ALMOST COMPLEX MANIFOLDS – ASSIGNMENT 4 SOLUTIONS

1. Show that a principal $U(n)$-bundle P admits a reduction of structure group to $SU(n)$ if and only if $c_1(P) = 0$.

Solution: There is a short exact sequence $1 \to SU(n) \xrightarrow{i} U(n) \to S^1 \to 1$ so we have $U(n)/SU(n) = S^1$ and hence there is a fiber bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & BSU(n) \\ & & \big\downarrow{\scriptstyle Bi} \\ & & BU(n) \end{array}$$

which is classified by a map $BU(n) \to BS^1 = BK(\mathbb{Z}, 1) = K(\mathbb{Z}, 2)$ and therefore corresponds to an element $k_2 \in H^2(BU(n); \mathbb{Z})$. This is the Moore-Postnikov tower of the map $Bi : BSU(n) \to BU(n)$ and k_2 is the only non-trivial k-invariant.

Note that $\pi_1(BSU(n)) \cong \pi_0(SU(n)) = 0$, and $\pi_2(BSU(n)) \cong \pi_1(SU(n)) = 0$ so $H_1(BSU(n); \mathbb{Z}) = 0$ and $H_2(BSU(n); \mathbb{Z}) = 0$ by the Hurewicz Theorem. Using the Universal Coefficient Theorem, we see that $H^2(BSU(n); \mathbb{Z}) \cong \text{Hom}(H_2(BSU(n); \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}(H_1(BSU(n); \mathbb{Z}), \mathbb{Z}) = 0$. So $(Bi)^* : H^2(BU(n); \mathbb{Z}) \to H^2(BSU(n); \mathbb{Z})$ is the zero map. As k_2 generates the kernel of $(Bi)^* : H^2(BU(n); \mathbb{Z}) \to H^2(BSU(n); \mathbb{Z})$, namely $H^2(BU(n); \mathbb{Z})$, we see that $k_2 = \pm c_1$ (we can arrange for it to be c_1).

A principal $U(n)$-bundle $P \to X$ admits a reduction of structure group to $SU(n)$ if and only if its classifying map $\phi_P : X \to BU(n)$ lifts to a classifying map $\widetilde{\phi_P} : X \to BSU(n)$.

$$\begin{array}{ccccc} & \overset{\widetilde{\phi_P}}{\dashrightarrow} & BSU(n) & & \\ & & \big\downarrow{\scriptstyle Bi} & & \\ X & \xrightarrow{\phi_P} & BU(n) & \xrightarrow{c_1} & K(\mathbb{Z}, 2) \end{array}$$

Such a lift exists if and only if $\phi_P^* k_2 = \phi_P^* c_1 = c_1(P)$ vanishes.

2. Let M be a $2n$-dimensional almost complex manifold. In question 6 of assignment 3, you were asked to show that there are two potential obstructions to TM admitting a complex line subbundle. Show that the first one vanishes.

Solution: Recall that TM admits a complex line subbundle if and only if $P_{U(n)}(TM)$ admits a reduction of structure group to $U(n-1) \times U(1)$. The two potential obstructions are $\gamma_3 \in H^3(M; \mathbb{Z})$ and $\gamma_{2n} \in H^{2n}(M; \mathbb{Z})$. We have $U(n)/(U(n-1) \times U(1)) = \mathbb{CP}^{n-1}$, so the Moore-Postnikov tower of

$B(U(n-1)\times U(1)) \to BU(n)$ takes the form

$$\begin{array}{ccccc}
 & & B(U(n-1)\times U(1)) & & \\
 & & \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
K(\pi_m(\mathbb{CP}^{n-1}), m) & \longrightarrow & E_{m+1} & & \\
 & & \downarrow & & \\
 & & E_m & \xrightarrow{k_{m+1}} & K(\pi_m(\mathbb{CP}^{n-1}), m+1) \\
 & & \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
 & & BU(n) & &
\end{array}$$

