2/n} & \text{if } Y(M_1), Y(M_2) \leq 0 \\ \min\{Y(M_1), Y(M_2)\} & \text{otherwise.} \end{cases}$$

As we have seen previously, metrics with special curvature properties (namely constant non-positive scalar curvature and Einstein) can be used to compute the value of the Yamabe constant of a conformal class. One might hope that a metric with even more restrictive curvature properties could be used to compute the Yamabe invariant directly.

Conjecture 1.15. *Let M be a closed smooth manifold which admits a metric g of constant sectional curvature. Then $Y(M) = \mathcal{E}(g)$.*

As a metric of constant sectional curvature is Einstein, we have $Y(M) \geq Y(M, [g]) = \mathcal{E}(g)$ by Theorem 1.8.

If $\pi : M' \to M$ is a k-sheeted covering, and g is a Riemannian metric on M, then

$$\mathcal{E}(\pi^* g) = \frac{\int_{M'} s_{\pi^* g} d\mu_{\pi^* g}}{(\int_{M'} d\mu_{\pi^* g})^{\frac{n-2}{n}}} = \frac{\int_{M'} \pi^*(s_g d\mu_g)}{(\int_{M'} \pi^* d\mu_g)^{\frac{n-2}{n}}} = \frac{k \int_M s_g d\mu_g}{(k \int_M d\mu_g)^{\frac{n-2}{n}}} = k^{2/n} \mathcal{E}(g).$$

So, if G is a finite group of order k acting by isometries on S^n, the above conjecture asserts that $Y(S^n/G) = k^{-2/n} Y(S^n)$. The only case with $k > 1$ which has been verified is that of $\mathbb{RP}^3$ which was computed by Bray & Neves [18]. Using the inverse mean curvature flow, they were able to show that $Y(\mathbb{RP}^2 \times S^1) = Y(\mathbb{RP}^3) = 2^{-2/3} Y(S^3) = 6\pi^{4/3}$.

Using the above computation, if $\mathcal{E}(g) > 0$ and $k > 1$, then $\mathcal{E}(\pi^* g) > \mathcal{E}(g)$. More generally, Aubin's Lemma (see Lemma 2 and Theorem 8 of [9], or Lemma 3.6 of [4]) states that if $Y(M, [g]) > 0$ and $k > 1$, then $Y(M', [\pi^* g]) > Y(M, [g])$; if $Y(M, [g]) \leq 0$, then Propostion 1.5 shows that $Y(M', [\pi^* g]) = k^{2/n} Y(M, [g])$. Akutagawa & Neves [4] used Aubin's Lemma to show that for any non-negative integers k, ℓ, m, n, we have

$Y(k(\mathbb{RP}^3)\#\ell(\mathbb{RP}^2\times S^1)\#m(S^2\times S^1)\#n(S^2\tilde{\times}S^1)) = Y(\mathbb{RP}^3)$ provided $k+\ell\geq 1$; here $S^2\tilde{\times}S^1$ denotes the non-orientable S^2-bundle over S^1. If $k+\ell=0$, that is $k=\ell=0$, it follows from Theorem 1.14 that $Y(m(S^2\times S^1)\#n(S^2\tilde{\times}S^1)) = Y(S^3)$ as $Y(S^2\times S^1) = Y(S^2\tilde{\times}S^1) = Y(S^3)$.

Although very little progress has been made on Conjecture 1.15 in the positive case, it has been solved for contant sectional curvature zero manifolds, i.e. flat manifolds. This follows immediately once one knows that such manifolds do not admit positive scalar curvature metrics; see section 3.2.

In the hyperbolic case, that is constant negative sectional curvature, Conjecture 1.15 has been resolved in dimension three. In [6], Anderson shows that Perelman's solution of Thurston's Geometrization Conjecture [82], [83], [84] can be used to show that for a closed hyperbolic three-manifold M, we have $Y(M) = -6\,\mathrm{Vol}(M,g_{-1})^{2/3} = \mathcal{E}(g_{-1})$ where g_{-1} is the metric of constant sectional curvature -1. More generally, if M is a three-manifold with $Y(M)\leq 0$, see Theorem 3.24 for when this occurs, we have $Y(M) = -6\,\mathrm{Vol}(H,g_{-1})^{2/3}$ where H is the hyperbolic part of M (with respect to the splitting given by the geometrization conjecture).

The Yamabe invariant of surfaces is easily computed thanks to the Gauss-Bonnet Theorem. For three-manifolds, the above collection of results shows that much is known, although far from everything. In dimension four, the existence of Seiberg-Witten theory and its implications regarding positive scalar curvature metrics is a special feature, see section 3.3 for more details. LeBrun in [65] used Seiberg-Witten theory to compute the value of the Yamabe invariant for most Kähler surfaces, and in [64], he showed that $Y(\mathbb{CP}^2) = 12\sqrt{2}\pi$; see chapter 4 for a more in-depth discussion. Similar techniques were also used to compute many other examples, see [100] and [101].

In dimensions five and above, very little is known. For example, the only values of the Yamabe invariant that have been computed are 0 and $Y(S^n)$. In the simply connected case, we have the following.

Theorem 1.16. (Petean [85]) *Let M be a closed simply connected manifold with $\dim M \geq 5$. Then $Y(M) \geq 0$.*

Note that the above statement is also true of simply connected manifolds in dimensions two and three, but it is false in dimension four, see Example 4.2.

As we have already seen, there is much we don't know about the Yamabe invariant. We end this chapter with a few questions which, as far as I am aware, remain unanswered.

Questions 1.17.

- Is there a description of those manifolds M with $Y(M) = Y(S^n)$?
- Are there two homeomorphic smooth manifolds M_1 and M_2 such that $Y(M_1)$ and $Y(M_2)$ are positive but not equal?
- Is there a countable collection of smooth manifolds $\{M_i\}$ such that they are all homeomorphic but they all have different Yamabe invariants? If such an example exists, it must be four-dimensional as lower-dimensional manifolds have a unique smooth structure by Radó [88] and Moise [74], while higher-dimensional manifolds only admit finitely many smooth structures, see page 221 of [93].
- Are there infinitely many values of the Yamabe invariant in each dimension?
- Are there only finitely many positive values of the Yamabe invariant in each dimension?

2. COMPLEX SURFACES

We begin by recalling some definitions.

Let X be a compact, connected, complex manifold and denote its canonical bundle by K_X. For $d \geq 0$, the d^{th} *plurigenus* of X is $P_d(X) := \dim H^0(X, K_X^d)$. The *Kodaira dimension* of X, denoted $\kappa(X)$, is defined to be $-\infty$ if all the plurigenera are zero, otherwise we set

$$\kappa(X) = \limsup \frac{\log P_d}{\log d}.$$

The Kodaira dimension satisfies $\kappa(X) \in \{-\infty, 0, 1, \ldots, \dim X\}$ and $\kappa(X \times Y) = \kappa(X) + \kappa(Y)$, see Theorem 10.8 and Proposition 10.8 of [46]. A complex manifold with $\kappa(X) = \dim X$ is said to be of *general type*.

If $Z \subset X$ is a complex submanifold of codimension k, then the *blowup* of X along Z is a complex manifold $\tilde{X}$ equipped with a map $\pi : \tilde{X} \to X$ such that $\pi|_{\pi^{-1}(X \setminus Z)}$ is a biholomorphism, and $E := \pi^{-1}(Z)$ is a complex submanifold of codimension 1. Moreover, $\pi|_E : E \to Z$ is a $\mathbb{CP}^{k-1}$-bundle over Z, namely the projectivisation of the normal bundle of Z in X. When Z is a point, the blowup is orientedly diffeomorphic to $X \# \overline{\mathbb{CP}^n}$, see Proposition 2.5.8 of [44]. The plurigenera are invariant under blowups, and hence so is the Kodaira dimension.

We say that a compact complex manifold X is *projective* if there is an embedding $X \to \mathbb{CP}^N$ for some N.

2.1 Kodaira-Enriques Classification

Let us now restrict our attention to dimension two, that is, complex surfaces.

We say that X is *minimal* if X cannot be obtained as the blowup of another complex surface. If Y can be obtained from a minimal complex surface X by a series of blowups, we say that X is a *minimal model* for Y. As b_2 increases by one after a blowup, it follows that every surface has a minimal model. If $\kappa(Y) \geq 0$, then Y has a unique minimal model, see Proposition III.4.6 of [10]. The same is true if $\kappa(Y) = -\infty$ and $b_1(Y)$ is odd, see page 262 of [10]. There are cases where the minimal model is not unique. For example, both $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2$ are minimal models for the blowup of $\mathbb{CP}^2$ at two points.

We now state the *Kodaira-Enriques classification* of complex surfaces, see Theorem blah of [10] for example.

Theorem 2.1. *Let Y be a complex surface. Then Y has a minimal model which belongs to exactly one of the following ten classes: rational, ruled, class VII, K3, Enriques, two-dimensional tori, hyperelliptic, Kodaira, properly elliptic, general type.*

A surface is *rational* if it projective and birational to $\mathbb{CP}^2$. Aside from $\mathbb{CP}^2$ itself, the only other minimal rational surfaces are the Hirzebruch surfaces $\Sigma_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$ where $n = 0$ or $n \geq 2$, see Proposition VI.3.3 of [10]. We exclude $n = 1$ as Σ_1 is biholomorphic to the blowup of $\mathbb{CP}^2$ at a point, which is not minimal.

A surface is *ruled* if it is the total space of a holomorphic $\mathbb{CP}^1$-bundle over a curve of positive genus with structure group $PGL(2, \mathbb{C})$. Every ruled surface is the projectivisation of a rank two holomorphic vector bundle, see Proposition V.4.1 of [10]. It follows that ruled surfaces are projective.

A surface is said to be of *class VII*, or is a *class VII surface*, if it has Kodaira dimension $-\infty$ and $b_1 = 1$. These surfaces will be discussed more in the next section.

A *K3 surface* is a compact complex surface with trivial canonical bundle and $b_1 = 0$. It follows from the adjunction formula and the Lefschetz Hyperplane Theorem that a smooth quartic in $\mathbb{CP}^3$ is an example of a K3 surface, moreover it is simply connected. Any two K3 surfaces are deformation equivalent, see Theorem 7.1.1 of [45], so there is only one diffeomorphism type and it is simply connected. A generic K3 surface is not projective.

An *Enriques surface* is a compact complex surface X with $b_1(X) = 0$ such that $K_X \otimes K_X$ is holomorphically trivial, but K_X is not. Every Enriques surface is double covered by a K3 surface, see Proposition VIII.17 of [11]. All Enriques surfaces are projective and any two are deformation equivalent, see chapter V, section 23 and Theorem VIII.18.5 of [10] respectively. Therefore there is only one diffeomorphism type and $\pi_1(X) \cong \mathbb{Z}_2$.

A *torus* is a complex manifold of the form $\mathbb{C}^n/\Lambda$ where Λ is a lattice in $\mathbb{C}^n$, i.e. a free abelian subgroup of rank $2n$. For $n > 1$, a generic torus is not projective, see pages 36-37 of [77].

A *hyperelliptic surface* (also known as a *bi-elliptic surface*) is a surface with $b_1 = 2$ which admits a holomorphic submersion over an elliptic curve with an elliptic curve as typical fibre. Every hyperelliptic surface is a quotient of a product of elliptic curves by a free action of $\mathbb{Z}_2$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, or $\mathbb{Z}_6$, see pages 36-37 of [17]. It follows that every hyperelliptic surface is projective, see Theorem IV.6.8 of [10].

A *Kodaira surface* is a complex surface of Kodaira dimension 0 with odd b_1. A *primary Kodaira surface* is a Kodaira surface with trivial canonical bundle, while a *secondary Kodaira surface* is a finite quotient of a primary Kodaira surface.

An *elliptic surface* is a surface which admits a holomorphic submersion to a Riemann surface with an elliptic curve as a typical fibre. Just by taking the product of an elliptic curve with other Riemann surfaces, it is easy to see that elliptic surfaces can have Kodaira dimension $-\infty$, 0, and 1, although less trivial examples also exist, e.g. hyperelliptic surfaces. No elliptic surface can have Kodaira dimension 2, see Proposition V.12.5 of [10]. A *properly elliptic surface* is an elliptic surface which has Kodaira dimension one.

A surface is said to be of *general type* if it has Kodaira dimension 2. It follows from Corollary IV.6.5 of [10] that they are projective.

The above classes can be organised in the following table:

$\kappa(X)$	$b_1(X)$ even	$b_1(X)$ odd
$-\infty$	Rational, Ruled	Class VII
0	K3, Enriques, Tori, Hyperelliptic	Kodaira Surfaces
1	Properly Elliptic Surfaces	Properly Elliptic Surfaces
2	General Type	

The reason for introducing the parity of $b_1(X)$ as a distinguishing factor in the above table is the following theorem.

Theorem 2.2. *Let X be a compact, connected, complex surface. Then X admits a Kähler metric if and only if $b_1(X)$ is even.*

This was initially conjectured by Kodaira, see page 85 of [76]. The necessity of $b_1(X)$ even follows from the Hodge decomposition, while sufficiency is much more difficult. As rational, ruled, Enriques, hyperelliptic, and general type surfaces are all projective, the Kodaira-Enriques classification tells us that only three cases remain: K3 surfaces, tori, and properly elliptic surfaces (with b_1 even). Tori of every dimension admit Kähler metrics, for example, the Euclidean metric on $\mathbb{C}^n$ is invariant under translations and hence descends to any torus. The case of properly elliptic surfaces was proved independently by Miyaoka [73], and Harvey and Lawson [39]. The final case of K3 surfaces was dealt with by Siu [96]. A proof of Kodaira's conjecture which does not rely on the classification was later given by Buchdahl [21] and Lamari [59] independently.

2.2 *Class VII Surfaces*

Of all the classes in the Kodaira-Enriques classification, class VII surfaces are the least understood. Below we give an exposition of what is currently known.

Let X be a class VII surface. As $b_1(X) = 1$, we have $h^{1,0}(X) = 0$ and $h^{0,1}(X) = 1$, see Theorem 2.7 of [10]. As $\kappa(X) = -\infty$, we know that $h^{2,0}(X) = \dim H^0(X, K_X) = 0$ and hence $h^{0,2}(X) = 0$ by Serre duality. Moreover, we have $b^+(X) = 2h^{2,0}(X) = 0$ and hence $b^-(X) = b_2(X)$, see Theorem 2.7 of [10].

