ALMOST COMPLEX MANIFOLDS – ASSIGNMENT 2 SOLUTIONS

1. Let (M, J) be an almost complex manifold and set $N_J(X,Y) = [X,Y]+J[JX,Y]+J[X,JY]-[JX,JY]$. Show the following:

(a) $N_J(X,Y) = -N_J(Y,X)$**,**

(b) $N_J(JX,Y) = -JN_J(X,Y)$**, and**

(c) $N_J(fX,Y) = fN_J(X,Y)$ **for** $f \in C^\infty(M)$**.**

Solution: (a)

$$\begin{aligned} N_J(X,Y) &= [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] \\ &= -[Y,X] - J[Y,JX] - J[JY,X] + [JY,JX] \\ &= -[Y,X] - J[JY,X] - J[Y,JX] + [JY,JX] \\ &= -([Y,X] + J[JY,X] + J[Y,JX] - [JY,JX]) \\ &= -N_J(Y,X). \end{aligned}$$

(b)

$$\begin{aligned} N_J(JX,Y) &= [JX,Y] + J[J(JX),Y] + J[JX,JY] - [J(JX),JY] \\ &= [JX,Y] - J[X,Y] + J[JX,JY] + [X,JY] \\ &= -J(J[JX,Y] + [X,Y] - [JX,JY] + J[X,JY]) \\ &= -J([X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]) \\ &= -JN_J(X,Y). \end{aligned}$$

(c)

$$\begin{aligned} N_J(fX,Y) &= [fX,Y] + J[J(fX),Y] + J[fX,JY] - [J(fX),JY] \\ &= [fX,Y] + J[fJX,Y] + J[fX,JY] - [fJX,JY] \\ &= f[X,Y] - Y(f)X + J(f[JX,Y] - Y(f)JX) + J(f[X,JY] - (JY)(f)X) \\ &\quad - f[JX,JY] + (JY)(f)JX \\ &= f[X,Y] - Y(f)X + fJ[JX,Y] + Y(f)X + fJ[X,JY] - (JY)(f)JX \\ &\quad - f[JX,JY] + (JY)(f)JX \\ &= f[X,Y] + fJ[JX,Y] + fJ[X,JY] - f[JX,JY] \\ &= fN_J(X,Y). \end{aligned}$$

2. Consider the almost complex structure J on $\mathbb{R}^4$ given by

$$J = \begin{bmatrix} 0 & 1 & f & -g \\ -1 & 0 & g & f \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

where $f, g \in C^\infty(\mathbb{R}^4)$.

(a) Compute the Nijenhuis tensor of J. That is, compute $N_J\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right)$ for $j, k \in \{1,2,3,4\}$.

(b) For $a, b \in \mathbb{R}$, consider the function $h : \mathbb{C} \to \mathbb{C}$ given by $h_{a,b}(z) = f(x,y,a,b) + ig(x,y,a,b)$ where $z = x + iy$. Show that J is integrable if and only if $h_{a,b}$ is holomorphic for all $a, b \in \mathbb{R}$.

Solution: (a) Denote $\frac{\partial}{\partial x^j}$ by ∂_j.

By skew-symmetry, namely 1(a), we only have to calculate $N_J(\partial_j, \partial_k)$ for $1 \le j < k \le 4$. In addition, by 1(b) we have $N_J(\partial_1, \partial_k) = N_J(J\partial_2, \partial_k) = -JN_J(\partial_2, \partial_k)$, so we only need to calculate $N_J(\partial_j, \partial_k)$ for $2 \le j < k \le 4$, i.e. $(j,k) \in \{(2,3),(2,4),(3,4)\}$. Using the fact that $[\partial_j, \partial_k] = 0$ and $[hX, Y] = h[X,Y] - Y(h)X$ (and hence $[X, hY] = h[X,Y] + X(h)Y$ and $[h_1X, h_2Y] = h_1h_2[X,Y] + h_1X(h_2)Y - h_2Y(h_1)X$), we have

