THE YAMABE INVARIANT OF COMPLEX SURFACES

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Abstract. These are notes for a series of four online lectures delivered as part of the 6th Geometry-Topology Summer School hosted by the Feza Gürsey Institute.

Overview

Throughout these lectures, we will be focussing on compact complex surfaces X, analysing them from two different points of view.

First, we consider X as a smooth manifold, i.e. forget the complex structure. We can then ask questions about the types of Riemannian metrics this manifold can admit. In particular, there is a real-valued invariant called the Yamabe invariant of X, denoted $Y(X)$, which arises from such considerations.

On the other hand, we can analyse X as a complex manifold. In particular, there is an associated invariant called the Kodaira dimension of X, denoted $\kappa(X)$, which takes values in $\{-\infty, 0, 1, 2\}$.

Question: What is the relationship (if any) between these two quantities?

The reason for restricting our attention to complex dimension two will be made clear.

1. The Yamabe Invariant

Reference: Besse - Einstein Manifolds, chapter 4.

Let M be a closed smooth manifold of dimension $n \geq 2$, and denote the set of Riemannian metrics on M by $\operatorname{Riem}(M)$. The *total scalar curvature functional* $\operatorname{Riem}(M) \to \mathbb{R}$ is given by

$$g \mapsto \int_M s_g d\mu_g.$$

Here s_g denotes the scalar curvature of g, and $d\mu_g$ the Riemannian volume density.

This functional is also known as the Einstein-Hilbert functional. Hilbert showed that the Euler-Lagrange equations of a constant multiple of this functional are precisely the Einstein field equations (with cosmological constant zero) in general relativity. In more mathematical language, the critical points of the total scalar curvature functional are Ricci-flat metrics.

If $c > 0$, then we have $s_{cg} = c^{-1}s_g$ and $d\mu_{cg} = c^{\frac{n}{2}}d\mu_g$, so

$$\int_M s_{cg} d\mu_g = c^{\frac{n}{2}-1}\int_M s_g d\mu_g.$$

On the other hand

$$\operatorname{Vol}(M, cg) = \int_M d\mu_{cg} = c^{\frac{n}{2}}\int_M d\mu_g = c^{\frac{n}{2}}\operatorname{Vol}(M, g).$$

To remove the dependence on constant rescaling, we rescale the total scalar curvature functional by an appropriate power of the volume. Note that $\frac{n}{2} - 1 = \frac{n-2}{2} = \frac{n}{2}\frac{n-2}{n}$ so we consider the *normalised Einstein-Hilbert functional* $\mathcal{E} : \mathrm{Riem}(M) \to \mathbb{R}$ given by

$$\mathcal{E}(g) = \frac{\displaystyle\int_M s_g d\mu_g}{\mathrm{Vol}(M, g)^{\frac{n-2}{n}}}.$$

Equivalently, we could have restricted the total scalar functional to the set of Riemannian metrics on M with volume 1.

Example: When $n = 2$, we have

$$\mathcal{E}(g) = \int_M s_g d\mu_g = \int_M 2K_g d\mu_g = 4\pi\chi(M)$$

by the Gauss-Bonnet Theorem; here K_g denotes the Gaussian curvature of g.

For $n > 2$, the functional $\mathcal{E}$ is not constant. Moreover, Hilbert showed that it's critical points are precisely Einstein metrics (Besse Theorem 4.21). Recall, an *Einstein metric* is a Riemannian metric with $\mathrm{Ric}_g = \lambda g$ for some constant λ; in the unnormalised case, $\lambda = 0$. Taking the trace of both sides shows that $\lambda = \frac{1}{n}s_g$. The tensor $\mathrm{Ric} - \frac{1}{n}s_g g$ is called the *trace-free Ricci tensor* and is denoted by $\mathring{\mathrm{Ric}}$. If $n > 2$, $\mathring{\mathrm{Ric}}$ vanishes if and only if g is Einstein, while for $n = 2$, it always vanishes because $\mathrm{Ric}_g = K_g g = \frac{1}{2}s_g g$. Recall, in the unnormalised case, the critical points were metrics with vanishing Ricci curvature, while in the normalised case, the critical points are metrics with vanishing trace-free Ricci curvature. (Einstein three dim?)