2.2.1 *Hopf Surfaces*

A *Hopf manifold* is a complex manifold X of dimension $n \geq 2$ whose universal cover is biholomorphic to $\mathbb{C}^n \setminus \{0\}$. If $\pi_1(X) \cong \mathbb{Z}$, then X is called a *primary Hopf manifold*, otherwise it is called a *secondary Hopf manifold*. Sometimes a primary Hopf manifold is defined to be a quotient of $\mathbb{C}^n \setminus \{0\}$ by the infinite cyclic group generated by a contraction. These two definitions coincide, as shown by Kodaira in section of 10 of [56]; moreover, he showed that every secondary Hopf manifold is finitely covered by a primary Hopf manifold[1].

When $n = 2$, we have a normal form for a contraction on $\mathbb{C}^2$. Up to an automorphism of $\mathbb{C}^2$, every contraction is of the form $(z_1, z_2) \mapsto (\alpha_1 z_1 + \lambda z_2^m, \alpha_2 z_2)$ where m is a

[1] Kodaira proved these statements in the case of surfaces, but the proofs work verbatim for the general case.

positive integer and $\alpha_1, \alpha_2, \lambda \in \mathbb{C}$ are subject to the conditions $(\alpha_1 - \alpha_2^m)\lambda = 0$ and $0 < |\alpha_1| \le |\alpha_2| < 1$; see [60] and [98]. It follows that all primary Hopf surfaces are deformation equivalent to the quotient of $\mathbb{C}^2 \setminus \{0\}$ by the infinite cyclic group generated by the contraction $(z_1, z_2) \mapsto (\frac{1}{2}z_1, \frac{1}{2}z_2)$; this is Hopf's original example[2] [43] which is clearly diffeomorphic to $S^1 \times S^3$, so all primary Hopf surfaces are diffeomorphic to $S^1 \times S^3$. The diffeomorphism types of secondary Hopf surfaces have been classified by Kato in [50] and [52], see Theorem 4.5.

Note that $\mathbb{C}^* \times \{0\}$ is preserved by a contraction (in normal form), and the image of $\mathbb{C}^* \times \{0\}$ under the quotient map is isomorphic to $\mathbb{C}^*/\sim$ where $z \sim \alpha_1 z$, i.e. a one-dimensional torus. So every primary Hopf surface contains a curve; this is also true of secondary Hopf surfaces, see Theorem 32 of [56].

2.2.2 Inoue Surfaces

Inoue surfaces are minimal class VII surfaces introduced by Inoue in [47] and [48]. Unlike Hopf surfaces, they do not contain curves. We outline the construction of the four families of Inoue surfaces: S^+_M, S^-_M, $S^+_{N,p,q,r,t}$, and $S^-_{N,p,q,r}$.

Let $M \in SL(3, \mathbb{Z})$ be a matrix with one real eigenvalue $\alpha > 1$ and two complex conjugate eigenvalues $\beta, \bar{\beta}$ where $\mathrm{Im}(\beta) > 0$. Let (a_1, a_2, a_3), (b_1, b_2, b_3), and $(\overline{b_1}, \overline{b_2}, \overline{b_3})$ be eigenvectors of M corresponding to the eigenvalues α, β, and $\overline{\beta}$ respectively. By replacing (a_1, a_2, a_3) with its real part if necessary, we can assume it is real.

Proposition 2.3. *The vectors $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in \mathbb{R} \times \mathbb{C}$ are linearly independent over $\mathbb{R}$.*

Proof. Suppose $c_1(a_1, b_1) + c_2(a_2, b_2) + c_3(a_3, b_3) = (0, 0)$ where $c_1, c_2, c_3 \in \mathbb{R}$. Then $(c_1, c_2, c_3) \cdot (a_1, a_2, a_3) = 0$ and $(c_1, c_2, c_3) \cdot (b_1, b_2, b_3) = 0$. By conjugating the last equa-

[2] Hopf actually defined complex manifolds $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$ where the $\mathbb{Z}$-action is generated by $g : (z_1, \dots, z_n) \mapsto (2z_1, \dots, 2z_n)$. By taking $n = 2$ and replacing g by g^{-1}, we obtain the stated surface.

tion, we also have $(c_1, c_2, c_3) \cdot (\overline{b_1}, \overline{b_2}, \overline{b_3}) = 0$. As (a_1, a_2, a_3), (b_1, b_2, b_3), and $(\overline{b_1}, \overline{b_2}, \overline{b_3})$ are eigenvectors for distinct eigenvalues, they are linearly independent and therefore $(c_1, c_2, c_3) = (0, 0, 0)$. $\square$

Consider the following biholomorphisms of $\mathbb{H} \times \mathbb{C}$:

$$\begin{aligned} g_0(w, z) &= (\alpha w, \beta z) \\ g_i(w, z) &= (w + a_i, z + b_i), \quad i = 1, 2, 3. \end{aligned}$$

Let G_M^+ be the group generated by g_0, g_1, g_2, g_3 and let Γ^+ be the subgroup of G_M^+ generated by g_1, g_2, g_3; as g_1, g_2, g_3 commute with each other, we see that $\Gamma^+ \cong \mathbb{Z}^3$.

Note that the action of Γ^+ on (w, z) preserves $\mathrm{Im}(w)$ and $\{(w, z) \in \mathbb{H} \times \mathbb{C} \mid \mathrm{Im}(w) = b\}/\Gamma^+$ is diffeomorphic to $(\mathbb{R} \times \mathbb{C})/\Lambda^+$ where Λ^+ is the lattice generated by (a_1, b_1), (a_2, b_2), and (a_3, b_3); the fact that Λ^+ is a lattice follows from Proposition 2.3. Therefore $(\mathbb{R} \times \mathbb{C})/\Lambda^+$ is diffeomorphic to T^3 and hence $(\mathbb{H} \times \mathbb{C})/\Gamma^+$ is diffeomorphic to $(0, \infty) \times T^3$.

As (a_1, a_2, a_3) and (b_1, b_2, b_3) are eigenvectors of M for the eigenvalues α and β respectively, we have

$$(\alpha a_i, \beta b_i) = \sum_{j=1}^{3} m_{ij}(a_j, b_j).$$

It follows that the biholomorphism g_0 descends to the quotient $(\mathbb{H} \times \mathbb{C})/\Gamma^+$. Under the identification with $(0, \infty) \times T^3$, the induced map g_0 restricts to a diffeomorphism between $\{1\} \times T^3$ and $\{\alpha\} \times T^3$. So $S_M^+ := (\mathbb{H} \times \mathbb{C})/G_M^+ = ((\mathbb{H} \times \mathbb{C})/\Gamma^+)/\langle g_0 \rangle$ is diffeomorphic to a mapping torus of a diffeomorphism f of T^3. By tracing through the identifications and using the displayed equation above, it is not hard to show that f is just the map on T^3 induced by M^T.

Note that $\pi_1(S_M^+) = G_M^+$ and $H_1(S_M^+; \mathbb{Z}) \cong G_M^+/[G_M^+, G_M^+]$. For $i, j \in \{1, 2, 3\}$, we have

$[g_i, g_j] = 1$, while it follows from the displayed equation above that

$$
\begin{aligned}
[g_0, g_1] &= g_1^{m_{11}-1} g_2^{m_{12}} g_3^{m_{13}} \\
[g_0, g_2] &= g_1^{m_{21}} g_2^{m_{22}-1} g_3^{m_{23}} \\
[g_0, g_3] &= g_1^{m_{31}} g_2^{m_{32}} g_3^{m_{33}-1}.
\end{aligned}
$$

Recall that there are invertible matrices $S, T \in GL(3, \mathbb{Z})$ such that $S(M - I)T$ is diagonal; this is called the *Smith normal form* of $M - I$, and the entries on the diagonal, say e_1, e_2, e_3, are the *elementary divisors* of $M - I$. Note that $e_1, e_2, e_3 \neq 0$ otherwise there is $v \neq 0$ such that $S(M - I)Tv = 0$ which implies $M(Tv) = Tv$; this is impossible as $Tv \neq 0$ and 1 is not an eigenvalue of M. Using T, we can replace the generators g_1, g_2, g_3 with $\hat{g}_1, \hat{g}_2, \hat{g}_3$, and using S, we can reduce the commutator relations so that we obtain

$$
\begin{aligned}
H_1(S_M^+; \mathbb{Z}) &\cong \langle g_0, \hat{g}_1, \hat{g}_2, \hat{g}_3 \mid \hat{g}_1^{e_1} = \hat{g}_2^{e_2} = \hat{g}_3^{e_3} = [g_0, \hat{g}_i] = [\hat{g}_i, \hat{g}_j] = 1 \rangle \\
&\cong \mathbb{Z} \oplus (\mathbb{Z}/e_1\mathbb{Z}) \oplus (\mathbb{Z}/e_2\mathbb{Z}) \oplus (\mathbb{Z}/e_3\mathbb{Z}).
\end{aligned}
$$

Therefore $b_1(S_M^+) = 1$. As S_M^+ is diffeomorphic to a mapping torus of T^3, we have $\chi(S_M^+) = \chi(T^3)\chi(S^1) = 0$ and therefore $b_2(S_M^+) = 0$.

The Inoue surface S_M^- is defined similarly to S_M^+. Consider the following biholomorphisms of $\mathbb{H} \times \mathbb{C}$:

$$
\begin{aligned}
h_0(w, z) &= (\alpha w, \overline{\beta} z) \\
h_i(w, z) &= (w + a_i, z + \overline{b_i}), \quad i = 1, 2, 3.
\end{aligned}
$$

Let G_M^- be the group generated by h_0, h_1, h_2, h_3, and define $S_M^- := (\mathbb{H} \times \mathbb{C})/G_M^-$. The arguments above concerning S_M^+ can be made analagously for S_M^-. More directly, note that the map $\mathbb{H} \times \mathbb{C} \to \mathbb{H} \times \mathbb{C}$ given by $(w, z) \mapsto (w, \overline{z})$ descends to a diffeomorphism

$S_M^+ \to S_M^-$.

Remark 2.4. There is a little bit of confusion in the literature regarding the Inoue surfaces S_M^+ and S_M^-. In [47], for a matrix $M \in SL(3,\mathbb{Z})$ with one real eigenvalue $\alpha > 1$ and two complex conjugate eigenvalues $\beta, \bar{\beta}$, Inoue defined a complex surface S_M in the same way as S_M^+ was defined above. However, Inoue did not indicate how to distinguish between β and $\overline{\beta}$; note that we required $\mathrm{Im}(\beta) > 0$. So, depending on the naming of the eigenvalues, the surface S_M could be S_M^+ or S_M^-. In [48], Inoue separated the two cases and showed that they are not biholomorphic, or even deformation equivalent. In the literature which followed, the distinction between S_M^+ and S_M^- was not always observed. A potential reason for this oversight is that [47] appears in English while [48] appears in Japanese.

In what follows, we refer to results in [47] regarding S_M, so as explained above, these results apply to both S_M^+ and S_M^-.

Inoue proved that $H^0(S_M, \mathcal{O}(L)) = 0$ for every non-trivial holomorphic line bundle L, see the proof of Proposition 2 (i) in [47]. It follows that S_M does not contain any curves: if C is a curve in S_M, there is an associated holomorphic line bundle and a section s such that $s^{-1}(0) = C$. Moreover, we see that $P_d(S_M) = \dim H^0(S_M, K_{S_M}^d) = 0$ unless $K_{S_M}^d$ is trivial; Inoue showed that this cannot happen for $d \neq 0$, see Lemma 1 (iii) of [47]. It follows that S_M has Kodaira dimension $-\infty$.

Recently, Endo and Pajitnov defined higher dimensional analogues of Inoue surfaces of type S_M, see [29] and [30].

The classes $S_{N,p,q,r,t}^+$ and $S_{N,p,q,r}^-$ are defined in a similar way to S_M^+ and S_M^-. We will only briefly cover their properties.

Let $N \in SL(2,\mathbb{Z})$ be a matrix with two real eigenvalues α and $\frac{1}{\alpha}$ where $\alpha > 1 > \frac{1}{\alpha} > 0$. Let (a_1, a_2) and (b_1, b_2) be real eigenvectors of N for α and $\frac{1}{\alpha}$ respectively. Let $p, q, r \in \mathbb{Z}$

with $r \neq 0$ and $t \in \mathbb{C}$. Consider the following equation for (c_1, c_2):

$$(N - I)(c_1, c_2) = (d_1, d_2) + \frac{1}{r}(b_1 a_2 - b_2 a_1)(p, q)$$

where $d_i = \frac{1}{2} n_{i1}(n_{i1} - 1) a_1 b_1 + \frac{1}{2} n_{i2}(n_{i2} - 1) a_2 b_2 + n_{i1} n_{i2} b_1 a_2$. As N has eigenvalues $\alpha > 1 > \frac{1}{\alpha}$, the matrix $N - I$ has eigenvalues $\alpha - 1 > 0 > \frac{1}{\alpha} - 1$ and is therefore invertible, so there is a unique solution to the above equation.

Consider the following biholomorphisms of $\mathbb{H} \times \mathbb{C}$:

$$\begin{aligned}
g_0(w, z) &= (\alpha w, z + t) \\
g_i(w, z) &= (w + a_i, z + b_i w + c_i) \quad i = 1, 2 \\
g_3(w, z) &= \left(w, z + \frac{1}{r}(b_1 a_2 - b_2 a_1)\right).
\end{aligned}$$

Let $G^+_{N,p,q,r,t}$ be the group generated by g_0, g_1, g_2, g_3 and let Γ be the subgroup generated by g_1, g_2, g_3. The quotient $(\mathbb{H} \times \mathbb{C})/\Gamma$ is diffeomorphic to $(0, \infty) \times F$ where F is a circle bundle over a two-dimensional torus, and $S^+_{N,p,q,r,t} := (\mathbb{H} \times \mathbb{C})/G^+_{N,p,q,r,t} = ((\mathbb{H} \times \mathbb{C})/\Gamma)/\langle g_0 \rangle$ is a mapping torus of F.