The first non-zero homotopy group of $\mathbb{CP}^{n-1}$ is $\pi_2(\mathbb{CP}^{n-1}) \cong \mathbb{Z}$. Therefore the first stage of the tower is

$$\begin{array}{ccccc}
K(\mathbb{Z},2) & \longrightarrow & E_3 & & \\
 & & \downarrow & & \\
 & & BU(2) & \xrightarrow{k_3} & K(\mathbb{Z},3)
\end{array}$$

which is classified by an element $k_3 \in H^3(BU(2);\mathbb{Z})$. Since $H^3(BU(2);\mathbb{Z}) = 0$, we have $k_3 = 0$. Since γ_3 is the pullback of k_3, we see that $\gamma_3 = 0$.

3. Consider the Lie group G_2. It can be defined in many different ways. For example, as the simply connected Lie group with Lie algebra $\mathfrak{g}_2$, or as the subgroup of $GL^+(7,\mathbb{R})$ fixing a particular 3-form on $\mathbb{R}^7$.

Let $E \to X$ be a real oriented rank 7 bundle. Determine the primary obstruction to E admitting a reduction of structure group to G_2.

Solution: The Moore-Postnikov tower of $BG_2 \to BGL^+(7,\mathbb{R})$ takes the form

$$\begin{array}{ccccc} & & BG_2 & & \\ & & \downarrow & & \\ & & \vdots & & \\ & & \downarrow & & \\ K(\pi_m(GL^+(7,\mathbb{R})/G_2), m) & \longrightarrow & E_{m+1} & & \\ & & \downarrow & & \\ & & E_m & \xrightarrow{k_{m+1}} & K(\pi_m(GL^+(7,\mathbb{R})/G_2), m+1) \\ & & \downarrow & & \\ & & \vdots & & \\ & & \downarrow & & \\ & & BGL^+(7,\mathbb{R}) & & \end{array}$$

Note that $GL^+(7,\mathbb{R})/G_2$ is the image of $GL^+(7,\mathbb{R})$ under the quotient map, so $GL^+(7,\mathbb{R})/G_2$ is path-connected, and hence $\pi_0(GL^+(7,\mathbb{R})/G_2) = 0$. Applying the long exact sequence in homotopy groups to the fiber bundle $G_2 \to GL^+(7,\mathbb{R}) \to GL^+(7,\mathbb{R})/G_2$, we have

$$\cdots \to \pi_1(G_2) \to \pi_1(GL^+(7,\mathbb{R})) \to \pi_1(GL^+(7,\mathbb{R})/G_2) \to \pi_0(G_2) \to \dots$$

Since G_2 is simply connected, we have $\pi_1(G_2) = 0$ and $\pi_0(G_2) = 0$ so $\pi_1(GL^+(7,\mathbb{R})/G_2) \cong \pi_1(GL^+(7,\mathbb{R})) \cong \pi_1(SO(7)) \cong \mathbb{Z}_2$. Since the first non-zero homotopy group of $GL^+(7,\mathbb{R})/G_2$ is $\pi_1(GL^+(7,\mathbb{R})/G_2) \cong \mathbb{Z}_2$, the first stage of the tower is

$$\begin{array}{ccccc} K(\mathbb{Z}_2, 1) & \longrightarrow & E_2 & & \\ & & \downarrow & & \\ & & BGL^+(7,\mathbb{R}) & \xrightarrow{k_2} & K(\mathbb{Z}_2, 2) \end{array}$$

which is classified by an element $k_2 \in H^2(BGL^+(7,\mathbb{R}); \mathbb{Z}_2) \cong H^2(BSO(7); \mathbb{Z}_2) = \{0, w_2\}$.

Note that $\pi_1(BG_2) \cong \pi_0(G_2) = 0$ and $\pi_2(BG_2) \cong \pi_1(G_2) = 0$, so $H_1(BG_2; \mathbb{Z}) = 0$ and $H_2(BG_2; \mathbb{Z}) = 0$ by the Hurewicz Theorem. Using the Universal Coefficient Theorem, we see that $H^2(BG_2; \mathbb{Z}_2) \cong \mathrm{Hom}(H_2(BG_2; \mathbb{Z}), \mathbb{Z}_2) \oplus \mathrm{Ext}(H_1(BG_2; \mathbb{Z}), \mathbb{Z}_2) = 0$.

