ALMOST COMPLEX MANIFOLDS – PRESENTATION TOPICS

At the end of the semester, each student will give a short presentation (20 – 30 minutes). The proposed topics are listed below. I am also open to suggestions if you have something else in mind.

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1. When does the product of two spheres admit an almost complex structure?

We have already mentioned Borel and Serre's result about the existence of almost complex structures on spheres. What about the case $S^n \times S^m$? This breaks up into two cases, the first is m and n odd, and the second is m and n even. The answer in these two cases is significantly different. The first is straightforward, while the resolution in the second case requires the techniques used to prove Borel and Serre's result.

References:

- Staples - A short and elementary proof that a product of spheres is parallelizable if one of them is odd
- Datta and Subramanian – Nonexistence of almost complex structures on products of even-dimensional spheres.

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2. Classifying spaces via the Brown Representability Theorem

We will soon introduce the notion of classifying spaces in a very concrete way (as topological spaces, well-defined up to homotopy equivalence). The Brown representability theorem is a theorem in category theory which gives conditions under which a functor is representable. This provides a different approach for the existence of classifying spaces as a certain functor (which we will see) is representable by a classifying space.

References:

- Brown - Cohomology theories
- Putman - Classifying spaces and Brown representability
- Switzer - Algebraic Topology – Homotopy and Homology (chapter 9)

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3. The Gray-Hervella classification of almost hermitian manifolds

Let (M, J) be an almost complex manifold. Integrability is one possible condition one could impose on J, but in the presence of a compatible metric g, there are many other geometrically interesting conditions. Such a triple (M, J, g) is called an almost hermitian manifold. Gray and Hervella gave a classification of types of almost hermitian manifolds based on the properties of ∇J where ∇ is the Levi-Civita connection of g.

References:

- Gray and Hervella - The sixteen classes of almost hermitian manifolds and their linear invariants
- Falcitelli, Farinola, and Salamon - Almost-Hermitian geometry

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4. LeBrun's Theorem on S^6

As we have seen in the course, Borel and Serre proved that the only spheres which admit almost complex structures are S^0, S^2, and S^6. We know that the first two admit integrable almost complex structures, while it is currently unknown if S^6 does. A surprisingly short and elegant argument by LeBrun shows that S^6 does not admit an integrable almost complex structure compatible with the round metric.

References:

- LeBrun - Orthogonal complex structures on S^6
- Ferreira - Non-existence of orthogonal complex structures on the round 6-sphere

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5. Almost complex structures on homogeneous spaces

There are very few manifolds on which we can construct almost complex structures. One family of manifolds where the task is simplified is the case of homogeneous manifolds. Given a homogeneous manifold G/H, the conditions for the existence of a left-invariant almost complex structure can be expressed in terms of the Lie algebras of G and H. Moreover, the Nijenhuis tensor of such an almost complex structure can be computed efficiently.

References:

- Kobayashi and Nomizu - Foundations of Differential Geometry, Volume 2 (section X.6)

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6. Integrability of almost complex structures on non-compact manifolds, Gromov's h-principle results

The question of when a manifold admits a complex structure is a difficult one. First, one must determine whether the manifold admits an almost complex structure, and then if it does, whether one can find an integrable one. We have spent much of the course focussed on the first problem. The second problem has proved much more challenging. In the non-compact case however, Gromov used the h-principle to further our understanding in low dimensions.

References:

- Gromov - Partial Differential Relations (section 2.2.7)
- Haefliger - Lectures on the theorem of Gromov
- Landweber - Complex structures on open manifolds

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7. Generalized almost complex structures

Generalized geometry, introduced by Hitchin, aims to study complex manifolds and symplectic manifolds in a common framework. In this setup, almost complex structures and almost symplectic forms can both be viewed as examples of a generalized almost complex structure. Many of the techniques we have discussed in this course can be used to address the question of which manifolds admit a generalized almost complex structure.

References:

- Hitchin - Generalized Calabi-Yau manifolds
- Gualtieri - Generalized complex geometry