ALMOST COMPLEX MANIFOLDS – ASSIGNMENT 3 SOLUTIONS

1. Show that the twistor fibration of S^4, namely $S^2 \to \mathbb{CP}^3 \xrightarrow{p} S^4$, does not admit a section, and deduce that S^4 does not admit an almost complex structure.

Solution: Suppose $\sigma : S^4 \to \mathbb{CP}^3$ is a section, so $p \circ \sigma = \mathrm{id}_{S^4}$.

The fiber bundle $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ induces a long exact sequence in homotopy groups

$$\cdots \to \pi_k(S^1) \to \pi_k(S^{2n+1}) \to \pi_k(\mathbb{CP}^n) \to \pi_{k-1}(S^1) \to \dots$$

It follows that $\pi_2(\mathbb{CP}^n) \cong \pi_1(S^1) \cong \mathbb{Z}$, and $\pi_k(\mathbb{CP}^n) \cong \pi_k(S^{2n+1})$ for $k \neq 2$. Therefore $\sigma_* : \pi_4(S^4) \to \pi_4(\mathbb{CP}^3)$ is the zero map since $\pi_4(\mathbb{CP}^3) \cong \pi_4(S^7) = 0$, and hence $p_* \circ \sigma_*$ is the zero map. This contradicts the fact that $p_* \circ \sigma_* = (p \circ \sigma)_* = (\mathrm{id}_{S^4})_* = \mathrm{id}_{\pi_4(S^4)}$ which is not the zero map as $\pi_4(S^4) \cong \mathbb{Z} \neq 0$. So $p : \mathbb{CP}^3 \to S^4$ does not admit a section, and hence S^4 does not admit an almost complex structure.

2. Note that $SO(2n)/U(n)$ is a closed smooth manifold of dimension $n(n-1)$. For small values of n we have

$$SO(2)/U(1) = * = \mathbb{CP}^0$$
$$SO(4)/U(2) = S^2 = \mathbb{CP}^1$$
$$SO(6)/U(3) = \mathbb{CP}^3.$$

When $n = 4$, we have $n(n-1) = 12$. If the pattern were to continue, then we would be able to identify $SO(8)/U(4)$ with $\mathbb{CP}^6$. Show that $SO(8)/U(4)$ and $\mathbb{CP}^6$ are not homotopy equivalent.

Solution: Combining the fiber bundle $SO(2n)/U(n) \to SO(2n+2)/U(n+1) \to S^{2n}$ with the diffeomorphism between $SO(6)/U(3)$ and $\mathbb{CP}^3$, we have a fiber bundle $\mathbb{CP}^3 \to SO(8)/U(4) \to S^6$. This induces a long exact sequence in cohomology groups

$$\cdots \to \pi_6(\mathbb{CP}^3) \to \pi_6(SO(8)/U(4)) \to \pi_6(S^6) \to \pi_5(\mathbb{CP}^3) \to \dots$$

From the solution to question 1, we have $\pi_6(\mathbb{CP}^3) \cong \pi_6(S^7) = 0$ and $\pi_5(\mathbb{CP}^3) \cong \pi_5(S^7) = 0$, and hence $\pi_6(SO(8)/U(4)) \cong \pi_6(S^6) \cong \mathbb{Z}$. On the other hand $\pi_6(\mathbb{CP}^6) = \pi_6(S^{13}) = 0$. Therefore $SO(8)/U(4)$ and $\mathbb{CP}^6$ are not homotopy equivalent as $\pi_6(SO(8)/U(4)) \not\cong \pi_6(\mathbb{CP}^6)$.

3. Recall, a space X is called m-connected if $\pi_i(X) = 0$ for $i \leq m$.

(a) Show that $O(n)/O(n-k)$ is $(n-k-1)$-connected.

(b) Show that $U(n)/U(n-k)$ is $(2n-2k)$-connected and $\pi_{2n-2k+1}(U(n)/U(n-k)) \cong \mathbb{Z}$.