Note that while the value $\mathcal{E}(g)$ doesn't change under constant rescaling, that is no longer true of conformal rescaling. Recall, we say $\tilde{g}$ is *conformal* to g if there is a smooth positive function $u : M \to (0, \infty)$ with $\tilde{g} = ug$. This gives rise to an equivalence relation of $\mathrm{Riem}(M)$ and the equivalence classes are called *conformal classes*; we denote the conformal class of g by $[g]$. While the functional is not constant when restricted to a conformal class, we know the following:

Proposition 1.1. *If $\tilde{g}$ is conformal to g, then $\mathcal{E}(\tilde{g}) \geq -\|s_g\|_{\frac{n}{2}} = -\left(\int_M |s_g|^{\frac{n}{2}} d\mu_g\right)^{\frac{2}{n}}$. That is, $\mathcal{E}$ restricted to $[g]$ is bounded below.*

Proof. If $n = 2$, we have $\mathcal{E}(\tilde{g}) = \int_M s_{\tilde{g}} d\mu_g = 4\pi\chi(M) = \int_M s_g d\mu_g \geq -\int_M |s_g| d\mu_g$.

Now suppose $n > 2$. Let $\tilde{g} = u^{\frac{4}{n-2}} g$ where u is a positive smooth function. Then $s_{\tilde{g}} = u^{-\frac{n+2}{n-2}} L_g u$ where L_g is the conformal Laplacian given by $L_g u = 4\frac{n-1}{n-2}\Delta_g u + s_g u$; here Δ_g denotes the Laplace-de Rham operator (with non-negative spectrum) $\Delta_d = d^*d + dd^*$. As $d\mu_{\tilde{g}} = (u^{\frac{4}{n-2}})^{\frac{n}{2}} d\mu_g = u^{\frac{2n}{n-2}} d\mu_g$, we have

$$\begin{aligned}
\int_M s_{\tilde{g}} d\mu_{\tilde{g}} &= \int_M u^{-\frac{n+2}{n-2}} \left(4\frac{n-1}{n-2}\Delta_g u + s_g u\right) u^{\frac{2n}{n-2}} d\mu_g \\
&= \int_M 4\frac{n-1}{n-2} u\Delta_g u + s_g u^2 d\mu_g \\
&= \int_M 4\frac{n-1}{n-2} |du|^2 + s_g u^2 d\mu_g
\end{aligned}$$

$$\begin{aligned} &\geq \int_M s_g u^2 d\mu_g \\ &\geq -\int_M |s_g u^2| d\mu_g \\ &= -\|s_g u^2\|_1 \\ &\geq -\|s_g\|_{\frac{n}{2}} \|u^2\|_{\frac{n}{n-2}} \end{aligned}$$

where the final inequality is Hölder's inequality (with $p = \frac{n}{2}$ and $q = \frac{n}{n-2}$). Now note that

$$\|u^2\|_{\frac{n}{n-2}} = \left(\int_M u^{\frac{2n}{n-2}} d\mu_g\right)^{\frac{n-2}{n}} = \left(\int_M d\mu_{\tilde{g}}\right)^{\frac{n-2}{n}} = \mathrm{Vol}(M, \tilde{g})^{\frac{n-2}{n}}$$

so $\mathcal{E}(\tilde{g}) \geq -\|s_g\|_{\frac{n}{2}}$. □

With this information in hand, we define

$$Y(M, \mathcal{C}) = \inf_{g \in \mathcal{C}} \mathcal{E}(g)$$

to be the *Yamabe constant* of the conformal class $\mathcal{C}$. If g realises the infimum (i.e. $Y(M, \mathcal{C}) = \mathcal{E}(g)$), we call g a *Yamabe metric* (or a *Yamabe minimiser*). The critical points of $\mathcal{E}$ are Einstein metrics, but the critical points of the restricted functional $\mathcal{E}|_{\mathcal{C}}$ are the constant scalar curvature metrics in $\mathcal{C}$ (Besse Proposition 4.25). In particular, if g is a Yamabe metric, then it has constant scalar curvature.

Recall that the Yamabe problem ventured to establish that every conformal class contains constant scalar curvature metrics. This was achieved thanks to work of Yamabe, Trudinger, Aubin, and finally Schoen. In fact, they showed that every conformal class admits Yamabe metrics, i.e. the infimum above is always a minimum. If $Y(M, \mathcal{C}) \leq 0$, then every constant scalar curvature metric is a Yamabe minimiser, but this is no longer true if $Y(M, \mathcal{C}) > 0$. It is true however if $\mathcal{C}$ contains an Einstein metric (Obata '71).