By computing the commutators of the generators of $G^+_{N,p,q,r,t}$, it can be shown that $H_1(S^+_{N,p,q,r,t}; \mathbb{Z}) \cong G^+_{N,p,q,r,t}/[G^+_{N,p,q,r,t}, G^+_{N,p,q,r,t}] \cong \mathbb{Z} \oplus (\mathbb{Z}/e_1\mathbb{Z}) \oplus (\mathbb{Z}/e_2\mathbb{Z}) \oplus (\mathbb{Z}/r\mathbb{Z})$ where e_1, e_2 are the non-zero elementary divisors of $N - I$, so $b_1(S^+_{N,p,q,r,t}) = 1$. As $\chi(S^+_{N,p,q,r,t}) = \chi(F)\chi(S^1) = 0$, we see that $b_2(S^+_{N,p,q,r,t}) = 0$. Inoue showed that $S^+_{N,p,q,r,t}$ does not contain curves and has Kodaira dimension $-\infty$, see Proposition 3 (i) of [47].

Finally, let $N \in GL(2, \mathbb{Z})$ be a matrix with $\det N = -1$ and eigenvalues $\alpha > 1 > -1 > -\frac{1}{\alpha}$. Let (a_1, a_2) and (b_1, b_2) be real eigenvectors of N for α and $\frac{1}{\alpha}$ respectively. Let $p, q, r \in \mathbb{Z}$

with $r \neq 0$ and consider the following equation for (c_1, c_2):

$$-(N+I)(c_1, c_2) = (d_1, d_2) + \frac{1}{r}(b_1 a_2 - b_2 a_1)(p, q)$$

where d_i are as before. As N has eigenvalues $\alpha > -1 > -\frac{1}{\alpha}$, the matrix $N + I$ has eigenvalues $\alpha + 1 > 0 > -\frac{1}{\alpha} + 1$ and is therefore invertible, so there is a unique solution to the above equation.

Consider the following biholomorphisms of $\mathbb{H} \times \mathbb{C}$:

$$\begin{aligned}
g_0(w, z) &= (\alpha w, -z) \\
g_i(w, z) &= (w + a_i, z + b_i w + c_i) \quad i = 1, 2 \\
g_3(w, z) &= \left(w, z + \frac{1}{r}(b_1 a_2 - b_2 a_1)\right).
\end{aligned}$$

Let $G^-_{N,p,q,r}$ be the group generated by g_0, g_1, g_2, g_3. Then the subgroup generated by g_0^2, g_1, g_2, g_3 is isomorphic to $G^+_{N^2,p_1,q_1,r,0}$ for some $p_1, q_1 \in \mathbb{Z}$. As such, $S^-_{N,p,q,r} := (\mathbb{H} \times \mathbb{C})/G^-_{N,p,q,r}$ has $S^+_{N^2,p_1,q_1,r,0}$ as a double cover. It follows that $\chi(S^-_{N,p,q,r}) = 0$, $b_1(S^-_{N,p,q,r}) = 1$, and $b_2(S^-_{N,p,q,r}) = 0$. Moreover, the surface $S^-_{N,p,q,r}$ does not contain curves and has Kodaira dimension $-\infty$.

The following theorem was first claimed by Bogomolov in [15] and [16]. Later, Li, Yau, and Zheng [70] used a different method to prove the result, namely the Kobayashi-Hitchin correspondence, but their argument was incomplete. Shortly after, Teleman [104] gave a proof using similar ideas.

Theorem 2.5. *Every class VII surface with $b_2 = 0$ is biholomorphic to a Hopf surface or an Inoue surface.*

2.2.3 Class VII Surfaces with $b_2 > 0$

A *spherical shell* in an n-dimensional complex manifold is a connected open subset which is biholomorphic to an open neighbourhood of S^{2n-1} in $\mathbb{C}^n \smallsetminus \{0\}$. If the complement of a spherical shell is connected, then it is said to be a *global spherical shell*. Every complex manifold admits a spherical shell, but not necessarily a global spherical shell. For example, a compact connected Riemann surface admits a global spherical shell if and only if it is not biholomorphic to $\mathbb{CP}^1$. In higher dimensions, the existence of a global spherical shell is a much more restrictive condition.

Theorem 2.6. (Kato [51]) *Suppose that a compact complex manifold X of dimension $n \geq 2$ contains a global spherical shell. Then there is a complex family $\pi : \mathfrak{X} \to \mathbb{D}$ such that $\pi^{-1}(0) = X$ and for $t \neq 0$, each fibre $\pi^{-1}(t)$ is biholomorphic to a modification of a primary Hopf manifold at finitely many points.*

A *modification* is a proper surjective holomorphic map $f : X \to Y$ with a closed nowhere-dense subset $N \subset Y$ such that $f|_{X \smallsetminus f^{-1}(N)} : X \smallsetminus f^{-1}(N) \to Y \smallsetminus N$ is a biholomorphism; we also call X a modification of Y. In the context of the above theorem, the set N is finite. Note that a modification is a special type of bimeromorphic map.

If X is obtained from Y by blowing up a finite collection of points, then the blowdown map $X \to Y$ is a modification. More generally, one could blowup submanifolds of the exceptional divisors in X, and the composition of the blowdown maps is also a modification.

In the case of surfaces, every modification is just the composition of blowups of points, see Theorem 5.7 of [61]. So by Theorem 2.6, a surface with a global spherical shell, often called a *Kato surface*, is deformation equivalent to an iterated blowup of a primary Hopf surface and is therefore diffeomorphic to $(S^1 \times S^3)\#k\overline{\mathbb{CP}^2}$ for some k. Note that such a surface has fundamental group $\mathbb{Z}$ and hence cannot admit a Kähler metric.

More generally, an arbitrary bimeromorphic map is a composition of blowups and blowdowns of smooth submanifolds, see Theorem 0.3.1 of [1]. In particular, a bimeromorphic map induces an isomorphism on fundamental groups, so if X admits a global spherical shell and $\dim X > 1$, then $\pi_1(X) \cong \mathbb{Z}$ and hence X does not admit a Kähler metric.

Blownup primary Hopf manifolds contain global spherical shells. For example, consider $X = (\mathbb{C}^n \smallsetminus \{0\})/\sim$ where $z \sim \lambda z$ where $|\lambda| > 1$. Then $U_\varepsilon = \{z \in \mathbb{C}^n \mid 1 < \|z\| < 1 + \varepsilon\}$ projects to a global spherical shell in X provided $1 + \varepsilon < |\lambda|$. If X is blownup at finitely many points $p_1, \dots, p_k$, then one can choose ε small enough so that the projection of U_ε doesn't contain the points $p_1, \dots, p_k$, and this will give rise to a global spherical shell in the blowup. Not every complex manifold with a global spherical shell is a modification of a primary Hopf surface; as we will see shortly, examples already exist in dimension two.

By Theorem 2.5, aside from Hopf and Inoue surfaces, all other minimal class VII surfaces must have $b_2 > 0$. Examples of such surfaces have been constructed, for example Enoki surfaces [28], Inoue-Hirzebruch surfaces and half-Inoue surfaces [49] – note, Inoue-Hirzebruch surfaces are sometimes called parabolic and hyperbolic Inoue surfaces, such as in [78]. However, minimal class VII surfaces with $b_2 > 0$ remain unclassified. If the following were true, we would have a classification up to deformation by Theorem 2.6.

Conjecture 2.7. (Global Spherical Shell) *Every minimal class VII surface with $b_2 > 0$ contains a global spherical shell; i.e. it is a Kato surface.*

By analysing the moduli space of stable holomorphic bundles on class VII surfaces, Teleman has verified the conjecture for small values of b_2, namely $b_2 = 1$ and 2, see [105] and [106] respectively. In [107], Teleman has announced the case $b_2 = 3$ with the "long and technical" details to follow in an upcoming paper.

All the known examples of minimal class VII surfaces have been shown to possess global spherical shells. For Inoue-Hirzebruch surfaces and half-Inoue surfaces, this was shown by Kato in [51], while for Enoki surfaces, the existence of global spherical shells follows from [27].

3. OBSTRUCTIONS TO POSITIVE SCALAR CURVATURE

The following theorem follows from the work of Trudinger [108] and Aubin [7] on the Yamabe problem.

Theorem 3.1. *Every closed smooth manifold of dimension at least three admits a Riemannian metric of (constant) negative scalar curvature.*

In fact, Lohkamp [69] has shown that every such manifold admits a metric of negative Ricci curvature.

On the other hand, there are obstructions to the existence of positive scalar curvature metrics.

Theorem 3.2. (Lichnerowicz [68]) *Let M be a closed spin manifold. If M admits a metric of positive scalar curvature, then $\hat{A}(M) = 0$.*

In dimension four, it follows from the Hirzebruch Signature Theorem that $\hat{A}(M) = -\frac{1}{8}\sigma(M)$, so $\hat{A}(K3) = -\frac{1}{8}\sigma(K3) = 2 \neq 0$ and therefore a K3 surface does not admit a metric of positive scalar curvature. In the following sections, we outline some more obstructions to the existence of positive scalar curvature.

What then about zero scalar curvature metrics? Are there obstructions to their existence? The answer is heavily dependent upon whether the manifold admits metrics of positive scalar curvature.

Proposition 3.3. *On a closed manifold which does not admit a metric of positive scalar curvature, any metric of non-negative scalar curvature is Ricci-flat.*

This was initially proved by Bourguignon, while another proof was given by Kazdan and Warner, see Lemma 5.2 of [54]. Combining with Proposition 1.7, this immediately gives the following result.

Corollary 3.4. *Let M be a closed smooth manifold with $Y(M) = 0$. A Yamabe metric g realises the Yamabe invariant if and only if g is Ricci-flat.*

As the following theorem demonstrates, there are topological restrictions on those manifolds which admit Ricci-flat metrics, so Proposition 3.3 leads to an obstruction to the existence of zero scalar curvature metrics when no metrics of positive scalar curvature exist.

Theorem 3.5. (Fischer & Wolf [31]) *Let (M, g) be a closed Ricci-flat manifold of dimension n. Then there is a finite covering $\pi : T^k \times M_0$ where M_0 is a closed simply connected manifold of dimension $n-k$. Moreover, $\pi^* g = g_1 + g_2$ where g_1 is a flat metric on T^k and g_2 is a Ricci-flat metric on M_0.*

If g is a Ricci-flat metric on M, then g is flat if and only if M has contractible universal cover. Therefore, any zero scalar curvature metric on K3 is Ricci-flat but not flat. The existence of such a metric is guaranteed by Yau's solution of the Calabi conjecture [112]. Historically, this was the first example of a Ricci-flat metric on a closed manifold which was not flat.

If the manifold M does in fact admit metrics of positive scalar curvature, then the existence of zero scalar curvature metrics was proved by Kazdan & Warner as part of their work on the prescribed scalar curvature problem: which functions $f \in C^\infty(M)$ arise as the scalar curvatures of a Riemannian metric?

Theorem 3.6. (Kazdan & Warner [53], [54]) *Let $n \geq 3$. Every smooth closed connected n-dimensional manifold M falls into one of three types:*

I. those which admit a metric of nonnegative scalar curvature which is positive somewhere, *II. those which don't but admit a metric of zero scalar curvature,*

III. all other closed manifolds.

If M is in class I, then any $f \in C^\infty(M)$ is the scalar curvature of some metric. If M is in class II, then $f \in C^\infty(M)$ is the scalar curvature of some metric iff it's identically zero or negative somewhere. If M is in class III, then $f \in C^\infty(M)$ is the scalar curvature of some metric iff it's negative somewhere.

Manifolds in class I have positive Yamabe invariant, manifolds in class II have Yamabe invariant zero which is realised, and manifolds in class III have non-positive Yamabe invariant. In the case that the Yamabe invariant of a manifold in class III is zero, the Yamabe invariant is not realised.

So a necessary condition for a manifold to admit a positive scalar curvature metric is that it admit a zero scalar curvature metric. At the time of Kazdan and Warner's work, it was unknown whether this condition was also sufficient, in particular, they asked whether T^3 admits a metric of positive scalar curvature, see Question 2 of [54]. The next two sections outline independent attempts to answer the question of whether tori admit metrics of positive scalar curvature.

3.1 Minimal Hypersurface Technique

Let (M, g) be a Riemannian manifold, and let Σ be submanifold of M. The *second fundamental form* of Σ is the pairing $\mathrm{II} : T\Sigma \times T\Sigma \to \nu$ given by $\mathrm{II}(X, Y) = \mathrm{proj}_\nu(\nabla_X Y)$ where ν is the normal bundle of Σ in M; the projection is defined with respect to the splitting $TM \cong \nu \oplus \nu^\perp$ given by the Riemannian metric g and ∇ denotes the Levi-Civita connection of g. As ∇ is torsion-free, the second fundamental form is symmetric.

We call $H = \mathrm{tr}_g(\mathrm{II}) \in \Gamma(\nu)$ the *mean curvature* of Σ in M. The first variation of the area functional at Σ along $V \in \Gamma(\nu)$ is given by $-\int_\Sigma g(V, H) d\mu_g$, see equation (1.45) of [26]; we call a submanifold Σ with $H = 0$ a *minimal submanifold*. The second variation of the area functional at Σ along V is given by $-\int_\Sigma g(V, \mathcal{L}V) d\mu_g$ where $\mathcal{L}$ is the Jacobi operator, a second order self-adjoint operator, see equation (1.143) of [26]. The (*Morse*) *index* of a minimal submanifold Σ is the number of negative eigenvalues of $\mathcal{L}$; if the index is zero, then we call Σ a *stable* minimal submanifold.

In low dimensions, one way to produce stable minimal hypersurfaces is to use the following theorem which is a combination of many results in geometric measure theory from several authors, see Remark 3.4 of [62].

Theorem 3.7. *Let (M, g) be a closed orientable smooth n-dimensional Riemannian manifold with $n \leq 7$. For any non-zero homology class in $c \in H_{n-1}(M; \mathbb{Z})$, there is a smooth closed orientable hypersurface Σ with $[\Sigma] = c$ which minimises area amongst all such hypersurfaces. In particular, Σ is a stable minimal hypersurface.*

In higher dimensions, the techniques used to produce Σ instead produce a subset which is smooth away from a subset of codimension 7. Smale later showed that for $c \in H_{n-1}(M; \mathbb{Z})$ with $\dim M = 8$, there is an open dense set of metrics for which c is represented by a stable minimal hypersurface as in the above theorem, see Theorem 1.1 of [97].

Schoen and Yau proved that the property of admitting positive scalar curvature metrics is inherited by stable minimal hypersurfaces, see the proof of Theorem 1 of [91].