Solution: (a) We have a principal bundle $O(n-1) \to O(n) \to S^{n-1}$, namely the orthonormal frame bundle of S^{n-1}. Quotienting by $O(n-k)$ gives rise to the fiber bundle $O(n-1)/O(n-k) \to O(n)/O(n-k) \to S^{n-1}$. Setting $V_{k,n} = O(n)/O(n-k)$, we can rewrite the fiber bundle as $V_{k-1,n-1} \to V_{k,n} \to S^{n-1}$. Applying the long exact sequence in homotopy groups, we obtain

$$\cdots \to \pi_{i+1}(S^{n-1}) \to \pi_i(V_{k-1,n-1}) \to \pi_i(V_{k,n}) \to \pi_i(S^{n-1}) \to \dots$$

If $i+1 < n-1$, i.e. $i \leq n-3$, then $\pi_{i+1}(S^{n-1}) = \pi_i(S^{n-1}) = 0$, so $\pi_i(V_{k,n}) \cong \pi_i(V_{k-1,n-1})$. Replacing n by $n-j$ and k by $k-j$, where $j < k$, we see that $\pi_i(V_{k-j,n-j}) \cong \pi_i(V_{k-j-1,n-j-1})$ for $i \leq n-j-3$, and hence $\pi_i(V_{k,n}) \cong \pi_i(V_{k-1,n-1}) \cong \dots \cong \pi_i(V_{k-j-1,n-j-1})$ for $i \leq n-j-3$. In particular, when $j = k-2$, we have $\pi_i(V_{k,n}) \cong \pi_i(V_{1,n-k+1})$ for $i \leq n-k-1$. Note that

$V_{1,n-k+1} = O(n-k+1)/O(n-k+1-1) = O(n-k+1)/O(n-k) = S^{n-k}$, so for $i \leq n-k-1$, we have $\pi_i(V_{k,n}) = \pi_i(V_{1,n-k+1}) = \pi_i(S^{n-k}) = 0$. Therefore $V_{k,n} = O(n)/O(n-k)$ is $(n-k-1)$-connected.

(b) We have a principal bundle $U(n-1) \to U(n) \to S^{2n-1}$, namely .... Quotienting by $U(n-k)$ gives rise to the fiber bundle $U(n-1)/U(n-k) \to U(n)/U(n-k) \to S^{2n-1}$. Setting $V^{\mathbb{C}}_{k,n} = U(n)/U(n-k)$, we can rewrite the fiber bundle as $V^{\mathbb{C}}_{k-1,n-1} \to V^{\mathbb{C}}_{k,n} \to S^{2n-1}$. Applying the long exact sequence in homotopy groups, we obtain

$$\cdots \to \pi_{i+1}(S^{2n-1}) \to \pi_i(V^{\mathbb{C}}_{k-1,n-1}) \to \pi_i(V^{\mathbb{C}}_{k,n}) \to \pi_i(S^{2n-1}) \to \dots$$

If $i+1 < 2n-1$, i.e. $i \leq 2n-3$, then $\pi_{i+1}(S^{2n-1}) = \pi_i(S^{2n-1}) = 0$, so $\pi_i(V^{\mathbb{C}}_{k,n}) \cong \pi_i(V^{\mathbb{C}}_{k-1,n-1})$. Replacing n by $n-j$ and k by $k-j$, where $j < k$, we see that $\pi_i(V^{\mathbb{C}}_{k-j,n-j}) \cong \pi_i(V^{\mathbb{C}}_{k-j-1,n-j-1})$ for $i \leq 2n-2j-3$, and hence $\pi_i(V^{\mathbb{C}}_{k,n}) \cong \pi_i(V^{\mathbb{C}}_{k-1,n-1}) \cong \ldots \cong \pi_i(V^{\mathbb{C}}_{k-j-1,n-j-1})$ for $i \leq 2n-2j-3$. In particular, when $j = k-2$, we have $\pi_i(V^{\mathbb{C}}_{k,n}) \cong \pi_i(V^{\mathbb{C}}_{1,n-k+1})$ for $i \leq 2n-2k+1$. Note that $V^{\mathbb{C}}_{1,n-k+1} = U(n-k+1)/U(n-k+1-1) = U(n-k+1)/U(n-k) = S^{2n-2k+1}$, so for $i \leq 2n-2k+1$, we have $\pi_i(V_{k,n}) = \pi_i(V_{1,n-k+1}) = \pi_i(S^{2n-2k+1})$. In particular, $V^{\mathbb{C}}_{n,k} = U(n)/U(n-k)$ is $(2n-2k)$-connected and $\pi_{2n-2k+1}(V^{\mathbb{C}}_{n,k}) = \pi_{2n-2k+1}(U(n)/U(n-k)) = \pi_{2n-2k+1}(S^{2n-2k+1}) \cong \mathbb{Z}$.

4. Let M be a closed simply connected four-manifold and let $p \in M$.

(a) Show that $M \setminus \{p\}$ admits an almost complex structure.

(b) Show that $M \times S^2$ admits an almost complex structure.

