ALMOST COMPLEX MANIFOLDS – ASSIGNMENT 4

Due Monday April 15.

1. Show that a principal $U(n)$-bundle P admits a reduction of structure group to $SU(n)$ if and only if $c_1(P) = 0$.

2. Let M be a $2n$-dimensional almost complex manifold. In question 6 of assignment 3, you were asked to show that there are two potential obstructions to TM admitting a complex line subbundle. Show that the first one vanishes.

3. Consider the Lie group G_2. It can be defined in many different ways. For example, as the simply connected Lie group with Lie algebra $\mathfrak{g}_2$, or as the subgroup of $GL^+(7, \mathbb{R})$ fixing a particular 3-form on $\mathbb{R}^7$.

Let $E \to X$ be a real oriented rank 7 bundle. Determine the primary obstruction to E admitting a reduction of structure group to G_2.

4. Let M be an almost complex manifold.

(a) Show that $\bar{\mu}$ is $C^\infty(M)$-linear.

By (a), we have a vector bundle homomorphism $\bar{\mu} : \bigwedge^{1,0} T^*M \to \bigwedge^{0,2} T^*M$.

(b) Suppose that $\dim M = 4$ and $\bar{\mu}$ is a surjective bundle homomorphism. Show that $5\chi(M) + 6\sigma(M) = 0$.

5. Recall, we defined an almost complex structure on $S^6 \times \mathbb{O}$ which restricts to an almost complex structure on TS^6, and defines an almost complex structure on the normal bundle of S^6 in $\mathbb{O}$.

(a) Show that if $0 \to E \to F \to G \to 0$ is a short exact sequence of complex vector bundles over a CW complex, with F and G trivial, then $c(E) = 1$.

(b) Explain why $c(TS^6) \neq 1$ and why this does not contradict (a).

6. A manifold is called a *integral/rational homology sphere* if it has the same integral/rational cohomology groups as a sphere of the same dimension.

(a) Let M be a four-dimensional rational homology sphere. Show that M does not admit an almost complex structure (for either orientation).

(b) Let M be a four-dimensional integral homology sphere. Show that M^n does not admit an almost complex structure for any n (for either orientation).

7. For non-negative integers k and ℓ, let $M_{k,\ell} = k\mathbb{CP}^2 \# \ell\overline{\mathbb{CP}^2}$. For which k and ℓ does $M_{k,\ell}$ admit an almost complex structure inducing the given orientation?

Bonus Problems (for fun):

8. Let M be a $2n$-dimensional almost complex manifold. In question 6 of assignment 3, you were asked to show that there are two potential obstructions to TM admitting a complex line subbundle. In question 2 of this assignment, you are asked to show that the first one vanishes. Find the second obstruction.

9. Let X be a CW complex.
   (a) Show that there is a one-to-one correspondence between isomorphism classes of real line bundles on X and $H^1(X;\mathbb{Z}_2)$ given by the first Stiefel-Whitney class.
   (b) Show that there is a one-to-one correspondence between isomorphism classes of complex line bundles on X and $H^2(X;\mathbb{Z})$ given by the first Chern class.
   (c) Show that the map $\beta : H^1(X;\mathbb{Z}_2) \to H^2(X;\mathbb{Z})$ corresponds to complexificiation, i.e. $[\ell] \mapsto [\ell \otimes_{\mathbb{R}} \mathbb{C}]$.