ALMOST COMPLEX MANIFOLDS – ASSIGNMENT 1 SOLUTIONS

1. Let J be a linear complex structure on a real vector space V. Show that $\mathrm{tr}(J) = 0$ and $\det(J) = 1$.

Solution: One approach is to use the fact that for any choice of isomorphism $\varphi : V \to \mathbb{R}^{2n}$, and any endomorphism $A : V \to V$, we have $\mathrm{tr}(A) = \mathrm{tr}(\varphi A \varphi^{-1})$ and $\det(A) = \det(\varphi A \varphi^{-1})$. By choosing a complex basis for V, we get an isomorphism $\varphi : V \to \mathbb{R}^{2n}$ such that $\varphi J \varphi^{-1} = J_{2n}$. Then $\mathrm{tr}(J_{2n}) = 0$ and $\det(J_{2n}) = \left(\det \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)^n = 1^n = 1$.

Here's another approach. If $\{b_1, b_2, \dots, b_{2n}\}$ is a basis for V, then for an endomorphism $A : V \to V$ we have

$$\begin{aligned} \mathrm{tr}(A) b_1 \wedge b_2 \wedge \cdots \wedge b_{2n} &= \sum_{k=1}^{2n} b_1 \wedge b_2 \wedge \cdots \wedge b_{k-1} \wedge A b_k \wedge b_{k+1} \wedge \cdots \wedge b_{2n} \\ \det(A) b_1 \wedge b_2 \wedge \cdots \wedge b_{2n} &= A b_1 \wedge A b_2 \wedge \cdots \wedge A b_{2n}. \end{aligned}$$

Let $\{v_1, v_2, \dots, v_n\}$ be a complex basis for V, then $\{v_1, Jv_1, v_2, Jv_2, \dots, v_n, Jv_n\}$ is a real basis for V.

$$\begin{aligned} &\mathrm{tr}(J) v_1 \wedge Jv_1 \wedge v_2 \wedge Jv_2 \wedge \cdots \wedge v_n \wedge Jv_n \\ = &\sum_{k=1}^{n} v_1 \wedge Jv_1 \wedge v_2 \wedge Jv_2 \wedge \cdots \wedge v_{k-1} \wedge Jv_{k-1} \wedge J(v_k) \wedge Jv_k \wedge v_{k+1} \wedge Jv_{k+1} \wedge \cdots \wedge v_n \wedge Jv_n \\ + &\sum_{k=1}^{n} v_1 \wedge Jv_1 \wedge v_2 \wedge Jv_2 \wedge \cdots \wedge v_{k-1} \wedge Jv_{k-1} \wedge v_k \wedge J(Jv_k) \wedge v_{k+1} \wedge Jv_{k+1} \wedge \cdots \wedge v_n \wedge Jv_n \\ = &\, 0 + 0 \\ = &\, 0 \end{aligned}$$

so $\mathrm{tr}(J) = 0$, and

$$\begin{aligned} &\det(J) v_1 \wedge Jv_1 \wedge v_2 \wedge Jv_2 \wedge \cdots \wedge v_n \wedge Jv_n \\ = &\, J(v_1) \wedge J(Jv_1) \wedge J(v_2) \wedge J(Jv_2) \wedge \cdots \wedge J(v_n) \wedge J(Jv_n) \\ = &\, Jv_1 \wedge -v_1 \wedge Jv_2 \wedge -v_2 \wedge \cdots \wedge Jv_n \wedge -v_n \\ = &\, v_1 \wedge Jv_1 \wedge v_2 \wedge Jv_2 \wedge \cdots \wedge v_n \wedge Jv_n \end{aligned}$$

so $\det(J) = 1$.

2. Determine those n for which J_{2n} and $\widetilde{J}_{2n}$ induce the same orientation on $\mathbb{R}^{2n}$.

Solution: From lectures, we have $J_{2n} = \alpha^{-1} \widetilde{J}_{2n} \alpha$ where $\alpha(e_{2k}) = e_{n+k}$ and $\alpha(e_{2k-1}) = e_k$. So J_{2n} and $\widetilde{J}_{2n}$ induce the same orientation on $\mathbb{R}^{2n}$ if and only if $\det(\alpha) > 0$.