Theorem 3.8. *Let (M, g) be a closed orientable Riemannian manifold with* $\dim M \geq 3$ *and Σ a closed orientable stable minimal hypersurface. If g has positive scalar curvature, then $g|_\Sigma$ is conformal to a positive scalar curvature metric on Σ, i.e.* $Y(M, [g|_\Sigma]) > 0$.

Schoen and Yau used the combination of these two results to show that the torus T^n does not admit metrics of positive scalar curvature for $n \leq 7$; by Smale's result, the proof extends to $n = 8$. Moreover, the same is true for any manifold which admits a map of non-zero degree to a torus (in these dimensions). Recently, Schoen and Yau [92] have announced that the conclusion is true in any dimension. They show this by dealing with the singular sets of hypersurfaces in higher dimensions using what they call *minimal k-slicings*.

3.2 Enlargeable Manifolds

If (M, g) is a complete Riemannian n-dimensional manifold with $\mathrm{Ric}_g > (n-1)\kappa > 0$, then Myers' Theorem states that the diameter is bounded above by $\pi\kappa^{-1/2}$ and is therefore compact. Applying Myers' Theorem to the universal cover shows that it is compact and hence $\pi_1(M)$ is finite. As the scalar curvature is the trace of the Ricci curvature, one might hope that the assumption of its positivity would similarly have implications for the diameter and hence fundamental group – this is not the case. For any Riemannian manifold (M, g_M), the product $M \times S^2$ admits metrics of the form $g_M + rg_{\text{round}}$ which, for r small enough, have positive scalar curvature. Moreover, their diameter is at least as large as that of (M, g_M) which can be arbitrary and $\pi_1(M \times S^2) \cong \pi_1(M)$ which could be any finitely presented group.

As the scalar curvature is the trace of the Ricci curvature, its positivity could result from having a small number of directions of positive Ricci curvature which are large in comparison to the Ricci curvature of the other directions. The proof of Myers' Theorem shows that if the Ricci curvature is positive in a given direction, then the length of a minimal geodesic in that direction is bounded above. So if the scalar curvature is positive, there must be some direction in which the manifold is relatively 'small'. Note, this is exactly what is occuring in the case of $M \times S^2$, the Ricci curvature is positive in the S^2 directions which are relatively 'small'. If $\pi : M' \to M$ is a covering, then $\pi^* g$ also has positive scalar curvature, so we can apply the same logic to $(M', \pi^* g)$. One would expect that if a manifold admits a positive scalar curvature metric, then its covering spaces can't be 'large' in every direction; note, the coverings of $M \times S^2$ are of the form $M' \times S^2$ and the S^2 directions remain 'small'. Enlargeability is an attempt to make these notions precise.

Recall that a map $f : (M, g) \to (N, h)$ between Riemannian manifolds is said to be *ε-contracting* if $\|df(v)\|_h \leq \varepsilon \|v\|_g$. Moreover, it is said to be *constant at infinity* if it is constant outside a compact set. If N is compact, then such maps have a notion of degree.

A compact Riemannian manifold of dimension n is called *enlargeable* if for every $\varepsilon > 0$, there exists an orientable Riemannian covering space which admits an ε-contracting map onto (S^n, g_{round}) which is constant at infinity and is of non-zero degree. If the coverings can be taken to be finite coverings for every ε, then it is called *compactly enlargeable*.

Example 3.9. Consider the torus T^n equipped with its standard flat metric g. The universal cover is $\mathbb{R}^n$ equipped with the Euclidean metric. Consider the map $f_D : \mathbb{R}^n \to S^n$ constructed as follows: for $\|x\| \geq D$, map x to $(0, \dots, 0, -1)$, while for the open disc centred at 0 of radius D, map radial lines through the origin to great circles through $(0, \dots, 0, 1)$ – intuitively, the disc 'wraps' around the sphere. This is a map which is constant at infinity of degree one and is ε-contracting for some ε inversely proportional to D. Varying D gives the required family of maps to show that (T^n, g) is enlargeable.

Note, the construction of the maps in the above example only requires the existence of discs of arbitrary large radius which implies the manifold is 'large' in every direction. One could view the property of containing such discs as a heuristic for the enlargeability hypothesis. Note that we didn't need to pass to the universal cover of T^n to find larger and larger discs. If $T^n = \mathbb{R}^n/\mathbb{Z}^n$, then it is covered by $\mathbb{R}^n/(k\mathbb{Z})^n$; by taking k large enough, we can find a disc of any radius. It follows that (T^n, g) is in fact compactly enlargeable. The distinction between these two notions of enlargeability is not understood. In particular, the following is unknown.

Question 3.10. Is every enlargeable manifold compactly enlargeable?

The collection of enlargeable manifolds is extensive, see Theorems IV.5.3 and IV.5.4 of [63].

Theorem 3.11.

(a) *Enlargeability depends only on the homotopy type of the manifold. In particular, it is independent of the Riemannian metric.*

(b) *The product of enlargeable manifolds is enlargeable.*

(c) *Any manifold which admits a map of non-zero degree onto an enlargeable manifold is itself enlargeable.*

(d) *Every nilmanifold is enlargeable.*

(e) *Any manifold which admits a non-positive sectional curvature metric is enlargeable.*

In view of (a), we will say a compact manifold is, or isn't, enlargeable.

Proposition 3.12. *Let $\pi : M \to N$ be a finite covering of closed orientable smooth manifolds. Then M is enlargeable if and only if N is.*

Proof. As $\pi : M \to N$ is a map of non-zero degree, if N is enlargeable, so is M by Theorem 3.11 (c).

If M is enlargeable for some metric g, it is also enlargeable for $\pi^* g_0$ where g_0 is a metric on N by Theorem 3.11 (a). As every Riemannian covering space of $(M, \pi^* g_0)$ is also a Riemannian covering space of (N, g_0), we see that (N, g_0) is also enlargeable. $\square$

Example 3.13. Let N be an orientable circle bundle over T^2. Then N is the sphere bundle of some complex line bundle L which is classified by its first Chern class $c_1(L) \in$

$H^2(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z}$; let N_k denote the orientable circle bundle of L with $c_1(L) = k$. Note that $c_1(\overline{L}) = -c_1(L)$ and complex conjugation gives rise to a diffeomorphism between N_k and N_{-k}. If $c_1(L) = 0$, then L is trivial and $N_0 = T^3$. If $c_1(L) = 1$ and $c_1(L') = k > 0$, then $L' \cong L^{\otimes k}$. The map $N_1 \to N_k$ given by $z \mapsto z^{\otimes k}$ is a k-sheeted covering, so by Proposition 3.12, we see that N_k is enlargeable if and only if N_1 is. It follows from the Serre spectral sequence that $H_1(N_k; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}$. Now note that the Heisenberg manifold $M = H(3, \mathbb{R})/H(3, \mathbb{Z})$ is a circle bundle over a torus, and as $H_1(M; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, we see that $M = N_1$. The Heisenberg manifold is a nilmanifold, so by Proposition 3.11 (d), it is enlargeable and hence so is N_k. In conclusion, every orientable circle bundle over T^2 is enlargeable.

Theorem 3.14. (Gromov & Lawson [35]) *An enlargeable spin manifold does not admit metrics of positive scalar curvature.*

In particular, tori do not admit metrics of positive scalar curvature. Moreover, as every compact manifold is finitely covered by a torus, compact flat manifolds do not admit metrics of positive scalar curvature.

It should be noted that the spin hypothesis can be weakened. The proof of the above theorem shows that for ε small enough, the corresponding cover admits a twisted Dirac operator which, under the assumption of positive scalar curvature, has index zero, while under the assumption of enlargeability, has non-zero index. The spin condition is needed to form the Dirac operator, but this only takes place on some cover. So, for example, if M admits a metric of non-positive sectional curvature, it is enlargeable by Theorem 3.11 (e), but it may not be spin. However, its universal cover is $\mathbb{R}^n$ by the Cartan-Hadamard Theorem, which is spin, so we see that such manifolds do not admit metrics of positive scalar curvature.

Recently, Cecchini and Schick [24] announced that one can remove the spin hypothesis completely from the above theorem. Their proof relies on the minimal k-slicing technique introduced in the recent work of Schoen and Yau.

Although they will not be needed, we end by pointing out that there are other notions of enlargeability, namely weak enlargeability and $\hat{A}$-enlargeability, see sections IV.5 and IV.6 of [63].

3.3 Seiberg-Witten Theory

Let (M, g) be a closed orientable four-dimensional Riemannian manifold with $b^{+}(M) \geq 2$, equipped with a spin^c structure $\mathfrak{c}$, and denote the associated spinor bundles by $\mathbb{V}^{\pm}$. The perturbed *Seiberg-Witten equations* for a unitary connection ∇ on $\det(\mathbb{V}^{+}) \cong \det(\mathbb{V}^{-})$ and a positive spinor $\psi \in \Gamma(M, \mathbb{V}^{+})$ are

$$\begin{aligned} \partial\!\!\!/^{c}\psi &= 0 \\ F_{\nabla}^{+} &= \sigma(\psi) + ih \end{aligned}$$

where

- $\partial\!\!\!/^{c}$ is the twisted spin^c Dirac operator formed from $\mathfrak{c}$ and ∇,
- F_{∇}^{+} is the self-dual part of the curvature of ∇,
- $\sigma(\psi) = \psi \otimes \psi^{*} - \frac{1}{2}|\psi|^{2}\,\mathrm{id}$, and
- h is a self-dual harmonic two-form called a perturbation.

The Seiberg-Witten equations were introduced by Seiberg and Witten in [94] and [95]

We give an extremely brief account of the Seiberg-Witten invariant, see [75] for more details. For a generic metric g and a generic perturbation h, the moduli space of irreducible solutions modulo gauge, denoted $\mathcal{M}$, is a finite-dimensional manifold which can be equipped with an orientation. If $\dim \mathcal{M} = 0$, the signed count of the points is defined to be the Seiberg-Witten invariant of the spin^c structure $\mathfrak{c}$. If the moduli space is of higher dimension, then a different definition is required. The Seiberg-Witten invariant is independent of the perturbation and the metric. It's worth noting that much of the theory can be made to work when $b^+(M) = 1$, but the situation is more complicated.

The following theorem of Taubes shows that this invariant is not always zero.

Theorem 3.15. (Taubes [102]) *Let (M, ω) be a closed symplectic four-manifold with $b^+(M) \geq 2$. The spin^c structure induced by ω has Seiberg-Witten invariant equal to ± 1.*

Seiberg-Witten invariants are a useful tool for showing non-existence of positive scalar curvature thanks to the following result.

Theorem 3.16. *Let (M, g) be a closed orientable Riemannian four-manifold. If M has a non-zero Seiberg-Witten invariant and $b^+(M) \geq 2$, then M does not admit a positive scalar curvature metric.*

Combining these two results, we conclude the following.

Corollary 3.17. *A closed symplectic four-manifold M with $b^+(M) \geq 2$ does not admit a metric of positive scalar curvature.*

3.4 General Results

We end this chapter by discussing the some general results regarding positive scalar curvature.

Theorem 3.18. (Gromov & Lawson [36], Schoen & Yau [91]) *Let M be a closed smooth manifold which admits metrics of positive scalar curvature. If M' is obtained from M by a surgery of codimension at least three, then M' admits metrics of positive scalar curvature.*

Note that $M_1 \# M_2$ is obtained from $M_1 \sqcup M_2$ by a codimension n surgery, so the above theorem immediately gives the following result.

Corollary 3.19. *Let M_1 and M_2 be closed smooth manifolds which admit metrics of positive scalar curvature. If* $\dim M_1 = \dim M_2 \geq 3$*, then $M_1 \# M_2$ admits metrics of positive scalar curvature.*

Note that the relevant case of Theorem 1.14 can be viewed as a generalisation of the above corollary.

Gromov and Lawson used Theorem 3.18 to analyse which simply connected smooth manifolds of dimension at least five admit metrics of positive scalar curvature. In the non-spin case, they showed that every such manifold could be obtained from a set of representatives for generators of the oriented cobordism ring Ω_*^{SO} by a series of surgeries of codimension at least three. Moreover, they showed that the representatives admit positive scalar curvature metrics, and hence every simply connected smooth non-spin manifold of dimension at least five admits metrics of positive scalar curvature.

In the spin case, the corresponding statement was already known to be false: by Theorem 3.2, a necessary condition for the existence of a positive scalar curvature metric is the vanishing of the $\hat{A}$-genus. So for example, the simply connected eight-manifold $K3 \times K3$ does not admit a positive scalar curvature metric as $\hat{A}(K3 \times K3) = \hat{A}(K3)^2 = 2^2 = 4 \neq 0$. More generally, there is a ring homomorphism $\alpha : \Omega_*^{spin} \to KO_*(\mathrm{pt})$ which generalises the $\hat{A}$-genus. In particular, for $n > 0$ we have

$$KO_n(\mathrm{pt}) \cong \widetilde{KO}(S^n) \cong \pi_n(BO) \cong \pi_{n-1}(O) \cong \begin{cases} \mathbb{Z}_2 & n \equiv 1, 2 \bmod 8 \\ \mathbb{Z} & n \equiv 0 \bmod 4 \\ 0 & n \equiv 3, 5, 6, 7 \bmod 8 \end{cases}$$

and when $n \equiv 0 \bmod 4$, the homomorphism $\alpha : \Omega_n^{spin} \to \mathbb{Z}$ coincides with the $\hat{A}$-genus. Hitchin generalised Theorem 3.2 by showing that if X is a closed spin manifold which admits a metric of positive scalar curvature, then $\alpha(X) = 0$, see [42]. As $\Omega_n^{spin} \otimes \mathbb{Q} = \Omega_n^{SO} \otimes \mathbb{Q}$, Gromov and Lawson were able to show that if X is a simply connected spin manifold of dimension at least five with $\alpha(X) = 0$, then for some $k > 0$, the connected sum of k copies of X admits metrics of positive scalar curvature. Stolz later showed that $k = 1$ is sufficient.

Theorem 3.20. (Gromov & Lawson [36], Stolz [99]) *Let M be a simply connected closed smooth manifold with $\dim M \geq 5$. Then M admits a metric of positive scalar curvature if and only if M is not spin or M is spin and $\alpha(M) = 0$.*

Note, the corresponding statement in dimensions two and three is true as M must be a sphere, but it is false in dimension four. In particular, there are simply connected non-spin four-manifolds which do not admit metrics of positive scalar curvature, see Example 4.2. There are also simply connected spin four-manifolds with vanishing $\hat{A}$-genus which do not admit metrics of positive scalar curvature: apply Theorem A of [65] to the manifolds in Theorem 5.8 (a) of [103].

