ALMOST COMPLEX MANIFOLDS – ASSIGNMENT 2

Due Friday March 1.

1. Let (M, J) be an almost complex manifold and set $N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$. Show the following:

(a) $N_J(X,Y) = -N_J(Y,X)$,

(b) $N_J(JX,Y) = -JN_J(X,Y)$, and

(c) $N_J(fX,Y) = fN_J(X,Y)$ for $f \in C^\infty(M)$.

2. Consider the almost complex structure J on $\mathbb{R}^4$ given by

$$J = \begin{bmatrix} 0 & 1 & f & -g \\ -1 & 0 & g & f \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

where $f, g \in C^\infty(\mathbb{R}^4)$.

(a) Compute the Nijenhuis tensor of J. That is, compute $N_J\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right)$ for $j, k \in \{1, 2, 3, 4\}$.

(b) For $a, b \in \mathbb{R}$, consider the function $h : \mathbb{C} \to \mathbb{C}$ given by $h_{a,b}(z) = f(x, y, a, b) + ig(x, y, a, b)$ where $z = x + iy$. Show that J is integrable if and only if $h_{a,b}$ is holomorphic for all $a, b \in \mathbb{R}$.

3. Let (Y, J) be an almost complex manifold with a properly embedded submanifold X such that $T_xX \subseteq T_xY$ is a complex subspace for all $x \in X$. Show that J induces an almost complex structure J_X on X, and if J is integrable, so too is J_X.

4. Let (M, J) be an almost complex manifold.

(a) Show that $\bar{\mu}^2 = 0$.

Let $E^{p,q}(M) = H^{p,q}(\mathcal{E}^{\bullet,\bullet}(M), \bar{\mu}) := \dfrac{\ker(\bar{\mu} : \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p-1,q+2}(M))}{\operatorname{im}(\bar{\mu} : \mathcal{E}^{p+1,q-2}(M) \to \mathcal{E}^{p,q}(M))}$.

(b) Show that $\bar{\partial} : \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p,q+1}(M)$ descends to a well-defined map $\bar{\partial} : E^{p,q}(M) \to E^{p,q+1}(M)$, $[\alpha] \mapsto [\bar{\partial}\alpha]$.

(c) Show that $\bar{\partial} : E^{p,q}(M) \to E^{p,q+1}(M)$ satisfies $\bar{\partial}^2 = 0$.

5. Let α be a (p, q)-form on an almost complex manifold (M, J). Find a fomula for $\mu(\alpha)$ in terms of the Nijenhuis tensor N_J. Use this to show (directly) that if $\mu : \mathcal{E}^{0,1}(M) \to \mathcal{E}^{2,0}(M)$ is zero, then J is integrable.

6. Let (M, J) be an almost complex manifold. Show that if $\bar{\partial}^2 = 0$, then J is integrable. (Hint: compute $\bar{\partial}^2 f$ for a function f.)

7. Let E be a real rank $2n$ bundle equipped with an orientation and a bundle metric. Let $P_{SO(2n)}(E)$ denote the oriented orthonormal frame bundle of E, i.e. the induced principal $SO(2n)$-bundle which gives a reduction of structure group of $P_{GL(2n,\mathbb{R})}(E)$ to $SO(2n)$. Show that $P_{SO(2n)}(E)$ admits a reduction of structure group to $U(n)$ if and only if E admits an almost complex structure which induces the given orientation and is compatible with the given bundle metric.

Bonus problems (for fun):

8. Let M be an orientable 6-manifold which admits an immersion into $\mathbb{R}^7$. Calculate the Nijenhuis tensor for the almost complex structure on M we defined in class.

9. Let (M, J) be an almost complex manifold. Consider the map $\mathcal{L}_J = di_J - i_J d$ where $i_J : \mathcal{E}^m(M) \to \mathcal{E}^m(M)$ is given by

$$i_J(\alpha)(X_1, \dots, X_m) = \sum_{k=1}^{m} \alpha(X_1, \dots, X_{k-1}, JX_k, X_{k+1}, \dots, X_m).$$

Show that $\mathcal{L}_J^2 = 0$ if and only if $N_J = 0$.