Note that

$$\begin{aligned} &\alpha(e_1) \wedge \alpha(e_2) \wedge \alpha(e_3) \wedge \alpha(e_4) \wedge \alpha(e_5) \wedge \alpha(e_6) \wedge \cdots \wedge \alpha(e_{2n-1}) \wedge \alpha(e_{2n}) \\ = &\, e_1 \wedge e_{n+1} \wedge e_2 \wedge e_{n+2} \wedge e_3 \wedge e_{n+3} \wedge \cdots \wedge e_n \wedge e_{2n} \\ = &\, -e_1 \wedge e_2 \wedge e_{n+1} \wedge e_{n+2} \wedge e_3 \wedge e_{n+3} \wedge \cdots \wedge e_n \wedge e_{2n} \\ = &\, -e_1 \wedge e_2 \wedge e_3 \wedge e_{n+1} \wedge e_{n+2} \wedge e_{n+3} \wedge \cdots \wedge e_n \wedge e_{2n}. \end{aligned}$$

In general, we swap e_k with $e_{n+1} \wedge e_{n+2} \wedge \cdots \wedge e_{n+k-1}$ which contributes a factor of $(-1)^{k-1}$, so

$$\begin{aligned}
&\alpha(e_1) \wedge \alpha(e_2) \wedge \alpha(e_3) \wedge \alpha(e_4) \wedge \alpha(e_5) \wedge \alpha(e_6) \wedge \cdots \wedge \alpha(e_{2n-1}) \wedge \alpha(e_{2n}) \\
&= (-1)^1(-1)^2 \cdots (-1)^{n-1} e_1 \wedge e_2 \wedge e_3 \wedge \cdots \wedge e_{2n} \\
&= (-1)^{1+2+\cdots+(n-1)} e_1 \wedge e_2 \wedge e_3 \wedge \cdots \wedge e_{2n} \\
&= (-1)^{n(n-1)/2} e_1 \wedge e_2 \wedge e_3 \wedge \cdots \wedge e_{2n}.
\end{aligned}$$

So $\det(\alpha) > 0$ is equivalent to $n(n-1)/2 \in 2\mathbb{Z}$ and hence $n(n-1) \in 4\mathbb{Z}$. Since $n-1$ and n are consecutive integers, one is even and the other is odd, so the product is divisible by 4 if and only if one of the two numbers is divisible by 4. Therefore, $\det(\alpha) > 0$ if and only if $n \equiv 0, 1 \bmod 4$.

3. Let V be a real vector space and let $g : V \times V \to \mathbb{R}$ be a symmetric, non-degenerate, bilinear map. We can view g as a not necessarily positive-definite inner product on V. Suppose g has signature (r, s) and J is a linear complex structure on V compatible with g. Show that r and s are even.

Solution: Let V^+ be a maximal subspace on which g is positive-definite, so $r = \dim V^+$. For $v \in V^+$, note that $g(Jv, Jv) = g(v, v) \geq 0$. Since V^+ is maximal, we have $Jv \in V^+$ and hence $J|_{V^+} : V^+ \to V^+$ is a linear complex structure on V^+, so $r = \dim V^+$ is even. Arguing similarly with a maximal subspace on which g is negative-definite, we see that s is also even (alternatively, use $\dim V = r + s$).

4. Let V be an even-dimensional real vector space equipped with an inner product g. Without choosing an isomorphism between V and $\mathbb{R}^{2n}$, show that V admits a linear complex structure J which is compatible with g.