Proposition 1.2. *$Y(M, \mathcal{C})$ is positive if and only if $\mathcal{C}$ contains a positive scalar curvature metric.*

Proof. If $Y(M, \mathcal{C}) > 0$, then there is a Yamabe metric $g \in \mathcal{C}$ with $\mathcal{E}(g) = Y(M, \mathcal{C}) > 0$. So $\mathcal{C}$ contains a (constant) positive scalar curvature metric.

Suppose now that g is a positive scalar curvature metric and $\tilde{g} = u^{\frac{4}{n-2}} g$ is a conformal metric. From the proof above, we have

$$\int_M s_{\tilde{g}} d\mu_{\tilde{g}} = \int_M 4\frac{n-1}{n-2}|du|^2 + s_g u^2 d\mu_g \geq c\|u\|_{1,2}^2$$

where $c = \min\left\{4\frac{n-1}{n-2}, \min_M s_g\right\} > 0$ and $\|u\|_{1,2}$ is the $W^{1,2}(M)$ norm ($\|w\|_{1,2} = \left(\int_M |dw|^2 + w^2 d\mu_g\right)^{\frac{1}{2}}$). By the Sobolev embedding theorem, there is a continuous embedding $W^{1,2}(M) \to L^p(M)$ where $p = \frac{2n}{n-2}$ ($\frac{1}{p} = \frac{1}{2} - \frac{1}{n}$). So there is a constant $c' > 0$ (actually, $c' > 1$) such that $\|u\|_p \leq c'\|u\|_{1,2}$ and hence

$$\int_M s_{\tilde{g}} d\mu_{\tilde{g}} \geq c\|u\|_{1,2}^2 \geq C\|u\|_p^2 = C\left(\int_M u^{\frac{2n}{n-2}} d\mu_g\right)^{\frac{n-2}{n}} = C\,\mathrm{Vol}(M, \tilde{g})^{\frac{n-2}{n}}$$

where $C = \frac{c}{(c')^2}$. So $\mathcal{E}(\tilde{g}) \geq C$ and hence $Y(M, \mathcal{C}) \geq C > 0$. □

Every closed manifold of dimension at least three admits a metric with negative scalar curvature (even negative Ricci curvature), but the same is not true of positive scalar curvature metrics. Moreover, the restrictions on manifold which admits a metric with zero scalar curvature are related to the existence of a metric of positive scalar curvature. If M admits a psc metric, then it also admits a metric with scalar curvature zero; if M does not admit a psc metric, then any scalar-flat metric is Ricci-flat.

The values of $Y(M, \mathcal{C})$ are not arbitrary.

Theorem 1.3 (Aubin '76)**.** *$Y(M, \mathcal{C}) \leq Y(S^n, [g_{round}])$ with equality if and only if $(M, \mathcal{C})$ is conformally diffeomorphic to $(S^n, [g_{round}])$, i.e. there is a diffeomorphism $f : M \to S^n$ with $f^* g_{round} \in \mathcal{C}$.*

Example: Let g be a unit volume constant positive scalar curvature metric on S^n, with $s_g \equiv s$, and let g_0 be a unit volume flat metric on T^m. Consider the family of metrics $h_t = g + tg_0$ on $S^n \times T^m$ for $t > 0$. Note that $s_{h_t} = s_g + s_{tg_0} = s + t^{-1} s_{g_0} = s$ and $\mathrm{Vol}(S^n \times T^m, h_t) = \mathrm{Vol}(S^n, g_{\mathrm{round}}) \mathrm{Vol}(T^m, tg_0) = t^{\frac{m}{2}}$. So

$$\begin{aligned}
\mathcal{E}(h_t) &= \frac{\displaystyle\int_{S^n \times T^m} s_{h_t} d\mu_{h_t}}{\mathrm{Vol}(S^n \times T^m, h_t)^{\frac{n+m-2}{n+m}}} \\
&= \frac{s\,\mathrm{Vol}(S^n \times T^m, h_t)}{\mathrm{Vol}(S^n \times T^m, h_t)^{\frac{n+m-2}{n+m}}} \\
&= s\,\mathrm{Vol}(S^n \times T^m, h_t)^{\frac{2}{n+m}} \\
&= s(t^{\frac{m}{2}})^{\frac{2}{m+n}} \\
&= st^{\frac{m}{n+m}}.
\end{aligned}$$

