# ALMOST COMPLEX MANIFOLDS – ASSIGNMENT 1

**Due Friday February 9**.

1. Let $J$ be a linear complex structure on a real vector space $V$. Show that $\mathrm{tr}(J) = 0$ and $\det(J) = 1$.

2. Determine those $n$ for which $J_{2n}$ and $\widetilde{J}_{2n}$ induce the same orientation on $\mathbb{R}^{2n}$.

3. Let $V$ be a real vector space and let $g : V \times V \to \mathbb{R}$ be a symmetric, non-degenerate, bilinear map. We can view $g$ as a not necessarily positive-definite inner product on $V$. Suppose $g$ has signature $(r, s)$ and $J$ is a linear complex structure on $V$ compatible with $g$. Show that $r$ and $s$ are even.

4. Let $V$ be an even-dimensional real vector space equipped with an inner product $g$. Without choosing an isomorphism between $V$ and $\mathbb{R}^{2n}$, show that $V$ admits a linear complex structure $J$ which is compatible with $g$.

5. Let $V$ be a finite-dimensional real vector space such that there is a complex subspace $W \subseteq V_{\mathbb{C}}$ with $V_{\mathbb{C}} = W \oplus \overline{W}$. Show that there is a unique linear complex structure $J$ on $V$ such that $V^{1,0} = W$ and $V^{0,1} = \overline{W}$.

6. Let $\omega$ be a linear symplectic form with a compatible linear complex structure $J$. Show that the complex bilinear extension of $\omega$ is a $(1, 1)$-form.

7. We've seen that if $E \to B$ is any real vector bundle then $E \oplus E \to B$ admits an almost complex structure. Explain why the following jump in logic is erroneous: for any smooth manifold $M$, the product manifold $M \times M$ admits an almost complex structure. (Note, a counterexample alone does not count as an explanation, but I encourage you to find one anyway).

8. Let $p : F \to C$ and $\pi : E \to B$ be real vector bundles, and suppose that $\Phi : F \to E$ is a vector bundle isomorphism covering $\varphi : C \to B$. Recall, we showed that given an almost complex structure $J$ on $E$, one obtains an almost complex structure $J'$ on $F$.

   (a) Show that $F \cong \varphi^* E$.

   (b) From the above isomorphism and the almost complex structure on $E$, we obtain an almost complex structure $J''$ on $F$. Show that $J'' = J'$.

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**Bonus Problems (for fun):**

9. Let $V$ be a real vector space which admits two anticommuting linear complex structures. Show that $V$ inherits the structure of a module over the quaternions.

10. Let $V$ be a complex vector space. Establish the 2-out-of-3 property for $U(V, h)$ in direct analogy with the 2-out-of-3 property for the unitary group as in lectures (in this case, $h$ will be a hermitian form on a module over the quaternions). Define all the necessary structures, groups, and compatibility relations, and determine the necessary intersections.