Example 3.21. In contrast to Proposition 3.3, note that Ricci-flat metrics can exist on manifolds which admit positive scalar curvature metrics. If X is a smooth degree 5 hypersurface of $\mathbb{CP}^4$, then $c_1(X) = 0$, so X admits a Ricci-flat Kähler metric by Yau's solution of the Calabi conjecture [112]. Also note that X is simply connected by the Lefschetz Hyperplane Theorem, so by Theorem 3.20, the manifold X also admits metrics of positive scalar curvature because $\alpha(X) \in KO_6(\mathrm{pt}) \cong \pi_5(O) = 0$.

Let M be a simply connected closed smooth manifold of dimension at least five. By Theorem 1.16, the Yamabe invariant of M is non-negative. By Theorem 3.20, we see that $Y(M) = 0$ if and only if M is spin and $\alpha(M) \neq 0$. In this case, the Yamabe invariant is realised if and only if M admits a Ricci-flat metric by Proposition 3.3. The following theorem classifies those M for which such a metric exists.

Theorem 3.22. (Futaki [33]) *Let M be a simply connected, closed smooth manifold* *of dimension at least five. If M admits a Ricci-flat metric but does not admit a positive scalar curvature metric, then $M = M_1 \times \cdots \times M_l$ where M_i (or $\overline{M_i}$) admits a Ricci-flat Kähler metric or a metric with Spin(7) holonomy (in which case M is necessarily spin) and $\alpha(M) \neq 0$.*

3.5 Aspherical Manifolds

Let G be a group and n a positive integer. A topological space X is called an *Eilenberg-MacLane space* of type $K(G,n)$ if $\pi_n(X) \cong G$ and $\pi_i(X) = 0$ for $i \neq n$. Any two Eilenberg-MacLane spaces of type $K(G,n)$ are unique up to homotopy, see Proposition 4.30 of [40]. An Eilenberg-MacLane space X of type $K(G,1)$ is called *aspherical*. If X has the homotopy type of a CW complex, e.g. a manifold, then X is aspherical if and only if it its universal cover is contractible.

Proposition 3.23. *The fundamental group of an aspherical manifold is torsion-free.*

Proof. Let M be an aspherical manifold and suppose $\pi_1(M)$ is not torsion-free. Then there is a subgroup of $\pi_1(M)$ isomorphic to $\mathbb{Z}_k$ for some k. Hence there is a manifold M' with $\pi_1(M') \cong \mathbb{Z}_k$ and a covering $M' \to M$. As $\pi_i(M') \cong \pi_i(M)$ for $i > 1$, we see that M' is also an aspherical manifold. Now note that S^∞ is contractible and admits a free $\mathbb{Z}_k$-action. So both M' and $S^\infty/\mathbb{Z}_k$ are Eilenberg-MacLane spaces of type $K(\mathbb{Z}_k,1)$ and are therefore homotopy equivalent. But this is impossible as M' is a manifold and therefore has bounded homology, while $H_n(S^\infty/\mathbb{Z}_k;\mathbb{Z}_k) \cong \mathbb{Z}_k$ for all $n \geq 0$. □

It follows from the Gauss-Bonnet Theorem that the closed surfaces which do not admit a metric of positive scalar curvature are precisely the aspherical ones. In dimension three, aspherical manifolds again play a role in the classification of those manifolds which do not admit positive scalar curvature.

Theorem 3.24. *Let M be a closed orientable three-manifold. Then M admits a positive scalar curvature metric if and only if M does not contain an aspherical factor in its prime decomposition.*

The necessity of an abscence of aspherical factors was shown by Gromov & Lawson, see Theorem 8.1 of [37]. However the sufficiency required Thurston's Geometrization Conjecture which was resolved by Perelman. In particular, it wasn't known if every closed three-manifold with finite fundamental group is the quotient of S^3 by a finite subgroup of $SO(4)$; this was known as the Elliptization Conjecture. Once this had been established, it was clear that such manifolds must admit positive scalar curvature metrics (in particular, the round metric on S^3 descends).

In arbitrary dimensions, if a manifold admits a metric of non-positive sectional curvature, it is aspherical by the Cartan-Hadamard Theorem, and it fails to admit psc metrics by enlargeability. These observations naturally lead to the following:

Conjecture 3.25. *A closed aspherical manifold cannot admit a metric of positive scalar curvature. The same is true of any closed manifold which admits a map of non-zero degree onto an aspherical manifold.*

Note that if $f : M \to N$ is a map of non-zero degree, then $b_i(M) \geq b_i(N)$. In particular, the conjecture is true in dimension two. The first of the two statements holds in dimension 3 by Theorem 3.24. The second holds thanks to the following result which I learned from Dennis Sullivan.

Proposition 3.26. *Let M and N be closed oriented three-manifolds, with N aspherical. If $f : M \to N$ is a map of non-zero degree, then M contains an aspherical factor in its prime decomposition.*

Proof. Suppose that M does not have an aspherical factor in its prime decomposition, then all of its prime factors are either of the form S^3/G_i where G_i is finite, or $S^2 \times S^1$.

Therefore $\pi_1(M) \cong G_1 * \cdots * G_m * F_n$ where F_n denotes the free group of rank n.

If $\alpha : S^1 \to M$ is a representative of a generator γ of G_i, then $f \circ \alpha$ is a representative of $f_*\gamma \in \pi_1(M)$. As M is aspherical, its fundamental group is torsion-free by Proposition 3.23, so $f \circ \alpha$ is nullhomotopic. Therefore f extends to $M \cup_\alpha e^2$ where e^r denotes an r-cell. Repeating this process for every generator of $G_1, \dots, G_m$, we see that there is a CW complex M_1 with $\pi_1(M_1) \cong F_n$, an inclusion $i_1 : M \to M_1$, and a map $f_1 : M_1 \to N$ such that $f = f_1 \circ i_1$.

In a similar fashion, if $\beta : S^2 \to M_1$ is a representative for a generator of $\pi_2(M_1)$, then $f_1 \circ \beta$ is nullhomotopic as $\pi_2(N) = 0$, so f_1 extends to $M_1 \cup_\beta e^3$. Repeating this process for all the generators of $\pi_2(M_1)$, we see that there is a CW complex M_2 with $\pi_1(M_2) \cong F_n$ and $\pi_2(M_2) = 0$, an inclusion $i_2 : M \to M_2$, and a map $f_2 : M_2 \to N$ such that $f = f_2 \circ i_2$. Repeating this procedure for all higher homotopy groups, we see that there is an aspherical CW complex M_∞ with $\pi_1(M_\infty) = F_n$, an inclusion $i_\infty : M \to M_\infty$, and a map $f_\infty : M_\infty \to N$ such that $f = f_\infty \circ i_\infty$.

Therefore $f : M \to N$ factors through M_∞ which is homotopically equivalent to a bouquet of n circles. As $H_3(M_\infty; \mathbb{Z}) = 0$, the map f has degree zero. □

Further evidence for the conjecture is provided by a recent result of Schoen and Yau. Theorem 5.2 of [92] implies that if a closed manifold admits a map to a torus of non-zero degree, then it fails to admit metrics of positive scalar curvature.

In dimensions two and three, not only does Conjecture 3.25 hold, manifolds which do not admit metrics of positive scalar curvature are in fact characterised by the property of admitting a map of non-zero degree to an aspherical manifold. This is not the case in higher dimensions; that is, there are manifolds which do not admit positive scalar curvature metrics, but also do not admit a map of non-zero degree to an aspherical manifold.

Example 3.27. Let Θ_n denote the group of n-dimensional oriented homotopy spheres. For $n > 2$ with $n \equiv 1, 2 \bmod 8$, the map[1] $\alpha : \Theta_n \to KO_n(\mathrm{pt}) \cong \mathbb{Z}_2$ is surjective, see [72] and [3]. That is, there are exotic n-dimensional spheres Σ with $\alpha(\Sigma) \neq 0$; these are precisely the exotic spheres which do not bound spin manifolds. By Theorem 3.20, they do not admit metrics of positive scalar curvature, however, for any aspherical manifold M, every map $\Sigma \to M$ is nullhomotopic.

Note, by combining Proposition 1.11 and Theorem 1.16, we see that $Y(\Sigma) = 0$, and by Theorem 3.22, it is not realised.

One might be tempted to replace the non-zero degree map in Conjecture 3.25 with an essential map, i.e. a non-nullhomotopic map. While the conjecture would still hold in dimension two, it already fails in dimension three. For example, the map $f : S^2 \times S^1 \to T^3$ given by $(p, z) \mapsto (z, 1, 1)$ is essential because it induces a non-trivial map on the level of fundamental groups: after identifying $\pi_1(S^2 \times S^1)$ with $\mathbb{Z}$ and $\pi_1(T^3)$ with $\mathbb{Z}^3$, the map f_* is given by $m \mapsto (m, 0, 0)$. However, the manifold $S^2 \times S^1$ admits a metric of positive scalar curvature, e.g. $d\theta^2 + g_{\mathrm{round}}$.

[1] As homotopy spheres in these dimensions are simply connected, they have a unique spin structure, so there is no ambiguity in regarding Θ_n as a subset of Ω_n^{spin}.

4. THE YAMABE INVARIANT OF SOME NON-KÄHLER SURFACES

We begin by stating what is known for Kähler surfaces.

Using Seiberg-Witten theory, LeBrun established the following relationship between the Kodaira dimension of a Kähler surface and the sign of the Yamabe invariant of the underlying smooth four-manifold.

Theorem 4.1. (LeBrun [65]) *Let M be a compact connected Kähler surface. Then*

- $Y(M) > 0$ *if and only if* $\kappa(M) = -\infty$,
- $Y(M) = 0$ *if and only if* $\kappa(M) = 0, 1$, *and*
- $Y(M) < 0$ *if and only if* $\kappa(M) = 2$.

If $\kappa(M) \in \{0, 1\}$ and X is the minimal model of M, then $Y(X)$ is realised if and only if $\kappa(X) = 0$.

If $\kappa(M) = 2$ and X is the minimal model of M, then

$$Y(M) = Y(X) = -4\pi\sqrt{2c_1^2(X)} = -4\pi\sqrt{4\chi(X) + 6\sigma(X)}.$$

The higher-dimensional analogue of this theorem does not hold. For example, if X is a degree 5 hypersurface in $\mathbb{CP}^4$, then it has trivial canonical bundle and hence $\kappa(X) = 0$. However, it admits a metric of positive scalar curvature metric, see Example 3.21, and hence has positive Yamabe invariant by Proposition 1.11.

Note that very little is known in the $\kappa(M) = -\infty$ case. For example, while the Yamabe invariant of M is equal to the Yamabe invariant of its minimal model if $\kappa(M) \geq 0$, it is unclear whether this is also true when $\kappa(M) = -\infty$. One reason to be skeptical of such a result is the fact that minimal models are not unique for Kähler surfaces with $\kappa(M) = -\infty$, e.g. both $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ are minimal models for $\mathbb{CP}^2$ blownup at two points. LeBrun [64] computed the Yamabe invariant of $\mathbb{CP}^2$ and found that $Y(\mathbb{CP}^2) = \mathcal{E}(g_{FS}) = 12\sqrt{2}\pi$ where g_{FS} denotes the Fubini-Study metric. On the other hand, the Yamabe invariant of $S^2 \times S^2$, the smooth four-manifold underlying $\mathbb{CP}^1 \times \mathbb{CP}^1$, remains unknown, although $Y(S^2 \times S^2) > Y(S^2 \times S^2, g_{S^2} + g_{S^2}) = \mathcal{E}(g_{S^2} + g_{S^2}) = 16\pi$; see page 22 of [109]. Note that $12\sqrt{2} \approx 16.971 > 16$, so it could still be the case that $Y(S^2 \times S^2)$ is equal to $Y(\mathbb{CP}^2)$.

Example 4.2. Consider the following three smooth manifolds:

- $M_+ = 3\mathbb{CP}^2 \# 20\overline{\mathbb{CP}^2}$,
- M_0, the blowup of a $K3$ surface at one point, and
- M_-, the blowup of a[1] surface X of general type with $h^{2,0}(X) = 1$ and $c_1(X)^2 = 1$ at two points.

Note that M_+ and M_0 are simply connected with odd intersection form of signature $(b^+, b^-) = (3, 20)$; this is also true of M_-. The fact that M_- is simply connected follows from the fact that X is simply connected, see Proposition 13 of [23]. As M_- is obtained as a blowup, it has odd intersection form. Moreover, we have $b^+(M_-) = b^+(X) = 2h^{2,0}(X) + 1 = 3$ while $b^-(M_-) = b^-(X) + 2$ and from the equation $1 = c_1^2(X) = 2\chi(X) + 3\sigma(X)$, it follows that $b^-(X) = 18$; see Theorem IV.2.7 (ii) and Theorem I.3.1 of [10] respectively.

[1] The existence of such a surface X is not obvious. See [23] and the references therein.

By Freedman's Theorem [32], all three manifolds are homeomorphic. However, no two are diffeomorphic as can be seen by considering their Yamabe invariants. First, as $\mathbb{CP}^2$ admits metrics of positive scalar curvature, so does M_+ by Corollary 3.19 and hence $Y(M_+) > 0$ by Proposition 1.11. In fact, as $Y(\mathbb{CP}^2) = 12\sqrt{2}\pi$, we have $Y(M_+) \geq 12\sqrt{2}\pi$ by Theorem 1.14. On the other hand, by Theorem 4.1 we have $Y(M_0) = 0$ and $Y(M_-) = -4\sqrt{2}\pi < 0$.

We now turn to the non-Kähler world, beginning with Kodaira dimension $-\infty$. That is, class VII surfaces.

4.1 Class VII Surfaces

Up to diffeomorphism, the known non-Kähler surfaces of Kodaira dimension $-\infty$ belong to four families:

1. $(S^1 \times S^3)\#k\overline{\mathbb{CP}^2}$ where $k \geq 0$,
2. $X\#k\overline{\mathbb{CP}^2}$ where X is a secondary Hopf surface and $k \geq 0$,
3. $X\#k\overline{\mathbb{CP}^2}$ where X is an Inoue surface of type S^+_M or S^-_M and $k \geq 0$, or
4. $X\#k\overline{\mathbb{CP}^2}$ where X is an Inoue surface of type $S^+_{N,p,q,r,t}$ or $S^-_{N,p,q,r}$ and $k \geq 0$.