$$\begin{aligned}
N_J(\partial_2, \partial_3) &= [\partial_2, \partial_3] + J[J\partial_2, \partial_3] + J[\partial_2, J\partial_3] - [J\partial_2, J\partial_3] \\
&= J[\partial_1, \partial_3] + J[\partial_2, f\partial_1 + g\partial_2 + \partial_4] - [\partial_1, f\partial_1 + g\partial_2 + \partial_4] \\
&= J[\partial_2, f\partial_1] + J[\partial_2, g\partial_2] + J[\partial_2, \partial_4] - [\partial_1, f\partial_1] - [\partial_1, g\partial_2] - [\partial_1, \partial_4] \\
&= J(f[\partial_2, \partial_1] + \partial_2(f)\partial_1) + J(g[\partial_2, \partial_2] + \partial_2(g)\partial_2) - f[\partial_1, \partial_1] - \partial_1(f)\partial_1 - g[\partial_1, \partial_2] - \partial_1(g)\partial_2 \\
&= \partial_2(f)J\partial_1 + \partial_2(g)J\partial_2 - \partial_1(f)\partial_1 - \partial_1(g)\partial_2 \\
&= -\partial_2(f)\partial_2 + \partial_2(g)\partial_1 - \partial_1(f)\partial_1 - \partial_1(g)\partial_2 \\
&= [\partial_2(g) - \partial_1(f)]\partial_1 - [\partial_1(g) + \partial_2(f)]\partial_2
\end{aligned}$$

$$\begin{aligned}
N_J(\partial_2, \partial_4) &= [\partial_2, \partial_4] + J[J\partial_2, \partial_4] + J[\partial_2, J\partial_4] - [J\partial_2, J\partial_4] \\
&= J[\partial_1, \partial_4] + J[\partial_2, -g\partial_1 + f\partial_2 - \partial_3] - [\partial_1, -g\partial_1 + f\partial_2 - \partial_3] \\
&= -J[\partial_2, g\partial_1] + J[\partial_2, f\partial_2] - J[\partial_2, \partial_3] + [\partial_1, g\partial_1] - [\partial_1, f\partial_2] + [\partial_1, \partial_3] \\
&= -J(g[\partial_2, \partial_1] + \partial_2(g)\partial_1) + J(f[\partial_2, \partial_2] + \partial_2(f)\partial_2) + g[\partial_1, \partial_1] + \partial_1(g)\partial_1 - f[\partial_1, \partial_2] - \partial_1(f)\partial_2 \\
&= -\partial_2(g)J\partial_1 + \partial_2(f)J\partial_2 + \partial_1(g)\partial_1 - \partial_1(f)\partial_2 \\
&= \partial_2(g)\partial_2 + \partial_2(f)\partial_1 + \partial_1(g)\partial_1 - \partial_1(f)\partial_2 \\
&= [\partial_1(g) + \partial_2(f)]\partial_1 + [\partial_2(g) - \partial_1(f)]\partial_2
\end{aligned}$$