For t large enough, we have $\mathcal{E}(h_t) > Y(S^{n+m}, [g_{\mathrm{round}}])$ and hence $\mathcal{E}(h_t) \neq Y(S^n \times T^m, [h_t])$. That is, h_t is a constant scalar curvature metric which is not a Yamabe minimiser; note that $Y(S^n \times T^m, [h_t]) > 0$ as $s_{h_t} = s > 0$.

With Aubin's result at our disposal, we define the *Yamabe invariant* of M by

$$Y(M) = \sup_{\mathcal{C}} Y(M, \mathcal{C}) = \sup_{\mathcal{C}} \inf_{g \in \mathcal{C}} \mathcal{E}(g).$$

This is a diffeomorphism invariant. Note that $Y(M) > 0$ if and only if M admits a positive scalar curvature metric; it follows that $Y(M)$ is not a homeomorphism invariant. We can therefore view $Y(M)$ as a real-valued refinement of the $\mathbb{Z}_2$-valued invariant P given by

$$P(M) = \begin{cases} 1 & M \text{ admits a positive scalar curvature metric} \\ 0 & M \text{ does not admit a positive scalar curvature metric.} \end{cases}$$

That is, if $P(M_1) \neq P(M_2)$, then $Y(M_1) \neq Y(M_2)$. However, there are pairs of manifolds M_1, M_2 with $P(M_1) = P(M_2)$ and $Y(M_1) \neq Y(M_2)$, i.e. the Yamabe invariant can distinguish manifolds that the invariant P cannot.

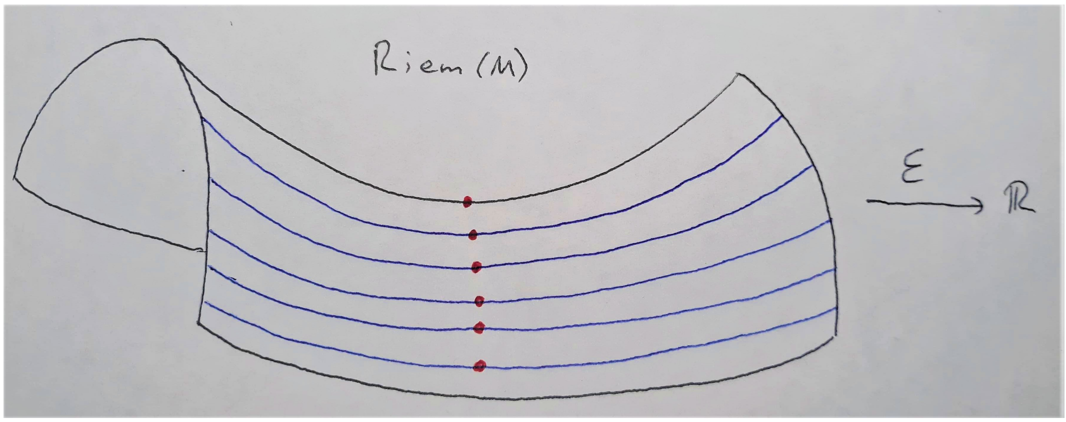
If $Y(M) \leq 0$, then it follows from aforementioned results that it is equal to the supremum of the values of unit-volume constant scalar curvature metrics on M. This is no longer true if $Y(M) > 0$ as the previous example demonstrates; it is true if we restrict to Yamabe metrics though.

Example: By Aubin's theorem, we have $Y(S^n, \mathcal{C}) \leq Y(S^n, [g_{\text{round}}])$ for all conformal classes $\mathcal{C}$, so $Y(M) = \sup Y(S^n, \mathcal{C}) = Y(S^n, [g_{\text{round}}])$. As g_{round} is an Einstein metric, we have $Y(S^n, [g_{\text{round}}]) = \mathcal{E}(g_{\text{round}})$ by Obata's theorem. Therefore

$$Y(S^n) = Y(S^n, [g_{\text{round}}]) = \mathcal{E}(g_{\text{round}}) = n(n-1)\,\text{Vol}(S^n, g_{\text{round}})^{\frac{n}{2}}.$$

We say a Riemannian metric g *realises* the Yamabe invariant if g is a Yamabe metric and $Y(M) = \mathcal{E}(g)$, i.e. $Y(M) = Y(M, [g])$. For example, g_{round} realises $Y(S^n)$.