Note, there are infinitely many possibilities for X in families 2, 3, and 4. If the Global Spherical Shell Conjecture is true, the above list is complete. In particular, any class VII surface whose minimal model has $b_2 > 0$ would belong to the first family.

First we record what is known about the Yamabe invariant for the first two families (i.e. Hopf surfaces and their blowups), before moving on to the main result of this thesis, which is the computation of the Yamabe invariant of the manifolds in the third and fourth families.

4.1.1 Hopf Surfaces

As $S^1 \times S^3$ and $\overline{\mathbb{CP}^2}$ admit metrics of positive scalar curvature, so do the manifolds $(S^1 \times S^3)\#k\overline{\mathbb{CP}^2}$ by Corollary 3.19, and hence they have positive Yamabe invariant by Proposition 1.11. Better still, as the Yamabe invariant doesn't depend on orientation, we have $Y(\overline{\mathbb{CP}^2}) = Y(\mathbb{CP}^2) = 12\sqrt{2}\pi$ while $Y(S^1 \times S^3) = Y(S^4) = 8\sqrt{6}\pi$, so by Theorem 1.14, the Yamabe invariant of a primary Hopf surface blownup up at $k > 0$ points is at least $12\sqrt{2}\pi$. For small values of k, we also obtain an upper bound from the following theorem.

Theorem 4.3. (Gursky & LeBrun [38]) *Let $k \in \{1,2,3\}$ and let m be any natural number. Then*

$$12\sqrt{2}\pi \leq Y(k\mathbb{CP}^2\#m(S^1 \times S^3)) \leq 4\pi\sqrt{2k+16}.$$

As $S^1 \times S^3$ admits an orientation-reversing diffeomorphism, we see that $k\overline{\mathbb{CP}^2}\#m(S^1 \times S^3)$ and $k\mathbb{CP}^2\#m(S^1 \times S^3)$ are diffeomorphic. In particular, we have the following corollary.

Corollary 4.4. *Let M be a primary Hopf surface blownup at k points where $k \in \{1,2,3\}$. Then*

$$12\sqrt{2}\pi \leq Y(M) \leq 4\pi\sqrt{2k+16}.$$

For $k \in \{1,2,3\}$, we have $4\pi\sqrt{2k+16} < 4\pi\sqrt{2(4)+16} = 4\pi\sqrt{24} = 8\sqrt{6}\pi = Y(S^4)$, so the upper bound is non-trivial. As $Y(S^1 \times S^3) = Y(S^4)$, the upper bound also shows that $Y(X) \neq Y(S^1 \times S^3)$; that is, the Yamabe invariant of X does not coincide with the Yamabe invariant of its minimal model.

We now move on to secondary Hopf surfaces. The situation is more complicated than the primary case as there are many diffeomorphism types.

Theorem 4.5. (Kato [50], [52]) *For every Hopf surface X, there is a finite subgroup*

$H \subset U(2)$ which acts freely on S^3 such that X is diffeomorphic to one of the following manifolds:

1. *$S^1 \times (S^3/H)$*
2. *a mapping torus of $u : S^3/H \to S^3/H$ where $u \in \mathrm{Diff}(S^3/H)$ has order two or three.*

It follows from the Elliptization Conjecture that S^3/H is a spherical space form. In order to obtain an anologue of Corollary 4.4 for the corresponding secondary Hopf surfaces, we need an anologue of Theorem 4.3.

Theorem 4.6. (Gursky & LeBrun [38]) *Let $X_1, \ldots, X_m$ be three-dimensional spherical space forms, and let*

$$M = k\mathbb{CP}^2\#(S^1 \times X_1)\# \ldots \#(S^1 \times X_m)$$

for $k \in \{1, 2, 3\}$. Then

$$12\sqrt{2}\pi \leq Y(M) \leq 4\pi\sqrt{2k + 16}.$$

Again, as $S^1 \times X_i$ admits an orientation-reversing diffeomorphism, we obtain the following corollary.

Corollary 4.7. *Let M be the blowup of a secondary Hopf surface diffeomorphic to $S^1 \times (S^3/H)$ at k points where $k \in \{1, 2, 3\}$. Then*

$$12\sqrt{2}\pi \leq Y(M) \leq 4\pi\sqrt{2k + 16}.$$

A more careful reading of Kato's papers gives restrictions on the subgroups H and the diffeomorphisms u in Theorem 4.5. A detailed analysis of these results might lead to some restrictions on the Yamabe invariants of blowups of secondary Hopf surfaces, in particular, those diffeomorphic to mapping tori of S^3/H.

Question 4.8. Do all Hopf surfaces have positive Yamabe invariant?

4.1.2 Inoue Surfaces

In this section we prove the main result of this thesis: the Yamabe invariants of Inoue surfaces and their blowups are all zero. Moreover, the Yamabe invariant is not realised. We begin by first noting that work of Paternian and Petean implies that the Yamabe invariant is non-negative.

A $\mathcal{T}$*-structure* on a closed smooth manifold is a finite open covering $\{U_1, \ldots, U_N\}$ and a non-trivial torus action on each U_i such that each intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ is invariant under the torus actions on $U_{i_1}, \ldots, U_{i_k}$, and the torus actions commute. If all of the torus actions are locally free and their orbits on intersections are constant, then the $\mathcal{T}$-structure is called *polarised*. If the dimensions of the orbits is constant across all intersections, then the $\mathcal{T}$-structure is called *pure* and the dimension of the orbits is called the *rank*.

Example 4.9. Consider an Inoue surface X of type S_M. It is the mapping torus of the diffeomorphism of T^3 induced by M^T. If $p : X \to S^1$ is the projection, then let $U_1 = p^{-1}(S^1 \setminus \{1\})$ and $U_2 = p^{-1}(S^1 \setminus \{-1\})$. Note that U_1 and U_2 are both diffeomorphic to $(0,1) \times T^3$ so they admit effective torus actions acting by translations. Moreover, the intersection $U_1 \cap U_2$ is invariant under the torus actions, and as the diffeomorphism $T^3 \to T^3$ is linear, they commute and hence X has a $\mathcal{T}$-structure. Moreover, the orbits are always three-dimensional tori (the fibres of p), so X admits a pure polarised $\mathcal{T}$-structure of rank three.

Paternian and Petean showed that Inoue surfaces of type $S^+_{N,p,q,r,t}$ and $S^-_{N,p,q,r}$ have pure polarised $\mathcal{T}$-structures of rank one, see section 3.2 of [81]. In fact, they show a surface of type $S^+_{N,p,q,r,t}$ admits a locally free S^1-action.

The existence of a $\mathcal{T}$-structure has implications for the Yamabe invariant.

Theorem 4.10. (Paternian & Petean [80]) *If M admits a $\mathcal{T}$-structure, then $Y(M) \geq 0$.*

So for any Inoue surface X, we have $Y(X) \geq 0$. If M is the blowup of X at k points, then M is diffeomorphic to $X \# k\overline{\mathbb{CP}^2}$. As $Y(\overline{\mathbb{CP}^2}) = Y(\mathbb{CP}^2) = 12\sqrt{2}\pi > 0$, it follows from Theorem 1.14 that $Y(M) \geq 0$. Alternatively, note that $\mathbb{CP}^2$ admits a locally free S^1-action and hence a $\mathcal{T}$-structure, so $M = X \# k\overline{\mathbb{CP}^2}$ admits a $\mathcal{T}$-structure by Theorem 5.9 of [80].

Now we just need to show that M does not admit a positive scalar curvature metric. In order to do this, we will need some preliminary results.

Proposition 4.11. *Let F be a closed orientable n-dimensional manifold, and let $p : M_f \to S^1$ be the mapping torus of some orientation-preserving homeomorphism $f : F \to F$. The inclusion $F \hookrightarrow M_f$ induces an injection $H_n(F;\mathbb{Z}) \to H_n(M_f;\mathbb{Z})$.*

Proof. Let $U = p^{-1}(S^1 \setminus \{i\})$ and $V = p^{-1}(S^1 \setminus \{-i\})$. Applying Mayer-Vietoris, we have the following exact sequence in homology (with $\mathbb{Z}$ coefficients)

$$0 \to H_{n+1}(M_f) \to H_n(U \cap V) \xrightarrow{((k_U)_*,(k_V)_*)} H_n(U) \oplus H_n(V) \xrightarrow{(\ell_U)_* - (\ell_V)_*} H_n(M_f) \to \dots$$

where $k_U : U \cap V \to U$, $k_V : U \cap V \to V$, $\ell_U : U \to M_f$, and $\ell_V : V \to M_f$ are inclusion maps.

Note that $H_{n+1}(M_f) \cong \mathbb{Z}$, $H_n(U \cap V) \cong H_n(F \sqcup F) \cong \mathbb{Z}^2$, and $H_n(U) \cong H_n(V) \cong H_n(F) \cong \mathbb{Z}$. As each connected component of $U \cap V$ includes into both U and V, the map $\mathbb{Z}^2 \to \mathbb{Z}^2$ is multiplication by $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. The kernel of this map is the span of $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$, so by exactness, the map $\mathbb{Z} \to \mathbb{Z}^2$ is multiplication by $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$. So we have an exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]} \mathbb{Z}^2 \xrightarrow{\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]} \mathbb{Z}^2 \xrightarrow{\alpha} \mathbb{Z}^r \to \dots$$

where $r = b_n(M_f)$; recall that $H_n(M_f)$ is torsion-free.

Now identify F with $p^{-1}(1)$ and let $i : F \to M_f$ denote the corresponding inclusion. Note that i factors as $i = \ell_U \circ j$ where $j : F \to U$ is the natural inclusion. Consider the map $(j_*, 0) : H_n(F) \to H_n(U) \oplus H_n(V)$. Identifying $H_n(F)$ with $\mathbb{Z}$, this map corresponds to the map $\mathbb{Z} \to \mathbb{Z}^2$ which is multiplication by $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. As $((\ell_U)_* - (\ell_V)_*) \circ (j_*, 0) = (\ell_U)_* j_* - (\ell_V)_* 0 = (\ell_U \circ j)_* = i_*$, we have the commutative diagram

$$\begin{tikzcd}
H_n(F) \arrow[r, "{(j_*,0)}"] \arrow[rr, bend left, "i_*"] \arrow[d, "\cong"] & H_n(U) \oplus H_n(V) \arrow[r, "(\ell_U)_* - (\ell_V)_*"] \arrow[d, "\cong"] & H_n(M_f) \arrow[d, "\cong"] \\
\mathbb{Z} \arrow[r, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"] & \mathbb{Z}^2 \arrow[r, "\alpha"] & \mathbb{Z}^r.
\end{tikzcd}$$

Now note that, by exactness of the previous sequence, the kernel of α is the span of $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, so α is injective on the span of $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. Therefore $i_* : H_n(F) \to H_n(M_f)$ is injective. $\square$

Lemma 4.12. *Let M be a closed connected smooth manifold with a closed connected smooth hypersurface Σ. If Σ is non-orientable, or $[\Sigma] \in H_{n-1}(M;\mathbb{Z})$ is non-zero, then $M \setminus \Sigma$ is connected.*

Proof. Let U be a tubular neighbourhood of Σ in M, and let $V = M \setminus \Sigma$. Then by Mayer-Vietoris, we have

$$\cdots \to H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to H_0(M) \to 0.$$

Note that $H_0(U) \cong H_0(M) \cong \mathbb{Z}$ as Σ and M are connected, and $H_0(V) \cong \mathbb{Z}^k$ where k is the number of connected components of V. Also note that $U \cap V$ deformation retracts onto the orientation double cover of Σ, so if Σ is non-orientable we have

$$\cdots \to \mathbb{Z} \to \mathbb{Z}^{k+1} \to \mathbb{Z} \to 0$$

from which it immediately follows that $k = 1$, i.e. $V = M \setminus \Sigma$ is connected. If Σ is orientable, we instead have

$$\cdots \to \mathbb{Z}^2 \to \mathbb{Z}^{k+1} \to \mathbb{Z} \to 0$$

which implies that $k \leq 2$. Suppose that $k = 2$.

As Σ and M are orientable, the normal bundle is trivial, so there is a diffeomorphism $\phi : U \to (-1,1) \times \Sigma$ such that $\phi|_\Sigma$ is the inverse of the inclusion $i : \Sigma \to M$. Let M_- and M_+ be the connected components of $M \setminus \Sigma$ where $\phi(M_- \cap U) = (-1,0) \times \Sigma$ and $\phi(M_+ \cap U) = (0,1) \times \Sigma$. Let $f : (-1,1) \to \mathbb{R}$ be a non-decreasing smooth function such that $f|_{(-1,-\frac{1}{2})} \equiv 0$ and $f|_{(\frac{1}{2},1)} \equiv 1$. Let $\hat{f}$ be the function given by $\hat{f}(p) = f(\mathrm{pr}_1(\phi(p)))$ for $p \in U$, and extended by 0 on $M_- \setminus U$ and 1 on $M_+ \setminus U$. Then $d\hat{f}$ is a smooth one-form such that $d\hat{f}|_{M \setminus U} = 0$ and $(\phi^{-1})^*(d\hat{f}|_U) = g(t)dt$ where $g : (-1,1) \to \mathbb{R}$ is a non-negative smooth function such that $g|_{(-1,-\frac{1}{2}) \cup (\frac{1}{2},1)} \equiv 0$ and $\int_{-1}^{1} g(t)dt = 1$. It follows that for any $(n-1)$-form η on M, we have $\int_M \eta \wedge d\hat{f} = \int_\Sigma i^*\eta$, so $d\tilde{f}$ is a representative of the Poincaré dual of $[\Sigma]$ under the map $\Phi : H^1(M;\mathbb{Z}) \to H^1_{\mathrm{dR}}(M)$ given by change of coefficients followed by the de Rham isomorphism. As $H^1(M;\mathbb{Z})$ is torsion-free, the map Φ is injective, so $\mathrm{PD}([\Sigma]) = 0$ as $[d\hat{f}] = 0$ in $H^1_{\mathrm{dR}}(M)$. Finally, as $H_{n-1}(M;\mathbb{Z})$ is torsion-free, we see that $[\Sigma] = 0$. □

Theorem 4.13. *Inoue surfaces and their blowups do not admit positive scalar curvature metrics.*

Proof. As a blownup Inoue surface of type $S^-_{N,p,q,r}$ is double covered by a blownup Inoue surface of type $S^+_{N,p,q,r,t}$, we only need to consider X of type S^+_M, S^-_M, or $S^+_{N,p,q,r,t}$. Let $\pi : M \to X$ be the blowdown map and $p : X \to S^1$ the fibre bundle projection with fibre F, which is either a three-dimensional torus or a circle bundle over a two-dimensional torus.