So $(Bi)^* : H^2(BGL^+(7,\mathbb{R}); \mathbb{Z}_2) \to H^2(BG_2; \mathbb{Z}_2)$ is the zero map. As k_2 generates the kernel of $(Bi)^* : H^2(BGL^+(7,\mathbb{R}); \mathbb{Z}_2) \to H^2(BG_2; \mathbb{Z}_2)$, namely $H^2(BGL^+(7,\mathbb{R}); \mathbb{Z}_2) = \{0, w_2\}$, we see that $k_2 = w_2$.

The primary obstruction for a principal $GL^+(7,\mathbb{R})$-bundle $P \to X$ to admit a reduction of structure group to G_2 vanishes if and only if a classifying map $\phi_P : X \to BGL^+(7,\mathbb{R})$ of P lifts to a classifying map $\widetilde{\phi}_P : X \to E_2$.

$$\begin{array}{ccccc} & & E_2 & & \\ & \overset{\widetilde{\phi}_P}{\nearrow} & \downarrow Bi & & \\ X & \xrightarrow{\phi_P} & BGL^+(7,\mathbb{R}) & \xrightarrow{w_2} & K(\mathbb{Z}_2, 2) \end{array}$$

Such a lift exists if and only if $\phi_P^* k_2 = \phi_P^* w_2 = w_2(P)$ vanishes. That is, $w_2(P)$ is the primary obstruction for a principal $GL^+(7,\mathbb{R})$-bundle to admit a reduction of structure group to G_2.

4. Let M be an almost complex manifold.

(a) Show that $\bar{\mu}$ is $C^\infty(M)$-linear.

By (a), we have a vector bundle homomorphism $\bar{\mu} : \bigwedge^{1,0} T^*M \to \bigwedge^{0,2} T^*M$.

(b) Suppose that $\dim M = 4$ and $\bar{\mu}$ is a surjective bundle homomorphism. Show that $5\chi(M) + 6\sigma(M) = 0$.

Solution: (a) Let $\alpha \in \mathcal{E}^{p,q}(M)$, and $f \in C^\infty(M)$. Then

$$\begin{aligned} d(f\alpha) &= df \wedge \alpha + f d\alpha \\ &= (\partial f + \bar{\partial} f) \wedge \alpha + f(\mu\alpha + \partial\alpha + \bar{\partial}\alpha + \bar{\mu}\alpha) \\ &= \underbrace{f\mu\alpha}_{(p+2,q-1)} + \underbrace{(\partial f \wedge \alpha + f\partial\alpha)}_{(p+1,q)} + \underbrace{(\bar{\partial} f \wedge \alpha + f\bar{\partial}\alpha)}_{(p,q+1)} + \underbrace{f\bar{\mu}\alpha}_{(p-1,q+2)} \end{aligned}$$

Equating $(p-1, q+2)$-parts, we see that $\bar{\mu}(f\alpha) = f\bar{\mu}\alpha$, so $\bar{\mu} : \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p-1,q+2}(M)$ is $C^\infty(M)$-linear. It follows by linearity that $\bar{\mu} : \mathcal{E}^*(M)_\mathbb{C} \to \mathcal{E}^*(M)_\mathbb{C}$ is $C^\infty(M)$-linear.

(b) Since $\bar{\mu} : \bigwedge^{1,0} T^*M \to \bigwedge^{0,2} T^*M$ is surjective, it has constant rank, so its kernel K is a vector subbundle of $\bigwedge^{1,0} T^*M$ and we have a short exact sequence $0 \to K \to \bigwedge^{1,0} T^*M \to \bigwedge^{0,2} T^*M \to 0$. Choosing a hermitian bundle metric on $\bigwedge^{1,0} T^*M$, we obtain $\bigwedge^{1,0} T^*M \cong K \oplus \bigwedge^{0,2} T^*M$. Note that $\bigwedge^{1,0} T^*M \cong (T^{1,0}M)^* \cong (TM)^*$. Choosing a hermitian metric on M, this is in turn isomorphic to $\overline{TM}$. We also have $\bigwedge^{0,2} T^*M = \bigwedge^2 (T^{0,1}M)^* \cong \bigwedge^2 (\overline{TM})^* \cong \bigwedge^2 TM = \det(TM)$, so $\overline{TM} \cong K \oplus \det(TM)$.