$$\begin{aligned}
N_J(\partial_3, \partial_4) &= [\partial_3, \partial_4] + J[J\partial_3, \partial_4] + J[\partial_3, J\partial_4] - [J\partial_3, J\partial_4] \\
&= J[f\partial_1 + g\partial_2 + \partial_4, \partial_4] + J[\partial_3, -g\partial_1 + f\partial_2 - \partial_3] - [f\partial_1 + g\partial_2 + \partial_4, -g\partial_1 + f\partial_2 - \partial_3] \\
&= J[f\partial_1, \partial_4] + J[g\partial_2, \partial_4] + J[\partial_4, \partial_4] - J[\partial_3, g\partial_1] + J[\partial_3, f\partial_2] - J[\partial_3, \partial_3] + [f\partial_1, g\partial_1] \\
&\quad - [f\partial_1, f\partial_2] + [f\partial_1, \partial_3] + [g\partial_2, g\partial_1] - [g\partial_2, f\partial_2] + [g\partial_2, \partial_3] + [\partial_4, g\partial_1] - [\partial_4, f\partial_2] + [\partial_4, \partial_3] \\
&= J(f[\partial_1, \partial_4] - \partial_4(f)\partial_1) + J(g[\partial_2, \partial_4] - \partial_4(g)\partial_2) - J(g[\partial_3, \partial_1] + \partial_3(g)\partial_1) + J(f[\partial_3, \partial_2] \\
&\quad + \partial_3(f)\partial_2) + fg[\partial_1, \partial_1] + f\partial_1(g)\partial_1 - g\partial_1(f)\partial_1 - (f^2[\partial_1, \partial_2] + f\partial_1(f)\partial_2 - f\partial_2(f)\partial_1) \\
&\quad + f[\partial_1, \partial_3] - \partial_3(f)\partial_1 + g^2[\partial_2, \partial_1] + g\partial_2(g)\partial_1 - g\partial_1(g)\partial_2 - (gf[\partial_2, \partial_2] + g\partial_2(f)\partial_2 - f\partial_2(g)\partial_2) \\
&\quad + g[\partial_2, \partial_3] - \partial_3(g)\partial_2 + g[\partial_4, \partial_1] + \partial_4(g)\partial_1 - (f[\partial_4, \partial_2] + \partial_4(f)\partial_2) \\
&= -\partial_4(f)J\partial_1 - \partial_4(g)J\partial_2 - \partial_3(g)J\partial_1 + \partial_3(f)J\partial_2 + f\partial_1(g)\partial_1 - g\partial_1(f)\partial_1 - f\partial_1(f)\partial_2 + f\partial_2(f)\partial_1 \\
&\quad - \partial_3(f)\partial_1 + g\partial_2(g)\partial_1 - g\partial_1(g)\partial_2 - g\partial_2(f)\partial_2 + f\partial_2(g)\partial_2 - \partial_3(g)\partial_2 + \partial_4(g)\partial_1 - \partial_4(f)\partial_2 \\
&= \partial_4(f)\partial_2 - \partial_4(g)\partial_1 + \partial_3(g)\partial_2 + \partial_3(f)\partial_1 + f\partial_1(g)\partial_1 - g\partial_1(f)\partial_1 - f\partial_1(f)\partial_2 + f\partial_2(f)\partial_1 \\
&\quad - \partial_3(f)\partial_1 + g\partial_2(g)\partial_1 - g\partial_1(g)\partial_2 - g\partial_2(f)\partial_2 + f\partial_2(g)\partial_2 - \partial_3(g)\partial_2 + \partial_4(g)\partial_1 - \partial_4(f)\partial_2 \\
&= f\partial_1(g)\partial_1 - g\partial_1(f)\partial_1 - f\partial_1(f)\partial_2 + f\partial_2(f)\partial_1 + g\partial_2(g)\partial_1 - g\partial_1(g)\partial_2 - g\partial_2(f)\partial_2 + f\partial_2(g)\partial_2
\end{aligned}$$

$$= [f\partial_1(g) - g\partial_1(f) + f\partial_2(f) + g\partial_2(g)]\partial_1 + [-f\partial_1(f) - g\partial_1(g) - g\partial_2(f) + f\partial_2(g)]\partial_2$$
$$= [f(\partial_1(g) + \partial_2(f)) + g(-\partial_1(f) + \partial_2(g))]\partial_1 + [f(-\partial_1(f) + \partial_2(g)) + g(-\partial_1(g) - \partial_2(f))]\partial_2$$

Setting $A = \partial_1(g) + \partial_2(f)$ and $B = \partial_1(f) - \partial_2(g)$, we have

$$[N_J(\partial_j, \partial_k)] = \begin{bmatrix} 0 & 0 & A & B \\ 0 & 0 & -B & A \\ -A & B & 0 & fA - gB \\ -B & -A & -fA + gB & 0 \end{bmatrix} \partial_1 + \begin{bmatrix} 0 & 0 & -B & A \\ 0 & 0 & -A & -B \\ B & A & 0 & -fB - gA \\ -A & B & fB + gA & 0 \end{bmatrix} \partial_2.$$

(b) By the Newlander-Nirenberg theorem, J is integrable if and only if $N_J = 0$. By our computation of N_J in (a), we see that $N_J = 0$ if and only if $A = B = 0$.

Note that

$$\frac{\partial f}{\partial x}(x, y, a, b) = \frac{\partial}{\partial x}(f(x, y, a, b)) \qquad \frac{\partial g}{\partial x}(x, y, a, b) = \frac{\partial}{\partial x}(g(x, y, a, b))$$
$$\frac{\partial f}{\partial y}(x, y, a, b) = \frac{\partial}{\partial y}(f(x, y, a, b)) \qquad \frac{\partial g}{\partial y}(x, y, a, b) = \frac{\partial}{\partial y}(g(x, y, a, b)).$$

It follows that $A(x, y, a, b) = 0$ and $B(x, y, a, b) = 0$ are precisely the Cauchy-Riemann equations for the real and imaginary parts of $h_{a,b}$. Therefore $h_{a,b}$ is holomorphic for all $a, b \in \mathbb{R}$ if and only if $A = B = 0$ which is equivalent to integrability of J.