Conjecture: If g realises $Y(M)$, then g is an Einstein metric.



The only case where this conjecture is not yet verified is when $\frac{s_g}{n-1}$ is a positive eigenvalue of Δ_g (Besse Proposition 4.47).

If $\dim M = 2$, we have $Y(M) = 4\pi\chi(M)$. Beyond that, the Yamabe invariant is notoriously difficult to calculate, especially when it is positive. In particular, there are no formulae for the Yamabe invariant of a covering space or a product. There is a lower bound for the Yamabe invariant of a connected sum due to Kobayashi; in particular, if $Y(M_1), Y(M_2) \geq 0$, then $Y(M_1 \# M_2) \geq 0$.

One calculation, due to Schoen, shows that $Y(S^n \times S^1) = Y(S^{n+1})$. More generally, Kobayashi showed that any mapping torus of S^n has the same Yamabe invariant of S^{n+1}. By Aubin's theorem, the Yamabe invariant of such manifolds are not realised by any metric.

A surprising fact about the Yamabe invariant is the following:

Theorem 1.4 (Petean '00)**.** *Let M be a closed simply connected manifold with* $\dim M \geq 5$*. Then* $Y(M) \geq 0$.

In this range of dimensions, only two values of the Yamabe invariant are known: 0 and $Y(S^n)$. Note that the theorem is also true in dimensions 2 and 3, but as we will see, it is not true in dimension 4.

Finally, there are only countably many smooth closed manifolds, so range of Y is countable (but it is infinite).

2. Kodaira Dimension

The canonical bundle of a complex manifold X is the holomorphic line bundle $K_X = \bigwedge^n T^*X$. If X is compact and d is a positive integer, the d^{th} plurigenus of X is defined to be $P_d(X) := h^0(X, K_X^d) = \dim H^0(X, K_X^d) = \dim \Gamma(X, K_X^d)$; where K_X^d denotes the tensor product of d copies of K_X. Unlike in the smooth case, the vector space of holomorphic sections of a holomorphic vector bundle over a compact manifold is always finite-dimensional (no partitions of unity).

If $P_d(X) = 0$ for all $d > 0$, we say that X has Kodaira dimension $-\infty$ which we denote by $\kappa(X) = -\infty$.Otherwise, we define

$$\kappa(X) = \limsup_{d\to\infty} \frac{\log P_d}{\log d}.$$

Although not obvious from the definition, the result is an integer between 0 and n. Note that $\kappa(X) = \kappa > 0$ means that $P_d(X) = O(d^\kappa)$, i.e the sequence of numbers $P_d(X)$ grows like a polynomial in d of degree κ.

Here's a more geometric interpretation. Let L be a holomorphic line bundle with $\dim\Gamma(X, L) > 0$. There is a partial map $\varphi_L : X \dashrightarrow \mathbb{P}(\Gamma(X, L)^*)$ constructed as follows. It maps x to $\mathrm{ev}_x : \Gamma(X, L) \to \mathbb{C}$ where $\mathrm{ev}_x(s) = s(x) \in L_x \cong \mathbb{C}$. There is no canonical isomorphism $L_x \to \mathbb{C}$, but given any two isomorphisms ψ_1, ψ_2, there is $\alpha \in \mathbb{C}^*$ such that $\psi_2 = \alpha\psi_1$. Therefore ev_x is a well-defined element of $\mathbb{P}(\Gamma(X, L)^*)$ provided ev_x is not the zero map. The *base locus* of L, denoted $\mathrm{Bs}(L)$, consists of those x for which $\mathrm{ev}_x \equiv 0$; the domain of φ_L is $X \setminus \mathrm{Bs}(L)$. Alternatively, we can view $\mathbb{P}(\Gamma(X, L)^*)$ as the set of hyperplanes of $\Gamma(X, L)$, and the hyperplane associated to x is the kernel of ev_x.

If $\kappa(X) \neq -\infty$, then $\kappa(X)$ is equal to the largest dimension of the image of X under the maps $\varphi_{K_X^d}$.

