ALMOST COMPLEX MANIFOLDS – ASSIGNMENT 3

Due March 24.

1. Show that the twistor fibration of S^4, namely $S^2 \to \mathbb{CP}^3 \xrightarrow{p} S^4$, does not admit a section, and deduce that S^4 does not admit an almost complex structure.

2. Note that $SO(2n)/U(n)$ is a closed smooth manifold of dimension $n(n-1)$. For small values of n we have

$$\begin{aligned} SO(2)/U(1) &= * = \mathbb{CP}^0 \\ SO(4)/U(2) &= S^2 = \mathbb{CP}^1 \\ SO(6)/U(3) &= \mathbb{CP}^3. \end{aligned}$$

When $n = 4$, we have $n(n-1) = 12$. If the pattern were to continue, then we would be able to identify $SO(8)/U(4)$ with $\mathbb{CP}^6$. Show that $SO(8)/U(4)$ and $\mathbb{CP}^6$ are not homotopy equivalent.

3. Recall, a space X is called m-connected if $\pi_i(X) = 0$ for $i \leq m$.

   (a) Show that $O(n)/O(n-k)$ is $(n-k-1)$-connected.

   (b) Show that $U(n)/U(n-k)$ is $(2n-2k)$-connected and $\pi_{2n-2k+1}(U(n)/U(n-k)) \cong \mathbb{Z}$.

4. Let M be a closed simply connected four-manifold and let $p \in M$.

   (a) Show that $M \setminus \{p\}$ admits an almost complex structure.

   (b) Show that $M \times S^2$ admits an almost complex structure.

5. Let $E \to X$ be a rank m orientable vector bundle over a d-dimensional CW complex. If $m > d$, show that $E \cong E_0 \oplus \varepsilon^{m-d}$ where $\mathrm{rank}(E_0) = d$. Give an example to show that E_0 is not unique (up to isomorphism).

6. Let M be a $2n$-dimensional almost complex manifold. Show that there are two potential obstructions to TM admitting a complex line subbundle.

7. Show that a smooth manifold M admits an almost complex structure if and only if it admits a non-degenerate 2-form.

Bonus Problems (for fun):

8. Any continuous map can be extended to a fibration. More precisely, we have the following:

Theorem. *Given any continuous map $f : A \to B$, there is a fibration $p : E \to B$ and a homotopy equivalence $i : A \to E$ such that $f = p \circ i$.*

Use this to show that if X is a connected CW complex with $\pi_1(X) \cong \mathbb{Z}$ and $\pi_n(X) = 0$ for $n \geq 2$, then X is homotopy equivalent to S^1.

9. A surface bundle is a fiber bundle whose fiber is a surface. Consider $M_{k,\ell} = k\mathbb{CP}^2 \# \ell\overline{\mathbb{CP}^2}$. Show that if $M_{k,\ell}$ is the total space of a surface bundle, then $k = \ell = 1$.