Solution: Since V is even-dimensional, it admits a linear complex structure, say J_0. Fix an inner product g_0 which is compatible with J_0, i.e. $g_0(J_0v, J_0w) = g_0(v, w)$. Note that g and g_0 determine isomorphisms $\Phi_g : V \to V^*$ and $\Phi_{g_0} : V \to V^*$ given by $v \mapsto g(v, \cdot)$ and $v \mapsto g_0(v, \cdot)$ respectively. Let $P = \Phi_{g_0}^{-1}\Phi_g : V \to V$. Note that

$$g(v, w) = \Phi_g(v)(w) = (\Phi_{g_0}\Phi_{g_0}^{-1}\Phi_g)(v)(w) = (\Phi_{g_0}P)(v)(w) = \Phi_{g_0}(Pv)(w) = g_0(Pv, w).$$

Now note that $g_0(Pv, w) = g(v, w) = g(w, v) = g_0(Pw, v) = g_0(v, Pw)$ so $P^* = P$, i.e. P is symmetric with respect to g_0. Furthermore, we have $g_0(Pv, v) = g(v, v) > 0$ for v non-zero, so P is positive-definite with respect to g_0. Therefore P has a unique positive-definite square root Q which is symmetric with respect to g_0. Note that Q is invertible, and since $P = Q^2$, we have $PQ^{-1} = Q$.

Define $J = Q^{-1}J_0Q$. Note that $J^2 = Q^{-1}J_0QQ^{-1}J_0Q = Q^{-1}J_0J_0Q = -Q^{-1}Q = -\mathrm{id}_V$, so J is a linear complex structure on V. We also have

$$\begin{aligned}
g(Jv, Jw) &= g_0(PJv, Jw) \\
&= g_0(PQ^{-1}J_0Qv, Q^{-1}J_0Qw) \\
&= g_0(QJ_0Qv, Q^{-1}J_0Qw) \\
&= g_0(J_0Qv, Q^*Q^{-1}J_0Qw) \\
&= g_0(J_0Qv, QQ^{-1}J_0Qw) \\
&= g_0(J_0Qv, J_0Qw) \\
&= g_0(Qv, Qw) \\
&= g_0(Q^*Qv, w) \\
&= g_0(QQv, w) \\
&= g_0(Pv, w) \\
&= g(v, w),
\end{aligned}$$

so J is compatible with g.

5. Let V be a finite-dimensional real vector space such that there is a complex subspace $W \subseteq V_{\mathbb{C}}$ with $V_{\mathbb{C}} = W \oplus \overline{W}$. Show that there is a unique linear complex structure J on V such that $V^{1,0} = W$ and $V^{0,1} = \overline{W}$.

Solution: Note that $V \hookrightarrow V_{\mathbb{C}} = W \oplus \overline{W}$, so we can write $v = \pi_W(v) + \pi_{\overline{W}}(v)$ where $\pi_W : V_{\mathbb{C}} \to W$ and $\pi_{\overline{W}} : V_{\mathbb{C}} \to \overline{W}$ are the natural projections.

If J were a linear complex structure on V with $V^{1,0} = W$ and $V^{0,1} = \overline{W}$, then $\pi_W(v) = \pi_{V^{1,0}}(v) = \frac{1}{2}(v - iJv)$ and $\pi_{\overline{W}}(v) = \pi_{V^{0,1}}(v) = \frac{1}{2}(v + iJv)$, so $Jv = i(\pi_W(v) - \pi_{\overline{W}}(v))$. So given a splitting $V_{\mathbb{C}} = W \oplus \overline{W}$, for $v \in V$, set $Jv = i(\pi_W(v) - \pi_{\overline{W}}(v)) = i\pi_W(v) - i\pi_{\overline{W}}(v)$.

If $v \in V \subset V_{\mathbb{C}}$ then $\overline{v} = v$, so $\overline{\pi_W(v)} = \pi_{\overline{W}}(v)$ and $\overline{\pi_{\overline{W}}(v)} = \pi_W(v)$, so

$$\overline{Jv} = \overline{i\pi_W(v) - i\pi_{\overline{W}}(v)} = -i\overline{\pi_W(v)} + i\overline{\pi_{\overline{W}}(v)} = -i\pi_{\overline{W}}(v) + i\pi_W(v) = Jv,$$

and hence $Jv \in V$, so $J : V \to V$.