3. Let (Y, J) be an almost complex manifold with a properly embedded submanifold X such that $T_xX \subseteq T_xY$ is a complex subspace for all $x \in X$. Show that J induces an almost complex structure J_X on X, and if J is integrable, so too is J_X.

Solution: Let $i : X \to Y$ be the inclusion. Then $i_*(T_xX) \subseteq T_{i(x)}Y = T_xY$ is a complex subspace. Therefore, if $v \in T_xX$, then $i_*v \in i_*(T_xX)$ and $J(i_*v) \in i_*(T_xX)$. Since i is an embedding, i_* is injective, so there is a unique $w \in T_xX$ such that $J(i_*v) = i_*w$. We define $J_Xv = w$, so that $J(i_*v) = i_*(J_Xv)$. Note that $i_*(J_X^2v) = J(i_*(J_Xv)) = J^2(i_*v) = -i_*v = i_*(-v)$, so by the injectivity of i_*, we have $J_X^2 = -\operatorname{id}_{TX}$, and hence J_X is an almost complex structure on X.

Let U, V be vector fields on X. Since X is properly embedded in Y, we can extend U, V to vector fields $\widetilde{U}$, $\widetilde{V}$ on Y. Note that for $x \in X$, we have $i_*U_x = U_{i(x)} = U_x = \widetilde{U}_x$, so U and $\widetilde{U}$ are i-related, and similarly for V and $\widetilde{V}$. We also have $i_*(J_XU_x) = J(i_*U_x) = J\widetilde{U}_x$, so J_XU and $J\widetilde{U}$ are i-related, as are J_XV and $J\widetilde{V}$. So, for any $x \in X$, we have

$$\begin{aligned} i_*N_{J_X}(U, V)_x &= i_*[U, V]_x + i_*J_X[J_XU, V]_x + i_*J_X[U, J_XV]_x - i_*[J_XU, J_XV]_x \\ &= i_*[U, V]_x + Ji_*[J_XU, V]_x + Ji_*[U, J_XV]_x - i_*[J_XU, J_XV]_x \\ &= [\widetilde{U}, \widetilde{V}]_x + J[J\widetilde{U}, \widetilde{V}]_x + J[\widetilde{U}, J\widetilde{V}]_x - [J\widetilde{U}, J\widetilde{V}]_x \\ &= N_J(\widetilde{U}, \widetilde{V})_x \\ &= 0. \end{aligned}$$

Since i_* is injective, we see that $N_{J_X}(U, V)_x = 0$. Since U, V, and x were arbitrary, $N_{J_X} = 0$ and hence J_X is integrable.

4. Let (M, J) be an almost complex manifold.

(a) Show that $\bar{\mu}^2 = 0$.

Let $E^{p,q}(M) = H^{p,q}(\mathcal{E}^{\bullet,\bullet}(M), \bar{\mu}) := \dfrac{\ker(\bar{\mu} : \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p-1,q+2}(M))}{\operatorname{im}(\bar{\mu} : \mathcal{E}^{p+1,q-2}(M) \to \mathcal{E}^{p,q}(M))}$**.**

(b) Show that $\bar\partial : \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p,q+1}(M)$ descends to a well-defined map $\bar\partial : E^{p,q}(M) \to E^{p,q+1}(M)$, $[\alpha] \mapsto [\bar\partial\alpha]$.

(c) Show that $\bar\partial : E^{p,q}(M) \to E^{p,q+1}(M)$ satisfies $\bar\partial^2 = 0$.

Solution: (a) Since $d = \mu + \partial + \bar\partial + \bar\mu$, the operator d^2 is a sum of 16 terms. We can separate them into 7 operators

$$\begin{aligned}
\mu^2 &: \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p+4,q-2}(M)\\
\mu\partial + \partial\mu &: \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p+3,q-1}(M)\\
\partial^2 + \mu\bar\partial + \bar\partial\mu &: \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p+2,q}(M)\\
\mu\bar\mu + \bar\mu\mu + \partial\bar\partial + \bar\partial\partial &: \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p+1,q+1}(M)\\
\bar\partial^2 + \bar\mu\partial + \partial\bar\mu &: \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p,q+2}(M)\\
\bar\mu\bar\partial + \bar\partial\bar\mu &: \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p-1,q+3}(M)\\
\bar\mu^2 &: \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p-2,q+4}(M).
\end{aligned}$$

Since $d^2 = 0$, each of these operators is the zero map as well. In particular, we have $\bar\mu^2 = 0$.