Solution: (a) Let U be an open neighbourhood of p homeomorphic to a ball. Note that $U \cup (M \setminus \{p\}) = M$ and $U \cap (M \setminus \{p\}) = U \setminus \{p\}$ which is homotopy equivalent to S^3. Using the fact that U is contractible, the Mayer-Vietoris sequence gives

$$\cdots \to H^2(S^3;\mathbb{Z}) \to H^3(M;\mathbb{Z}) \to H^3(M \setminus \{p\};\mathbb{Z}) \to H^3(S^3;\mathbb{Z}) \to H^4(M;\mathbb{Z}) \to H^4(M \setminus \{p\};\mathbb{Z}) \to \dots$$

Note that $H^2(S^3;\mathbb{Z}) = 0$ and $H^4(M \setminus \{p\};\mathbb{Z}) = 0$, because $M \setminus \{p\}$ is a non-compact four-manifold. In addition, we have $H^3(S^3;\mathbb{Z}) \cong \mathbb{Z}$ and $H^4(M;\mathbb{Z})$ since M is orientable (because M is simply connected). Therefore, we obtain the exact sequence

$$0 \to H^3(M;\mathbb{Z}) \to H^3(M \setminus \{p\};\mathbb{Z}) \to \mathbb{Z} \to \mathbb{Z} \to 0$$

Every surjective group homomorphism $\mathbb{Z} \to \mathbb{Z}$ is an isomorphism (its either the $\mathrm{id}_{\mathbb{Z}}$ or $-\mathrm{id}_{\mathbb{Z}}$), so the map $H^3(M;\mathbb{Z}) \to H^3(M \setminus \{p\};\mathbb{Z})$ is an isomorphism. Since M is orientable, we have $H^3(M;\mathbb{Z}) \cong H_1(M;\mathbb{Z}) \cong \pi_1(M)^{\mathrm{ab}} = 0$ by Poincaré Duality and the Hurewicz Theorem respectively, so $H^3(M \setminus \{p\};\mathbb{Z}) = 0$.

The obstructions to the existence of an almost complex structure on $M \setminus \{p\}$ (inducing a given orientation) lie in $H^{k+1}(M \setminus \{p\};\pi_k(SO(4)/U(2))) \cong H^{k+1}(M \setminus \{p\};\pi_k(S^2))$. The primary obstruction lies in $H^3(M \setminus \{p\};\pi_2(S^2)) \cong H^3(M \setminus \{p\};\mathbb{Z}) = 0$ and therefore vanishes, as does the next obstruction as $H^4(M \setminus \{p\};\pi_3(S^2)) \cong H^4(M \setminus \{p\};\mathbb{Z}) = 0$ because $M \setminus \{p\}$ is non-compact. All the higher obstructions vanish as $H^{k+1}(M \setminus \{p\};\pi_k(S^2)) = 0$ for $k \geq 4$ because $\dim(M \setminus \{p\}) = 4$. Therefore M admits an almost complex structure (for both choices of orientation).

(b) The obstructions to $M \times S^2$ admitting an almost complex structure (inducing a given orientation) lie in $H^{k+1}(M \times S^2;\pi_k(SO(6)/U(3))) \cong H^{k+1}(M \times S^2;\pi_k(\mathbb{CP}^3))$. This group vanishes for $k > 5$ since $\dim(M \times S^2) = 6$. For $k \leq 5$, we have $\pi_k(\mathbb{CP}^3) = 0$ except for $k = 2$ where $\pi_2(\mathbb{CP}^3) \cong \mathbb{Z}$. Therefore the only obstruction to the existence of an almost complex structure on $M \times S^2$ lies in $H^3(M \times S^2;\mathbb{Z})$. By the Kunneth Theorem for cohomology (see Theorem 5.5.11 of Spanier's "Algebraic Topology" for example), we have a short exact sequence

$$0 \to \bigoplus_{i+j=3} H^i(M;\mathbb{Z}) \otimes H^j(S^2;\mathbb{Z}) \to H^3(M \times S^2;\mathbb{Z}) \to \bigoplus_{i+j=4} \mathrm{Tor}(H^i(M;\mathbb{Z}), H^j(S^2;\mathbb{Z})) \to 0$$

Since $H^j(S^2;\mathbb{Z})$ is torsion-free for every j, we have $\operatorname{Tor}(H^i(M;\mathbb{Z}), H^j(S^2;\mathbb{Z})) = 0$ for all i and j. From part (a), we have $H^3(M;\mathbb{Z}) = 0$, but also $H_1(M;\mathbb{Z}) = 0$, so $H^1(M;\mathbb{Z}) \cong \operatorname{Hom}(H_1(M;\mathbb{Z}),\mathbb{Z}) = 0$. Since $H^1(S^2;\mathbb{Z}) = 0$ and $H^3(S^2;\mathbb{Z}) = 0$, we see that $H^3(M \times S^2;\mathbb{Z}) = 0$ and hence $M \times S^2$ admits an almost complex structure (for both choices of orientation).