Therefore $-c_1(TM) = c_1(\overline{TM}) = c_1(K) + c_1(\det(TM)) = c_1(K) + c_1(TM)$, so $c_1(K) = -2c_1(TM)$. Now $c_2(TM) = c_2(\overline{TM}) = c_1(K)c_1(\det(TM)) = (-2c_1(TM))c_1(TM) = -2c_1(TM)^2$, so $c_2(TM) + 2c_1(TM)^2 = 0$, and hence

$$\begin{aligned} 0 &= \langle c_2(TM) + 2c_1(TM)^2, [M] \rangle \\ &= \langle e(TM), [M] \rangle + 2\langle c_1(TM)^2, [M] \rangle \\ &= \chi(M) + 2(2\chi(M) + 3\sigma(M)) \\ &= 5\chi(M) + 6\sigma(M). \end{aligned}$$

5. Recall, we defined an almost complex structure on $S^6 \times \mathbb{O}$ which restricts to an almost complex structure on TS^6, and defines an almost complex structure on the normal bundle of S^6 in $\mathbb{O}$.

(a) Show that if $0 \to E \to F \to G \to 0$ is a short exact sequence of complex vector bundles over a CW complex, with F and G trivial, then $c(E) = 1$.

(b) Explain why $c(TS^6) \neq 1$ and why this does not contradict (a).

Solution: (a) First note that a trivial bundle has trivial Chern classes since the classifying map of a trivial bundle is nullhomotopic and hence the induced pullback map on cohomology is the zero map.

Choosing a hermitian bundle metric on F, we obtain $F \cong E \oplus G$. By Cartan's formula we have $1 = c(F) = c(E)c(G) = c(E)$ since F and G are trivial.

(b) Note that $c_3(TS^6) = e(TS^6) \neq 0$ since $\langle e(TS^6), [S^6] \rangle = \chi(S^6) = 2 \neq 0$, hence $c(TS^6) \neq 1$ (in fact, $c(TS^6) = 1 + c_3(TS^6)$).

We have a short exact sequence of vector bundles over S^6 given by $0 \to TS^6 \to S^6 \times \mathbb{O} \to \nu \to 0$ where ν is the normal bundle. Note that the Euler vector field of $S^6 \subset \operatorname{Im} \mathbb{O}$, defines a nowhere-zero section of the complex line bundle ν, so ν is trivial. While $S^6 \times \mathbb{O}$ is trivial as a real bundle, it is not trivial as a complex vector bundle, which is why there is no contradiction to (a). To see that $S^6 \times \mathbb{O}$ is non-trivial as a complex vector bundle, note that by choosing a hermitian metric on $S^6 \times \mathbb{O}$, we obtain $S^6 \times \mathbb{O} \cong TS^6 \oplus \nu$, so $c(S^6 \times \mathbb{O}) = c(TS^6)c(\nu) = c(TS^6) \neq 1$.

6. A manifold is called a *integral/rational homology sphere* if it has the same integral/rational cohomology groups as a sphere of the same dimension.

(a) Let M be a four-dimensional rational homology sphere. Show that M does not admit an almost complex structure (for either orientation).

(b) Let M be a four-dimensional integral homology sphere. Show that M^n does not admit an almost complex structure for any n (for either orientation).

Solution: (a) Note that M is orientable as $H^4(M;\mathbb{Q}) \cong H^4(S^4;\mathbb{Q}) \cong \mathbb{Q}$.