(b) By the argument in part (a), we have $\bar\mu\bar\partial + \bar\partial\bar\mu = 0$. Let $\alpha \in \mathcal{E}^{p,q}(M)$ with $\bar\mu\alpha = 0$. Note that $\bar\partial\alpha \in \mathcal{E}^{p,q+1}(M)$ satisfies $\bar\mu\bar\partial\alpha = -\bar\partial\bar\mu\alpha = 0$, so $[\bar\partial\alpha] \in E^{p,q+1}(M)$ is defined.

Suppose now that $\beta \in \mathcal{E}^{p,q}(M)$ with $\bar\mu\beta = 0$ and $[\alpha] = [\beta]$. Then there is $\gamma \in \mathcal{E}^{p+1,q-2}(M)$ with $\alpha = \beta + \bar\mu\gamma$, so $\bar\partial\alpha = \bar\partial\beta + \bar\partial\bar\mu\gamma = \bar\partial\beta - \bar\mu\bar\partial\gamma$. Therefore $[\bar\partial\alpha] = [\bar\partial\beta - \bar\mu\bar\partial\gamma] = [\bar\partial\beta]$, so $\bar\partial : E^{p,q}(M) \to E^{p,q+1}(M)$ is well-defined.

(c) From the argument in part (a), we have $\bar\partial^2 + \bar\mu\partial + \partial\bar\mu = 0$, so for $\alpha \in \mathcal{E}^{p,q}(M)$ with $\bar\mu\alpha = 0$ we have $\bar\partial^2\alpha = -\bar\mu\partial\alpha - \partial\bar\mu\alpha = -\bar\mu\partial\alpha$, so $\bar\partial^2[\alpha] = \bar\partial[\bar\partial\alpha] = [\bar\partial^2\alpha] = [-\bar\mu\partial\alpha] = [0] = 0$.

5. Let α be a (p,q)-form on an almost complex manifold (M,J). Find a fomula for $\mu(\alpha)$ in terms of the Nijenhuis tensor N_J. Use this to show (directly) that if $\mu : \mathcal{E}^{0,1}(M) \to \mathcal{E}^{2,0}(M)$ is zero, then J is integrable.

Solution: Note that $\mu(\alpha)$ is the $(p+2, q-1)$ component of $d\alpha$, so for $X_1, \ldots, X_{p+1} \in \Gamma(M, T^{1,0}M)$ and $Y_1, \ldots, Y_q \in \Gamma(M, T^{0,1}M)$, then

$$\begin{aligned}
&\mu(\alpha)(X_1, \ldots, X_{p+2}, Y_1, \ldots, Y_{q-1})\\
&= (d\alpha)(X_1, \ldots, X_{p+2}, Y_1, \ldots, Y_{q-1})\\
&= \sum_{k=1}^{p+2} (-1)^{k-1} X_k(\alpha(X_1, \ldots, \widehat{X_k}, \ldots, X_{p+2}, Y_1, \ldots, Y_{q-1})\\
&\quad + \sum_{k=1}^{q-1} (-1)^{(p+2+k)-1} Y_k(\alpha(X_1, \ldots, X_{p+2}, Y_1, \ldots, \widehat{Y_k}, \ldots, Y_{q-1})\\
&\quad + \sum_{1 \le j < k \le p+2} (-1)^{j+k} \alpha([X_j, X_k], X_1, \ldots, \widehat{X_j}, \ldots, \widehat{X_k}, \ldots, X_{p+2}, Y_1, \ldots, Y_{q-1})\\
&\quad + \sum_{\substack{1 \le j \le p+2\\ 1 \le k \le q-1}} (-1)^{(p+2+j)+k} \alpha([X_j, Y_k], X_1, \ldots, \widehat{X_j}, \ldots, X_{p+2}, Y_1, \ldots, \widehat{Y_k}, \ldots, Y_{q-1})\\
&\quad + \sum_{1 \le j < k \le q-1} (-1)^{(p+2+j)+(p+2+k)} \alpha([Y_j, Y_k], X_1, \ldots, X_{p+2}, Y_1, \ldots, \widehat{Y_j}, \ldots, \widehat{Y_k}, \ldots, Y_{q-1}).
\end{aligned}$$