Since $Jv = i\pi_W(v) - i\pi_{\overline{W}}(v)$, we have $\pi_W(Jv) = i\pi_W(v)$ and $\pi_{\overline{W}}(Jv) = -i\pi_{\overline{W}}(v)$, so

$$J(Jv) = i\pi_W(Jv) - i\pi_{\overline{W}}(Jv) = i^2\pi_W(v) + i^2\pi_{\overline{W}}(v) = -\pi_W(v) - \pi_{\overline{W}}(v) = -v.$$

Therefore J is a linear complex structure on V. Let $J_{\mathbb{C}}$ be the complex linear extension of J. Note that

$$J_{\mathbb{C}}(iv) = iJ(v) = i(i\pi_W(v) - i\pi_{\overline{W}}(v)) = i(\pi_W(iv) - \pi_{\overline{W}}(iv)) = i\pi_W(iv) - i\pi_{\overline{W}}(iv).$$

It follows that $J_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$ is given by the same formula as J, namely $J_{\mathbb{C}}v = i\pi_W(v) - i\pi_{\overline{W}}(v)$.

Note that

$$\begin{aligned} V^{1,0} &= \{v \in V_{\mathbb{C}} \mid J_{\mathbb{C}}v = iv\} \\ &= \{v \in V_{\mathbb{C}} \mid i\pi_W(v) - i\pi_{\overline{W}}(v) = i\pi_W(v) + i\pi_{\overline{W}}(v)\} \\ &= \{v \in V_{\mathbb{C}} \mid \pi_{\overline{W}}(v) = 0\} \\ &= W \end{aligned}$$

and $V^{0,1} = \overline{V^{1,0}} = \overline{W}$.

To see that J is unique, suppose J' is another linear complex structure on V with $V^{1,0} = W$ and $V^{0,1} = \overline{W}$. Then $J_{\mathbb{C}}|_W = J_{\mathbb{C}}|_{V^{1,0}} = i\,\mathrm{id}_{V^{1,0}} = i\,\mathrm{id}_W$ and $J_{\mathbb{C}}|_{\overline{W}} = J_{\mathbb{C}}|_{V^{0,1}} = -i\,\mathrm{id}_{V^{0,1}} = -i\,\mathrm{id}_{\overline{W}}$, and likewise $J'_{\mathbb{C}}|_W = i\,\mathrm{id}_W$ and $J'_{\mathbb{C}}|_{\overline{W}} = -i\,\mathrm{id}_{\overline{W}}$, so $J_{\mathbb{C}} = J'_{\mathbb{C}}$. Therefore $J = J_{\mathbb{C}}|_V = J'_{\mathbb{C}}|_V = J'$.

6. Let ω be a linear symplectic form with a compatible linear complex structure J. Show that the complex bilinear extension of ω is a $(1,1)$-form.

Solution: Note that compatibility of J with ω is equivalent to $J^*\omega = \omega$. After extending complex bilinearly, we have $\omega = \omega^{2,0} + \omega^{1,1} + \omega^{0,2}$ where $\omega^{p,q}$ denotes the (p,q)-part of ω. Using the fact that $J^*\omega^{p,q} = i^{p-q}\omega$, we have

$$\begin{aligned} J^*\omega &= \omega \\ J^*\omega^{2,0} + J^*\omega^{1,1} + J^*\omega^{2,0} &= \omega^{2,0} + \omega^{1,1} + \omega^{0,2} \\ -\omega^{2,0} + \omega^{1,1} - \omega^{0,2} &= \omega^{2,0} + \omega^{1,1} + \omega^{0,2}. \end{aligned}$$

Equating (p,q)-parts, we see that $\omega^{2,0} = 0$ and $\omega^{0,2} = 0$, so $\omega = \omega^{1,1}$, i.e. ω is a $(1,1)$-form.

7. We've seen that if $E \to B$ is any real vector bundle then $E \oplus E \to B$ admits an almost complex structure. Explain why the following jump in logic is erroneous: for any smooth manifold M, the product manifold $M \times M$ admits an almost complex structure. (Note, a counterexample alone does not count as an explanation, but I encourage you to find one anyway).