Let g be a positive scalar curvature metric on M. As $[F] \neq 0$ by Proposition 4.11, there is a stable minimal hypersurface Σ with $[\Sigma] = [F]$ by Theorem 3.7. Let $\Sigma_1, \dots, \Sigma_\ell$ be the connected components of Σ. Recall that Σ minimises area amongst all hypersurfaces representing $[F]$, so we must have $[\Sigma_j] \neq 0$ otherwise $\Sigma_1 \sqcup \cdots \sqcup \Sigma_{j-1} \sqcup \Sigma_{j+1} \sqcup \cdots \sqcup \Sigma_\ell$ is a hypersurface with strictly smaller area which also represents $[F]$. As $b_3(M) = b_3(X) = 1$, there is a non-zero integer n_j such that $[\Sigma_j] = n_j[F]$.

As Σ_j and M are orientable, the normal bundle is trivial, so there is an embedding $\phi : (-1,1) \times \Sigma_j \to M$ such that $\phi(0,\cdot)$ is the inclusion $i : \Sigma_j \hookrightarrow M$. Note that $M \setminus \Sigma_j$ is a connected non-compact manifold by Lemma 4.12. Let $\phi_- : (-1,0) \times \Sigma_j \to M \setminus \Sigma_j$ and $\phi_+ : (0,1) \times \Sigma_j \to M \setminus \Sigma_j$ be the embeddings given by restricting ϕ appropriately. Let $Y = (\phi([0,1) \times \Sigma_j) \sqcup (M \setminus \Sigma_j) \sqcup \phi((-1,0] \times \Sigma_j))/\sim$ where $\phi_+(t,p) \sim \phi(t,p)$ for $(t,p) \in (0,1) \times \Sigma_j$ and $\phi_-(t,p) \sim \phi(t,p)$ for $(t,p) \in (-1,0) \times \Sigma_j$. Note that Y is a manifold with boundary $\partial Y = \Sigma_j \sqcup \Sigma_j$ and the interior of Y is diffeomorphic to $M \setminus \Sigma_j$.

Fix a diffeomorphism $f : \Sigma_j \times \{0,1\} \to \partial Y$. Now let $Z = (Y \times \mathbb{Z})/\sim$ where $(f(s,1), m) \sim (f(s,0), m+1)$ for all $s \in \Sigma_j$ and $m \in \mathbb{Z}$. Note that the self-map of $Y \times \mathbb{Z}$ given by $(y,m) \mapsto (y, m+1)$ descends to a fixed-point free diffeomorphism of Z. The quotient of Z by the $\mathbb{Z}$-action generated by this diffeomorphism is naturally identified with M, so we have a regular covering $\rho : Z \to M$ with group of deck transformations $\mathbb{Z}$. Hence, there is a short exact sequence

$$0 \to \pi_1(Z) \xrightarrow{\rho_*} \pi_1(M) \xrightarrow{\alpha} \mathbb{Z} \to 0.$$

Using the fact that X is a mapping torus, we can construct another such covering. First note that there is a regular covering $F \times \mathbb{R} \to X$ with deck transformation group $\mathbb{Z}$ generated by $(x,t) \mapsto (f(x), t+1)$ where f is the diffeomorphism which gives rise to the mapping torus X. There is a corresponding regular covering $\tau : M' \to M$ where M' is diffeomorphic to the connected sum of $F \times \mathbb{R}$ with a copy of $k\overline{\mathbb{CP}^2}$ at $(p, m + \frac{1}{2})$ for some fixed $p \in F$ and all $m \in \mathbb{Z}$. Hence, there is a short exact sequence

$$0 \to \pi_1(M') \xrightarrow{\tau_*} \pi_1(M) \xrightarrow{\beta} \mathbb{Z} \to 0.$$

As $b_1(M) = b_1(X) = 1$, we see that $\mathrm{Hom}(\pi_1(M), \mathbb{Z}) \cong H^1(M; \mathbb{Z}) \cong \mathbb{Z}$. As α and β are non-zero, there is $k \in \mathbb{Z} \setminus \{0\}$ such that $\alpha = k\beta$ or $\beta = k\alpha$; without loss of generality, suppose $\alpha = k\beta$. As X is aspherical and $\pi_1(M) \cong \pi_1(X)$, the group $\pi_1(M)$ is torsion-free by Proposition 3.23, so $\pi_1(Z) \cong \ker(\alpha) = \ker(k\beta) = \ker(\beta) \cong \pi_1(M')$. Therefore, the two coverings are isomorphic; let $\Psi : Z \to M'$ be an isomorphism of covering spaces.

As $\phi(0, s) = i(s)$, it follows from the construction of Y and Z that there is an inclusion $\iota : \Sigma_j \to Z$ such that $\rho \circ \iota = i$. So we have the following commutative diagram:

$$\begin{tikzcd}
\Sigma_j \arrow[r, "\iota"] \arrow[drr, "i"'] & Z \arrow[r, "\Psi"] \arrow[dr, "\rho"] & M' \arrow[d, "\tau"] \\
& & M
\end{tikzcd}$$

As $i_*[\Sigma_j] = n_j[F] \neq 0$, we see that $(\Psi \circ \iota)_*[\Sigma_j] \neq 0$. In fact, as τ_* is an isomorphism on H_3, we see that $(\Psi \circ \iota)_*[\Sigma_j] = n_j[F]$. Note that there is a map $g : M' \to F$ by first mapping to $F \times \mathbb{R}$ then projecting onto F, and g_* is an isomorphism on H_3. Therefore $g \circ \Psi \circ \iota : \Sigma_j \to F$ is a map of three-manifolds with $(g \circ \Psi \circ \iota)_*[\Sigma_j] = n_j[F]$; that is, the map has degree $n_j \neq 0$. As F is enlargeable, see Example 3.9 and Example 3.13, so is Σ_j by Theorem 3.11 (c). Moreover, as Σ_j is a closed orientable three-manifold, it is spin, and hence does not admit a metric of positive scalar curvature by Theorem 3.14. This contradicts Theorem 3.8, and hence M does not admit any metrics of positive scalar curvature. □

Remark 4.14. Note that Theorem 3.8 states that if g were a positive scalar curvature metric, then the restriction metric $g|_\Sigma$ would be conformal to a metric with positive scalar curvature. The above proof demonstrates that cannot occur, but actually shows much more. It shows that Σ does not admit any positive scalar curvature metric at all. Better still, no connected component of Σ admits a positive scalar curvature metric. In the language of [91], the proof above shows that M is not in the class C'_4.

Also note that instead of deducing that Σ_j is enlargeable in order to show it does not admit metrics of positive scalar curvature, we could have used Proposition 3.26.

The only question which remains is whether the Yamabe invariant is realised. By Corollary 3.4, it is realised if and only if M admits a Ricci-flat metric.

Theorem 4.15. (LeBrun [66]) *Let M be the underlying smooth 4-manifold of a compact complex surface. Then M admits an Einstein metric with $\lambda \geq 0$ if and only if it is diffeomorphic to one of the following: a del Pezzo surface, a K3 surface, an Enriques surface, a torus, or a hyper-elliptic surface.*

In particular, we have the following.

Corollary 4.16. *Non-Kähler surfaces do not admit Ricci-flat metrics.*

Therefore, the Yamabe invariants of Inoue surfaces and their blowups are never realised. Combining all the elements of this section, we finally arrive at the following theorem.

Theorem 4.17. *Inoue surfaces and their blowups have Yamabe invariant zero. Moreover, the Yamabe invariant is not realised.*

In particular, unlike the Kähler case, the sign of the Yamabe invariant of a non-Kähler surface is not determined by its Kodaira dimension.

4.2 Kodaira Surfaces

We take this opportunity to record the value of the Yamabe invariant of Kodaira surfaces and their blowups; the argument was outlined in [65]. Let M be the blowup of a Kodaira surface X at k points, then M is diffeomorphic to $X\#k\overline{\mathbb{CP}^2}$.

LeBrun showed that every elliptic surface has non-negative Yamabe invariant, see Corollary 1 of [65]. As every Kodaira surface is deformation equivalent to an elliptic surface, we see that $Y(X) \geq 0$. Again, as $Y(\overline{\mathbb{CP}^2}) = 12\sqrt{2}\pi > 0$, we have $Y(M) \geq 0$ by Theorem 1.14. The conclusion also follows from the fact that every elliptic surface has a $\mathcal{T}$-structure, see Theorem 2.4 of [81].

Proposition 4.18. *Every primary Kodaira surface admits a symplectic form.*

Proof. Let X be a primary Kodaira surface. As K_X is holomorphically trivial, there is a nowhere-zero holomorphic two-form α; let $\omega = 2\,\mathrm{Re}(\alpha) = \alpha + \overline{\alpha}$. As α is holomorphic, $\bar{\partial}\alpha = 0$ while $\partial\alpha = 0$ for bidegree reasons, so α is closed and hence so is ω. In local holomorphic coordinates $(U, (z^1, z^2))$, we have $\alpha|_U = f dz^1 \wedge dz^2$ where $f : U \to \mathbb{C}$ is a nowhere-zero holomorphic function. So $\omega^2|_U = \alpha \wedge \overline{\alpha}\,|_U = |f|^2 dz^1 \wedge dz^2 \wedge d\bar{z}^1 \wedge d\bar{z}^2$ which is nowhere-zero, hence ω is a non-degenerate closed two-form, i.e. a symplectic form. □

Note that M admits a symplectic form as it is diffeomorphic to the symplectic blowup of X at k points. As $b^+(M) = 2h^{2,0}(M) = 2$, see Theorem 2.7 (iii) of [10], it follows from Corollary 3.17 that a blownup primary Kodaira surface does not admit positive scalar curvature metrics. As blownup secondary Kodaira surfaces are finitely covered by blownup primary Kodaira surfaces, they also fail to admit positive scalar curvature metrics. Therefore $Y(M) = 0$. Again, by Corollary 3.4, if the Yamabe invariant were realised, it would be Ricci-flat, but this is impossible by Corollary 4.16.

Theorem 4.19. *Kodaira surfaces and their blowups have Yamabe invariant zero. Moreover, the Yamabe invariant is not realised.*

Note that the arguments used to show that the Yamabe invariants of Kodaira surfaces and their blowups are non-negative also apply to non-Kähler properly elliptic surfaces and their blowups. However, the argument to rule out the existence of positive scalar curvature metrics does not carry over; it was shown by Biquard that non-Kähler properly elliptic surfaces are never symplectic, see Theorem 8.2 of [14]. Biquard achieves this by computing the Seiberg-Witten invariants on such a surface, and showing that none of them are ± 1. However, in order to show that the Yamabe invariant is zero, it would be enough to know that there is a non-zero Seiberg-Witten invariant – it may be possible that a more careful analysis of Biquard's paper may shed light on the existence of such an invariant.

Question 4.20. Are the Yamabe invariants of non-Kähler properly elliptic surfaces and their blowups all zero?

BIBLIOGRAPHY

[1] Abramovich, D., Karu, K., Matsuki, K. and Wodarczyk, J., 2002. *Torification and factorization of birational maps*. Journal of the American Mathematical Society, 15(3), pp.531-572.

[2] Abramowitz, M. and Stegun, I.A., 1965. *Handbook of mathematical functions: with formulas, graphs, and mathematical tables* (Vol. 55). Courier Corporation.

[3] Adams, J.F., 1966. *On the groups $J(X)$ – IV*. Topology, 5(1), pp.21-71.

[4] Akutagawa, K. and Neves, A., 2007. *3-manifolds with Yamabe invariant greater than that of $\mathbb{RP}^3$*. Journal of Differential Geometry, 75(3), pp.359-386.

[5] Anderson, M.T., 2005. *On uniqueness and differentiability in the space of Yamabe metrics*. Communications in Contemporary Mathematics, 7(03), pp.299-310.

[6] Anderson, M.T., 2006. *Canonical Metrics on 3-Manifolds and 4-Manifolds*. Asian Journal of Mathematics, 10(1), pp.127-164.

[7] Aubin, T., 1970. *Métriques riemanniennes et courbure*. Journal of Differential Geometry, 4(4), pp.383-424.

[8] Aubin, T., 1976. *Équations différentielles non linéaires et probleme de Yamabe concernant la courbure scalaire*. J. Math. Pures Appl.(9), 55, pp.269-296.

[9] Aubin, T., 1976. *The scalar curvature*. In "Differential Geometry and Relativity" (pp. 5-18). Springer, Dordrecht.

[10] Barth, W., Hulek, K., Peters, C. and van de Ven, A., 2003. *Compact Complex Surfaces* (Vol. 4). Springer Science & Business Media.

[11] Beauville, A., 1996. *Complex algebraic surfaces* (No. 34). Cambridge University

Press.

[12] Berger, M., 1965. *Sur les variétés d'Einstein compactes.* Comptes Rendus de la IIIe Réunion du Groupement des Mathématiciens d'Expression Latine (Namur, 1965), pp.35-55.

[13] Besse, A.L., 2007. *Einstein manifolds.* Springer Science & Business Media.

[14] Biquard, O., 1998. *Les équations de SeibergWitten sur une surface complexe non Kählérienne.* Communications in Analysis and Geometry, 6(1), pp.173-197.

[15] Bogomolov, F., 1976. *Classification of surfaces of class VII_0 with $b_2 = 0$.* Mathematics of the USSR-Izvestija, 10(2), pp.255-269.

[16] Bogomolov, F., 1982. *Surfaces of class VII_0 and affine geometry.* Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya, 46(4), pp.710-761.

[17] Bombieri, E. and Mumford, D., 1977. *Enriques classification of surfaces in char. p, II*, Complex Analysis and Algebraic Geometry–Collection of papers dedicated to K. Kodaira, ed. WL Baily and T. Shioda, Iwanami and Cambridge Univ. Press (1977), pp.23-42.

[18] Bray, H.L. and Neves, A., 2004. *Classification of prime 3-Manifolds with σ-invariant greater than $\mathbb{RP}^3$.* Annals of mathematics, pp.407-424.