5. Let $E \to X$ be a rank m orientable vector bundle over a d-dimensional CW complex. If $m > d$, show that $E \cong E_0 \oplus \varepsilon^{m-d}$ where $\operatorname{rank}(E_0) = d$. Give an example to show that E_0 is not unique (up to isomorphism).

Solution: The bundle E splits as $E_0 \oplus \varepsilon^{m-d}$ if and only if $P_{O(m)}(E)$ admits a reduction of structure group to $O(d)$ which is equivalent to the bundle $P_{O(m)}(E)/O(d) \to X$ admitting a section. The latter is a fiber bundle with fiber $O(m)/O(d) = O(m)/O(m-(m-d))$ which is N-connected for $N = m-(m-d)-1 = d-1$ by question 3 (a). The obstructions to the existence of a section lie in $H^{k+1}(X;\pi_k(O(m)/O(d)))$ which vanishes because either $k \leq d-1$, in which case $\pi_k(O(m)/O(d)) = 0$, or $k \geq d$, in which case it vanishes because X is d-dimensional. Therefore every such bundle E is isomorphic to $E_0 \oplus \varepsilon^{m-d}$ for some $E_0 \to X$ with $\operatorname{rank} E_0 = d$.

To see that E_0 is not unique, let $E = \varepsilon^{d+1}$ be the trivial rank $d+1$ bundle over S^d. We have $\varepsilon^{d+1} \cong \varepsilon^d \oplus \varepsilon^1$, but also $\varepsilon^{d+1} \cong TS^d \oplus \varepsilon^1$. For $d \neq 0, 1, 3, 7$, we have $TS^d \ncong \varepsilon^d$, so it serves as a counterexample to the uniqueness of E_0.

6. Let M be a $2n$-dimensional almost complex manifold. Show that there are two potential obstructions to TM admitting a complex line subbundle.

Solution: The complex vector bundle TM admits a complex line subbundle if and only if $P_{U(n)}(TM)$ admits a reduction of structure group to $U(n-1)\times U(1)$ which is equivalent to the bundle $P_{U(n)}(TM)/(U(n-1)\times U(1)) \to M$ admitting a section. The latter is a fiber bundle with fiber $U(n)/(U(n-1)\times U(1)) = \mathbb{CP}^{n-1}$. Therefore, the obstructions to the existence of a complex line subbundle of TM lie in $H^{k+1}(M;\pi_k(\mathbb{CP}^{n-1}))$. From the solution to question 1, we have $\pi_2(\mathbb{CP}^{n-1}) \cong \mathbb{Z}$ and $\pi_k(\mathbb{CP}^{n-1}) = \pi_k(S^{2n-1})$ for $k \neq 2$. Therefore, the primary obstruction lies in $H^3(M;\mathbb{Z})$ and the next obstruction lies in $H^{2n}(M;\pi_{2n-1}(\mathbb{CP}^{n-1})) \cong H^{2n}(M;\pi_{2n-1}(S^{2n-1})) \cong H^{2n}(M;\mathbb{Z})$. All the higher obstructions vanish because $\dim M = 2n$.

7. Show that a smooth manifold M admits an almost complex structure if and only if it admits a non-degenerate 2-form.

Solution: Suppose M admits an almost complex structure J. Then we can find a compatible Riemannian metric g and construct the 2-form ω given by $\omega(u,v) = g(Ju,v)$. Note that for $u \neq 0$, we have $\omega(u,Ju) = g(Ju,Ju) = g(u,u) = \|u\|^2 > 0$, so ω is non-degenerate.

Conversely, suppose M admits a non-degenerate 2-form ω and let $\dim M = 2n$. Then $P_{GL(2n,\mathbb{R})}(TM)$ admits a reduction of structure group to $Sp(2n,\mathbb{R})$, namely $P_{Sp(2n,\mathbb{R})}(TM) = \{f \in P_{GL(2n,\mathbb{R})}(TM) \mid f^*\omega = e^1 \wedge e^2 + \cdots + e^{2n-1} \wedge e^{2n}\}$. Since $U(n)$ is a maximal compact subgroup of $Sp(2n,\mathbb{R})$, the quotient $Sp(2n,\mathbb{R})/U(n)$ is contractible, so $P_{Sp(2n,\mathbb{R})}(TM)$ admits a reduction of structure group to $U(n)$. Therefore $P_{GL(2n,\mathbb{R})}(TM)$ admits a reduction of structure group to $U(n)$ and hence M admits an almost complex structure.