Solution: At a point, $(p, q) \in M \times M$, we have $T_{p,q}(M \times M) \cong T_pM \oplus T_qM$, but the tangent bundle of $M \times M$ is not, in general, of the form $E \oplus E$.

More precisely, we have $T(M \times M) \cong \pi_1^*TM \oplus \pi_2^*TM$, but $\pi_1^*TM \not\cong \pi_2^*TM$ in general. To see this, fix $q \in M$ and consider the map $\sigma : M \to M \times M$ given by $\sigma(m) = (m, q)$, then $\sigma^*\pi_1^*TM \cong (\pi_1 \circ \sigma)^*TM \cong \mathrm{id}_M^* TM \cong TM$ while $\sigma^*\pi_2^*TM \cong (\pi_2 \circ \sigma)^*TM \cong c_q^*TM \cong \varepsilon^n$ where $c_q : M \to M$ denotes the constant map with value q and ε^n denotes the trivial real bundle of rank n. So if $\pi_1^*TM \cong \pi_2^*TM$, then TM is trivial; conversely, if TM is trivial, then $\pi_1^*TM \cong \varepsilon^n \cong \pi_2^*TM$.

So, unless M is parallelisable, the bundle $T(M \times M)$ is not of the form $E \oplus E$, so $M \times M$ does not necessarily admit an almost complex structure.

8. Let $p : F \to C$ and $\pi : E \to B$ be real vector bundles, and suppose that $\Phi : F \to E$ is a vector bundle isomorphism covering $\varphi : C \to B$. Recall, we showed that given an almost complex structure J on E, one obtains an almost complex structure J' on F.

- **(a) Show that $F \cong \varphi^*E$.**
- **(b) From the above isomorphism and the almost complex structure on E, we obtain an almost complex structure J'' on F. Show that $J'' = J'$.**

Solution: (a) Recall that there is a commutative diagram

$$\begin{array}{ccc} \varphi^*E & \xrightarrow{\mathrm{pr}_2} & E \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ C & \xrightarrow[\varphi]{} & B \end{array}$$

where $\varphi^*E = \{(c, e) \in C \times E \mid \varphi(c) = \pi(e)\}$. On the other hand, we have a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\Phi} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ C & \xrightarrow[\varphi]{} & B \end{array}$$

Define $\Psi : F \to \varphi^*E$ by $\Psi(f) = (p(f), \Phi(f))$. Note that $\Psi(f) \in \varphi^*E$ because $\varphi(p(f)) = \pi(\Phi(f))$ by commutativity of the second diagram. Since p and Φ are continuous, so is Ψ, and since Φ is an isomorphism on fibers, so too is Ψ. Therefore $\Psi : F \to \varphi^*E$ is an isomorphism of vector bundles.

(b) First note that $J' : F \to F$ is defined by $J' := \Phi^{-1}J\Phi$. On the other hand, J induces an almost complex structure $\widehat{J}$ on φ^*E given by $\widehat{J}(c, e) = (c, Je)$. Using the isomorphism $\Psi : F \to \varphi^*E$, this induces an almost complex structure J'' on F defined by $J'' := \Psi^{-1}\widehat{J}\Psi$. Unravelling the definitions, this yields

$$\begin{aligned} J''f &= (\Psi^{-1}\widehat{J}\Psi)(f) \\ &= (\Psi^{-1}\widehat{J})(p(f), \Phi(f)) \\ &= \Psi^{-1}(p(f), J\Phi(f)) \\ &= \Psi^{-1}(p(f), \Phi\Phi^{-1}J\Phi(f)) \\ &= \Psi^{-1}(p(f), \Phi(J'f)) \\ &= \Psi^{-1}(p(J'f), \Phi(J'f)) \\ &= \Psi^{-1}(\Psi(J'f)) \\ &= J'f, \end{aligned}$$

so $J'' = J'$.