Since α is a (p,q)-form, the summands in the first, second, fourth, and fifth sums vanish. Now note that

$$\begin{aligned}&\alpha([X_j,X_k],X_1,\dots,\widehat{X_j},\dots,\widehat{X_k},\dots,X_{p+2},Y_1,\dots,Y_{q-1})\\ =&\,\alpha([X_j,X_k]^{1,0}+[X_j,X_k]^{0,1},X_1,\dots,\widehat{X_j},\dots,\widehat{X_k},\dots,X_{p+2},Y_1,\dots,Y_{q-1})\\ =&\,\alpha([X_j,X_k]^{1,0},X_1,\dots,\widehat{X_j},\dots,\widehat{X_k},\dots,X_{p+2},Y_1,\dots,Y_{q-1})\\ &+\alpha([X_j,X_k]^{0,1},X_1,\dots,\widehat{X_j},\dots,\widehat{X_k},\dots,X_{p+2},Y_1,\dots,Y_{q-1}).\end{aligned}$$

Again, because α is a (p,q)-form, the first term vanishes. Writing $X_j = U_J - iJU_j$ and $X_k = U_k - iJU_k$, we see that $[X_j,X_k]^{0,1} = [U_j - iJU_j, U_k - iJU_k]^{0,1} = N_J(U_j,U_k) + iJN_J(U_j,U_k) = 2N_J(U_j,U_k)^{0,1}$. Therefore,

$$\begin{aligned}&\mu(\alpha)(X_1,\dots,X_{p+2},Y_1,\dots,Y_{q-1})\\ =&\sum_{1\le j<k\le p+2}(-1)^{j+k}\alpha(2N_J(U_j,U_k)^{0,1},X_1,\dots,\widehat{X_j},\dots,\widehat{X_k},\dots,X_{p+2},Y_1,\dots,Y_{q-1}).\end{aligned}$$

When $\alpha\in\mathcal{E}^{0,1}(M)$, we have

$$\mu(\alpha)(X_1,X_2) = -\alpha(2N_J(U_1,U_2)^{0,1}) = -2\alpha(N_J(U_1,U_2)^{0,1}) = -2\alpha(N_J(U_1,U_2)).$$

Let g be a bundle metric on $T^{0,1}M$ and consider $\alpha\in\mathcal{E}^{0,1}(M)$ given by $\alpha(X) = g(X^{0,1}, N_J(U_1,U_2)^{0,1})$. Note that

$$\mu(\alpha)(X_1,X_2) = -2\alpha(N_J(U_1,U_2)) = -2g(N_J(U_1,U_2)^{0,1},N_J(U_1,U_2)^{0,1}) = -2\|N_J(U_1,U_2)^{0,1}\|^2.$$

Therefore, if $\mu:\mathcal{E}^{0,1}(M)\to\mathcal{E}^{0,2}(M)$ is the zero map, then $N_J(U_1,U_2)^{0,1}=0$. Since $N_J(U_1,U_2)$ is real, we also have $N_J(U_1,U_2)^{1,0} = \overline{N_J(\partial_1,\partial_2)^{0,1}} = 0$ and hence $N_J(U_1,U_2)=0$.

6. Let (M,J) be an almost complex manifold. Show that if $\bar\partial^2=0$, then J is integrable. (Hint: compute $\bar\partial^2 f$ for a function f.)

Solution: Let $Y_1,Y_2\in\Gamma(M,T^{0,1}M)$. Note that $\bar\partial^2 f = \bar\partial(\bar\partial f)$ is the $(0,2)$ component of $d(\bar\partial f)$, so

$$\begin{aligned}(\bar\partial^2 f)(Y_1,Y_2) &= \bar\partial(\bar\partial f)(Y_1,Y_2)\\ &= d(\bar\partial f)(Y_1,Y_2)\\ &= Y_1(\bar\partial f)(Y_2) - Y_2(\bar\partial f)(Y_1) - (\bar\partial f)([Y_1,Y_2])\end{aligned}$$

Since $\bar\partial f$ is a $(0,1)$-form, we have $(\bar\partial f)([Y_1,Y_2]) = (\bar\partial f)([Y_1,Y_2]^{1,0}+[Y_1,Y_2]^{0,1}) = (\bar\partial f)([Y_1,Y_2]^{0,1})$. Moreover, since ∂f is the $(0,1)$ component of df, we have

$$\begin{aligned}(\bar\partial^2 f)(Y_1,Y_2) &= Y_1(\bar\partial f)(Y_2) - Y_2(\bar\partial f)(Y_1) - (\bar\partial f)([Y_1,Y_2]^{0,1})\\ &= Y_1(df)(Y_2) - Y_2(df)(Y_1) - (df)([Y_1,Y_2]^{0,1})\\ &= Y_1Y_2f - Y_2Y_1f - [Y_1,Y_2]^{0,1}f\\ &= [Y_1,Y_2]f - [Y_1,Y_2]^{0,1}f\\ &= [Y_1,Y_2]^{1,0}f.\end{aligned}$$