Since $H^2(M;\mathbb{Q}) = H^2(S^4;\mathbb{Q}) = 0$, the signature of M is zero (regardless of orientation). As $H^i(M;\mathbb{Q}) \cong H^i(S^4;\mathbb{Q})$, we see that $\chi(M) = \sum_{k=0}^4 (-1)^k b_k(M) = \sum_{k=0}^4 (-1)^k b_k(S^4) = \chi(S^4) = 2$. As $\chi(M) \not\equiv -\sigma(M) \bmod 4$, we see that M does not admit an almost complex structure (for either orientation).

(b) We can use induction to show that M^n has torsion-free integral cohomology which vanishes in degrees which are not a multiple of 4. Since M has the same integral cohomology of S^4, the claim is true for $n = 1$. Suppose it is true for M^k, then $M^{k+1} = M^k \times M$ so by the Künneth Theorem, we have a short exact sequence

$$0 \to \bigoplus_{i+j=d} H^i(M^k;\mathbb{Z}) \otimes H^j(M;\mathbb{Z}) \to H^d(M^{k+1};\mathbb{Z}) \to \bigoplus_{i+j=d+1} \operatorname{Tor}(H^i(M^k;\mathbb{Z}), H^j(M;\mathbb{Z})) \to 0.$$

Since M has torsion-free cohomology, the Tor term vanishes. Therefore

$$\begin{aligned} H^d(M^{k+1};\mathbb{Z}) &\cong \bigoplus_{i+j=d} H^i(M^k;\mathbb{Z}) \otimes H^j(M;\mathbb{Z}) \\ &\cong \bigoplus_{i+j=d} H^i(M^k;\mathbb{Z}) \otimes H^j(S^4;\mathbb{Z}) \\ &\cong H^d(M^k;\mathbb{Z}) \oplus H^{d-4}(M^k;\mathbb{Z}). \end{aligned}$$

By the inductive hypothesis, M^k has torsion-free integral cohomology, so $H^d(M^{k+1};\mathbb{Z})$ is torsion-free. In addition, the integral cohomology of M^k vanishes in degrees which are not a multiple of 4, so $H^d(M^{k+1};\mathbb{Z})$ can only be non-zero if d is a multiple of 4. By induction, the claim is proved for M^n.

As $H^4(M;\mathbb{Z}) \cong H^4(S^4;\mathbb{Z}) \cong \mathbb{Z}$, we see that M is orientable[1]. Since $H^2(M;\mathbb{Z}) \cong H^2(S^4;\mathbb{Z}) = 0$, the signature of M is zero and hence $\langle \frac{1}{3}p_1(TM), [M]\rangle = 0$. Therefore $p_1(TM) = 0$, so $p(TM) = 1$. As the integral cohomology of M^n has no 2-torsion,

$$p(T(M^n)) = p(\pi_1^*TM \oplus \cdots \oplus \pi_n^*TM) = p(\pi_1^*TM)\dots p(\pi_n^*TM) = \pi_1^*p(TM)\dots\pi_n^*p(TM) = 1$$

where $\pi_i : M^n \to M$ denotes projection onto the i^{th} factor.

Suppose now that M^n admits an almost complex structure. We can use induction to show that all the Chern classes vanish. Since $H^2(M^n;\mathbb{Z}) = 0$, we have $c_1(T(M^n)) = 0$. Suppose now that $c_i(T(M^n)) = 0$ for $i < k$. If k is odd, then $c_k(T(M^n)) = 0$ since $H^{2k}(M^n;\mathbb{Z}) = 0$. If k is even, say $k = 2r$, then we have

$$p_r(T(M^n)) = c_r(T(M^n))^2 - 2c_{r-1}(T(M^n))c_{r+1}(T(M^n)) + \cdots + 2(-1)^r c_{2r}(T(M^n)) = 2(-1)^r c_k(T(M^n)).$$

Since $p_r(T(M^n)) = 0$ and the cohomology of M^n has no 2-torsion, we conclude that $c_k(T(M^n)) = 0$. Therefore all the Chern classes of M^n vanish, in particular, $c_{2n}(T(M^n)) = 0$. However $c_{2n}(T(M^n)) = e(T(M^n))$ and $\langle e(T(M^n)), [M^n]\rangle = \chi(M^n) = \chi(M)^n = \chi(S^4)^n = 2^n \neq 0$, so $c_{2n}(T(M^n)) \neq 0$ which is a contradiction. Therefore M^n does not admit an almost complex structure (for either orientation).