[19] Brendle, S., 2008. *Blow-up phenomena for the Yamabe equation.* Journal of the American Mathematical Society, 21(4), pp.951-979.

[20] Brendle, S. and Marques, F.C., 2009. *Blow-up phenomena for the Yamabe equation II.* Journal of Differential Geometry, 81(2), pp.225-250.

[21] Buchdahl, N., 1999. *On compact Kähler surfaces.* In Annales de linstitut Fourier, volume 49, pp.287-302.

[22] Cairns, S.S., 1935. *Triangulation of the manifold of class one.* Bulletin of the American Mathematical Society, 41(8), pp.549-552.

[23] Catanese, F., 1979. *Surfaces with $K^2 = p_g = 1$ and their period mapping.* In Algebraic geometry (pp. 1-29). Springer, Berlin, Heidelberg.

[24] Cecchini, S. and Schick, T., 2018. *Enlargeable metrics on nonspin manifolds.* arXiv preprint `arXiv:1810.02116`.

[25] Cheeger, J., 1970. *Finiteness theorems for Riemannian manifolds*. American Journal of Mathematics, 92(1), pp.61-74.

[26] Colding, T.H. and Minicozzi, W.P., 2011. *A course in minimal surfaces* (Vol. 121). American Mathematical Soc.

[27] Dloussky, G., Oeljeklaus, K. and Toma, M., 2003. *Class VII_0 surfaces with b_2 curves*. Tohoku Mathematical Journal, Second Series, 55(2), pp.283-309.

[28] Enoki, I., 1981. *Surfaces of class VII_0 with curves*. Tohoku Mathematical Journal, Second Series, 33(4), pp.453-492.

[29] Endo, H, and Pajitnov, A, 2019. *On generalized Inoue manifolds*. arXiv preprint `arXiv:1903.08030`.

[30] Endo, H. and Pajitnov, A., 2019. *On a generalization of Inoue and Oeljeklaus-Toma manifolds*. arXiv preprint `arXiv:1906.07401`.

[31] Fischer, A.E. and Wolf, J.A., 1975. *The structure of compact Ricci-flat Riemannian manifolds*. Journal of Differential Geometry, 10(2), pp.277-288.

[32] Freedman, M.H., 1982. *The topology of four-dimensional manifolds*. J. Differential Geom, 17(3), pp.357-453.

[33] Futaki, A., 1993. *Scalar-flat closed manifolds not admitting positive scalar curvature metrics*. Inventiones mathematicae, 112(1), pp.23-29.

[34] Gromov, M., 1982. *Volume and bounded cohomology*. Publications Mathématiques de l'IHÉS, 56, pp.5-99.

[35] Gromov, M. and Lawson, H.B., 1980. *Spin and scalar curvature in the presence of a fundamental group. I*. Ann. of Math, 111(2), pp.209-230.

[36] Gromov, M. and Lawson, H.B., 1980. *The classification of simply connected manifolds of positive scalar curvature*. Annals of Mathematics, pp.423-434.

[37] Gromov, M. and Lawson, H.B., 1983. *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*. Publications Mathmatiques de l'IHÉS, 58, pp.83-196.

[38] Gursky, M. and LeBrun, C, 1998. *Yamabe Invariants and $Spin^c$ Structures*. Geometric & Functional Analysis GAFA, 8(6), pp. 965-977.

[39] Harvey, R. and Lawson, H.B., 1983. *An intrinsic characterization of Kähler manifolds*. Inventiones mathematicae, 74(2), pp.169-198.

[40] Hatcher, A., 2002. *Algebraic Topology*. Cambridge University Press.

[41] Hitchin, N., 1974. *On compact four-dimensional Einstein manifolds*, J. Diff. Geom. 9, pp. 435-442.

[42] Hitchin, N., 1974. *Harmonic Spinors*. Adv. in Math., 14, pp.1-55.

[43] Hopf, H., 1948. *Zur topologie der komplexen Mannigfaltigkeiten*. In Studies and Essays Presented to R. Courant on his 60th Birthday, pp.167-185.

[44] Huybrechts, D., 2006. *Complex Geometry: An Introduction*. Springer Science & Business Media.

[45] Huybrechts, D., 2016. *Lectures on K3 surfaces* (Vol. 158). Cambridge University Press.

[46] Iitaka, S., 1982. *Algebraic Geometry: An Introduction to Birational Geometry of Algebraic Varieties*, volume 76 of Graduate Texts in Mathematics.

[47] Inoue, M., 1974. *On surfaces of class* VII_0. Inventiones mathematicae, 24(4): pp.269-310.

[48] Inoue, M., 1975. *An example of an analytic surface*. Sugaku, 27, pp.358-364.

[49] Inoue, M., 1977. *New surfaces with no meromorphic functions. II.* Complex analysis and algebraic geometry, pp.91-106.

[50] Kato, M., 1975. *Topology of Hopf surfaces*. Journal of the Mathematical Society of Japan, 27(2), pp.222-238.

[51] Kato, M., 1978. *Compact complex manifolds containing "global" spherical shells. I*. Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977), pp.45-84.

[52] Kato, M., 1989. *Erratum to "Topology of Hopf surfaces"*. Journal of the Mathematical Society of Japan, 41(1), pp.173-174.

[53] Kazdan, J.L. and Warner, F.W., 1975. *Scalar curvature and conformal deformation of Riemannian structure*. Journal of Differential Geometry, 10(1), pp.113-134.

[54] Kazdan, J. and Warner, F., 1975. *Prescribing curvatures.* In Proceedings of Symposia in Pure Mathematics (Vol. 27, pp. 309-319).

[55] Kobayashi, O., 1987. *Scalar curvature of a metric with unit volume.* Mathematische Annalen, 279(2), pp.253-265.

[56] Kodaira, K., 1966. *On the structure of compact complex analytic surfaces, II.* American Journal of Mathematics, 88(3), pp.682-721.

[57] Kotschick, D., 1998. *On the Gromov-Hitchin-Thorpe inequality.* Comptes Rendus de l'Acadmie des Sciences-Series I-Mathematics, 326(6), pp.727-731.

[58] Kotschick, D., 2012. *Entropies, volumes, and Einstein metrics.* In Global differential geometry (pp. 39-54). Springer, Berlin, Heidelberg.

[59] Lamari, A., 1999. *Courants kählériens et surfaces compactes.* In Annales de linstitut Fourier, volume 49, pp.263-285.

[60] Lattès, S., 1911. *Sur les formes réduites des transformations ponctuelles dans le domaine d'un point double.* Bulletin de la Société Mathématique de France, 39, pp.309-345.

[61] Laufer, H.B., 1971. *Normal two-dimensional singularities.* Princeton University Press.

[62] Lawson, H.B., 1974. *Minimal varieties in real and complex geometry* (Vol. 57). Les Presses de l'Université de Montrél.

[63] Lawson, H.B. and Michelsohn, M.L., 1989. *Spin geometry* (pms-38) (Vol. 38). Princeton university press.

[64] LeBrun, C., 1997. *Yamabe constants and perturbed Seiberg-Witten equations.* Comm. Anal. Geom., 5, pp.535-553.

[65] LeBrun, C., 1999. *Kodaira dimension and the Yamabe problem.* Communications in Analysis and Geometry, 7, pp.135-156.

[66] LeBrun, C., 2009. *Einstein metrics, complex surfaces, and symplectic 4-manifolds.* In Mathematical Proceedings of the Cambridge Philosophical Society (Vol. 147, No. 1, pp. 1-8). Cambridge University Press.

[67] Lee, J.M., 2006. *Riemannian manifolds: an introduction to curvature* (Vol. 176).

Springer Science & Business Media.

[68] Lichnerowicz, A., 1963. *Spineurs harmoniques*. CR Acad. Sci. Paris Série AB, 257, pp.7-9.

[69] Lohkamp, J., 1994. *Metrics of negative Ricci curvature*. Annals of Mathematics, 140(3), pp.655-683.

[70] Li, J., Yau, S.T. and Zheng, F., 1990. *A simple proof of Bogomolov's Theorem on class VII_0 surfaces with* $b_2 = 0$. Illinois Journal of Mathematics, 34(2), pp.217-220.

[71] Macbeth, H.R., 2017. *Conformal classes realizing the Yamabe invariant*. International Mathematics Research Notices, 2019(5), pp.1333-1349.

[72] Milnor, J., 1965. *Remarks concerning spin manifolds*. In Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse) (pp. 55-62). Princeton Univ. Press.

[73] Miyaoka, Y., 1974. *Kähler metrics on elliptic surfaces*. Proceedings of the Japan Academy, 50(8), pp.533-536.

[74] Moise, E.E., 1952. *Affine structures in* 3*-manifolds: V. The triangulation theorem and Hauptvermutung*. Annals of mathematics, pp.96-114.

[75] Morgan, J.W., 1996. *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds* (Vol. 44). Princeton University Press.

[76] Morrow, J.A. and Kodaira, K., 2006. *Complex manifolds* (Vol. 355). American Mathematical Soc.

[77] Mumford, D., 1970. *Abelian varieties*. Tata Studies in Mathematics.

[78] Nakamura, I., 1984. *On surfaces of class VII_0 with curves*. Inventiones mathematicae, 78(3), pp.393-443.

[79] Obata, M., 1971. *The conjectures on conformal transformations of Riemannian manifolds*. Bulletin of the American Mathematical Society, 77(2), pp.265-270.

[80] Paternain, G.P. and Petean, J., 2003. *Minimal entropy and collapsing with curvature bounded from below*. Inventiones mathematicae, 151(2), pp.415-450.

[81] Paternain, G.P. and Petean, J., 2004. *Entropy and collapsing of compact complex*

surfaces. Proceedings of the London Mathematical Society, 89(3), pp.763-786.

[82] Perelman, G., 2002. *The entropy formula for the Ricci flow and its geometric applications*. arXiv preprint `arXiv:math/0211159`.

[83] Perelman, G., 2003. *Ricci flow with surgery on three-manifolds*. arXiv preprint `arXiv:math/0303109`.

[84] Perelman, G., 2003. *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*. arXiv preprint `arXiv:math/0307245`.

[85] Petean, J, 2000. *The Yamabe invariant of simply connected manifolds*. J. Reine Angew. Math. 523, pp.225-231.

[86] Peters, S., 1984. *Cheeger's finiteness theorem for diffeomorphism classes of Riemannian manifolds*. Journal für die reine und angewandte Mathematik, 349, pp.77-82.

[87] Pollack, D., 1993. *Nonuniqueness and high energy solutions for a conformally invariant scalar equation*. Communications in Analysis and Geometry, 1(3), pp.347-414.

[88] Radó, T., 1925. *Über den begriff der Riemannschen fläche*. Acta Litt. Sci. Szeged, 2, pp.101-121.

[89] Schoen, R., 1984. *Conformal deformation of a Riemannian metric to constant scalar curvature*. Journal of Differential Geometry, 20(2), pp.479-495.

[90] Schoen, R., 1989. *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*. In "Topics in calculus of variations" (pp.120-154). Springer, Berlin, Heidelberg.

[91] Schoen, R. and Yau, S.T., 1979. *On the structure of manifolds with positive scalar curvature*. Manuscripta mathematica, 28(1), pp.159-183.

[92] Schoen, R. and Yau, S.T., 2017. *Positive scalar curvature and minimal hypersurface singularities*. arXiv preprint `arXiv:1704.05490`.

[93] Scorpan, A., 2005. *The Wild World of 4-Manifolds*. American Mathematical Soc..

[94] Seiberg, N. and Witten, E., 1994. *Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*. Nuclear

Physics B, 426(1), pp.19-52.

[95] Seiberg, N. and Witten, E., 1994. *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*. Nuclear Physics B, 431(3), pp.484-550.

[96] Siu, Y.T., 1983. *Every K3 surface is Kähler*. Inventiones mathematicae, 73(1), pp.139-150.

[97] Smale, N., 1993. *Generic regularity of homologically area minimizing hypersurfaces in eight dimensional manifolds*. Communications in Analysis and Geometry, 1(2), pp.217-228.

[98] Sternberg, S., 1957. *Local contractions and a theorem of Poincaré*. American Journal of Mathematics, 79(4), pp.809-824.

[99] Stolz, S., 1992. *Simply connected manifolds of positive scalar curvature*. Annals of Mathematics, 136(3), pp.511-540.

[100] Sung, C., 2009. *Surgery, Yamabe invariant, and Seiberg-Witten theory*. Journal of Geometry and Physics, 59(2), pp.246-255.

[101] Şuvaina, I., 2015. *The Yamabe invariant of a class of symplectic 4-manifolds*. Journal of Geometry and Physics, 91, pp.54-65.

[102] Taubes, C.H., 1994. *The Seiberg-Witten invariants and symplectic forms*. Mathematical Research Letters, 1(6), pp.809-822.

[103] Teicher, M., 1999. *Hirzebruch surfaces: degenerations, related braid monodromy, Galois covers*. Contemporary Mathematics, 241, pp.305-326.

[104] Teleman, A., 1994. *Projectively flat surfaces and Bogomolov's theorem on class VII_0 surfaces*. International Journal of Mathematics, 5(02), pp.253-264.

[105] Teleman, A., 2005. *Donaldson theory on non-Kählerian surfaces and class VII surfaces with $b_2 = 1$*. Inventiones mathematicae, 162(3), pp.493-521.

[106] Teleman, A., 2010. *Instantons and curves on class VII surfaces*. Annals of Mathematics, 172, pp.1749-1804.

[107] Teleman, A., 2018. *Donaldson theory in non-Kählerian geometry*, in "Modern Geometry: A Celebration of the Work of Simon Donaldson", Proceedings of Symposia in Pure Mathematics, Vol. 99, pp.363-392.

[108] Trudinger, N., 1968. *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds.* Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 22(2): pp.265-274.

[109] Viaclovsky, J.A., 2014. Critical metrics for Riemannian curvature functionals. arXiv preprint `arXiv:1405.6080`.

[110] Whitehead, J.H.C., 1940. *On C^1-complexes.* Annals of Mathematics, pp.809-824.

[111] Yamabe, H., 1960. *On a deformation of Riemannian structures on compact manifolds.* Osaka Mathematical Journal, 12(1): 21-37.

[112] Yau, S.T., 1978. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampére equation, I.* Communications on pure and applied mathematics, 31(3), pp.339-411.