So if $\bar\partial^2=0$, then $[Y_1,Y_2]^{1,0}f=0$ for all $f\in C^\infty(M)$, so $[Y_1,Y_2]^{0,1}=0$. Since Y_1 and Y_2 are arbitrary, we see that $\Gamma(M,T^{0,1}M)$ is closed under Lie bracket, and hence J is integrable.

7. Let E be a real rank $2n$ bundle equipped with an orientation and a bundle metric. Let $P_{SO(2n)}(E)$ denote the oriented orthonormal frame bundle of E, i.e. the induced principal $SO(2n)$-bundle which gives a reduction of structure group of $P_{GL(2n,\mathbb{R})}(E)$ to $SO(2n)$. Show that $P_{SO(2n)}(E)$ admits a reduction of structure group to $U(n)$ if and only if E admits an almost complex structure which induces the given orientation and is compatible with the given bundle metric.

Solution: Suppose E admits an almost complex structure J which induces the given orientation and is compatible with the given bundle metric. Since J is compatible with the given bundle metric, it gives rise to a hermitian bundle metric on the complex vector bundle (E, J). Then the unitary frame bundle $P_{U(n)}(E) \subset P_{SO(2n)}(E)$ is a principal $U(n)$-bundle, so $P_{SO(2n)}(E)$ admits a reduction of structure group to $U(n)$.

Conversely, let $P_{U(n)} \hookrightarrow P_{SO(2n)}(E)$ be a reduction of structure group. If $v \in E_x$ and $f \in P_{U(n)}$ is a frame for E_x, we set $Jv = fJ_{2n}f^{-1}(v)$. To see this is well-defined, note that if $f' \in P_{U(n)}$, then $f' = f \circ A$ for some $A \in U(n)$ so

$$f' \circ J_{2n} \circ (f')^{-1} = (f \circ A) \circ J_{2n} \circ (f \circ A)^{-1} = f \circ (A \circ J_{2n} \circ A^{-1}) \circ f^{-1} = f \circ J_{2n} \circ f^{-1}.$$

Clearly $J^2 = -\,\mathrm{id}_E$. If V is a non-empty open set over which E is trivial (as a complex vector bundle), then $P_{U(n)}(E|_V) = P_{U(n)}(E)|_V$ is trivial, so one can find a continuous section $\sigma : V \to P_{U(n)}(E|_V)$. Then $J|_V = \sigma \circ J_{2n} \circ \psi$ where $\psi(v) = \sigma(v)^{-1}$. Since σ is continuous, so too is ψ, and hence J is a continuous section of $\mathrm{End}(E)$.

To see that J induces the given orientation, note that if $\{e_1, \ldots, e_{2n}\}$ is the standard basis for $\mathbb{R}^{2n}$, and $v_j := f(e_j)$, then $\{v_1, \ldots, v_{2n}\}$ is an oriented basis for E_x (since f preserves orientation). As $v_{2j} = f(e_{2j}) = f(J_{2n}e_{2j-1}) = f(J_{2n}f^{-1}(v_{2j-1})) = Jv_{2j-1}$, we see that $\{v_1, Jv_1, v_3, Jv_3, \ldots, v_{2n-1}, Jv_{2n-1}\}$ is an oriented basis for E_x, and hence J induces the correct orientation.

Finally, denote the bundle metric on E by g and let g_{Eucl} denote the standard inner product on $\mathbb{R}^{2n}$. Then we have

$$\begin{aligned} g(Ju_1, Ju_2) &= g(f(J_{2n}f^{-1}u_1), f(J_{2n}f^{-1}u_2)) \\ &= g_{\mathrm{Eucl}}(J_{2n}f^{-1}u_1, J_{2n}f^{-1}u_2) \\ &= g_{\mathrm{Eucl}}(f^{-1}u_1, f^{-1}u_2) \\ &= g(u_1, u_2), \end{aligned}$$

so J is compatible with g.