7. For non-negative integers k and ℓ, let $M_{k,\ell} = k\mathbb{CP}^2 \# \ell\overline{\mathbb{CP}^2}$. For which k and ℓ does $M_{k,\ell}$ admit an almost complex structure inducing the given orientation?

[1] Since every integral homology sphere is a rational homology sphere, this already follows from (a).

Solution: Under the isomorphism $H^2(M\#N;\mathbb{Z}_2)\cong H^2(M;\mathbb{Z}_2)\oplus H^2(N;\mathbb{Z}_2)$, we have $w_2(T(M\#N))\mapsto w_2(TM)+w_2(TN)$. Therefore, under the isomorphism $H^2(M_{k,\ell};\mathbb{Z}_2)\cong\mathbb{Z}_2^{k+\ell}$, the element $w_2(M_{k,\ell})$ corresponds to $(1,\dots,1)$. It follows that if c is an integral lift of $w_2(M_{k,\ell})$, then under the isomorphism $H^2(M_{k,\ell};\mathbb{Z})\cong\mathbb{Z}^{k+\ell}$, the element c corresponds to $(2a_1+1,\dots,2a_k+1,2b_1+1,\dots,2b_\ell+1)$ for some $a_1,\dots,a_k,b_1,\dots,b_\ell\in\mathbb{Z}$. Since $M_{k,\ell}$ has intersection form $\operatorname{diag}(\underbrace{1,\dots,1}_{k},\underbrace{-1,\dots,-1}_{\ell})$, we see that

$$\begin{aligned}\langle c^2,[M_{k,\ell}]\rangle &= (2a_1+1)^2+\cdots+(2a_k+1)^2-(2b_1+1)^2-\cdots-(2b_\ell+1)^2\\ &= 4a_1(a_1+1)+\cdots+4a_k(a_k+1)-4b_1(b_1+1)-\cdots-4b_\ell(b_\ell+1)+k-\ell.\end{aligned}$$

On the other hand $\chi(M_{k,\ell})=2+k+\ell$ and $\sigma=k-\ell$, so $2\chi(M_{k,\ell})+3\sigma(M_{k,\ell})=2(2+k+\ell)+3(k-\ell)=5k-\ell+4$. Therefore $\langle c^2,[M_{k,\ell}]\rangle=2\chi(M_{k,\ell})+3\sigma(M_{k,\ell})$ if and only if

$$\begin{gathered}4a_1(a_1+1)+\cdots+4a_k(a_k+1)-4b_1(b_1+1)-\cdots-4b_\ell(b_\ell+1)+k-\ell=5k-\ell+4\\ 4a_1(a_1+1)+\cdots+4a_k(a_k+1)-4b_1(b_1+1)-\cdots-4b_\ell(b_\ell+1)=4k+4\\ a_1(a_1+1)+\cdots+a_k(a_k+1)-b_1(b_1+1)-\cdots-b_\ell(b_\ell+1)=k+1\end{gathered}$$

Since the product of consecutive integers is always even, we see that the left hand side is even, and hence k is necessarily odd.

To see that k odd is sufficient, write $k+1=2d$ and note that $d\le k$. Now $a_1=\cdots=a_d=1$ and $a_{d+1}=\cdots=a_k=b_1=\cdots=b_\ell=0$ is a solution to the equation above, and hence we obtain an integral lift $c\in H^2(M_{k,\ell};\mathbb{Z})$ of $w_2(M_{k,\ell})$ satisfying $\langle c^2,[M_{k,\ell}]\rangle=2\chi(M_{k,\ell})+3\sigma(M_{k,\ell})$. Therefore $M_{k,\ell}$ admits an almost complex structure inducing the given orientation.