The Kodaira dimension satisfies $\kappa(X \times Y) = \kappa(X) + \kappa(Y)$; this is why we chose $-\infty$ as a value.

Example: Let's calculate the Kodaira dimension of Riemann surfaces.

The *degree* of a holomorphic line bundle $L \to C$ where C is a compact Riemann surface is $\deg(L) = \int_X c_1(L)$. If $\deg(L) < 0$, then $h^0(X, L) = 0$, i.e. L does not admit any holomorphic sections other than the zero section.

If $g = 0$, then $\deg(K_X^d) = d\deg(K_X) = d(-\chi(X)) = -2d < 0$, so $P_d(X) = 0$ for all $d > 0$ and hence $\kappa(X) = -\infty$.

If $g = 1$, we have $K_X \cong \mathcal{O}_X$ so $\Gamma(X, K_X^d) = \Gamma(X, \mathcal{O}_X^d) = \Gamma(X, \mathcal{O}_X) = \mathcal{O}(X) \cong \mathbb{C}$ for all d. Therefore $P_d(X) = 1$ for all $d > 0$, so $\kappa(X) = 0$.

If $g \geq 2$ and $d > 1$, then $\deg(K_X^{1-d}) = (1-d)(2g-2) < 0$ so $\Gamma(X, K_X^{1-d}) = \{0\}$. By Riemann-Roch, we have $P_d(X) = h^0(X, K_X^d) = \deg(K_X^d) - g + 1 = d(-\chi(X)) - g + 1 = d(2g-2) - (g-1) = (2d-1)(g-1)$ which grows linearly in d, so $\kappa(X) = 1$. On the other hand, $P_1(X) = h^0(X, K_X) = \deg(K_X) - g + 1 + h^0(X, K_X \otimes K_X^{-1}) = -\chi(X) - g + 1 + h^0(X, \mathcal{O}_X) = 2g - 2 - g + 1 + 1 = g$.

Summarising, we have

$\kappa(X)$	$-\infty$	0	1
genus	0	1	≥ 2

3. Complex Surfaces

Reference: Barth, Hulek, Peters, & Van de Ven - Compact Complex Surfaces

The Kodaira dimension leads to a classification of compact complex surfaces. First, there's an important construction on complex manifolds which first appears in complex dimension 2.

If Y is a complex submanifold of X of codimension k, then the *blowup of X along Y* is a complex manifold $\mathrm{Bl}_Y(X)$ equipped with a holomorphic map $\pi : \mathrm{Bl}_Y(X) \to X$ such that $\pi|_{\pi^{-1}(X\setminus Y)}$ is a biholomorphism, and $E = \pi^{-1}(Y)$ is a complex submanifold of codimension 1; we call π the blowdown map. Moreover, $\pi|_E : E \to Y$ is a $\mathbb{CP}^{k-1}$-bundle over Y, namely the projectivisation of the normal bundle of Y in X.

If $k = 1$, then $\mathrm{Bl}_Y(X) = X$ and $\pi = \mathrm{id}_X$. If $Y = \{x\}$ is a point, then there is an orientation-preserving diffeomorphism between $\mathrm{Bl}_x(X)$ and $X\#\overline{\mathbb{CP}^n}$ where $\overline{\mathbb{CP}^n}$ indicates $\mathbb{CP}^n$ equipped with its non-standard orientation, i.e. not the orientation induced by the complex structure.

In complex dimension two, we can only blowup points. We say X is minimal if it cannot be obtained by blowing up another surface. We say X is a *minimal model* for X' if X is minimal and X' can be obtained from X by a series of blowups.

Finally, the Kodaira dimension is invariant under blowups, i.e. $\kappa(\mathrm{Bl}_x(X)) = \kappa(X)$.

Theorem 3.1 (Kodaira-Enriques Classification)**.** *Every connected compact complex surface has a minimal model in exactly one of these families*

$\kappa(X)$	$b_1(X)$ *even*	$b_1(X)$ *odd*
$-\infty$	*Rational, Ruled*	*Class VII*
0	*Tori, K3, Hyperelliptic, Enriques*	*Primary Kodaira, Secondary Kodaira*
1	*Properly Elliptic*	*Properly Elliptic*
2	*General Type*	

A *rational surface* is a projective surface birational to $\mathbb{CP}^2$ (i.e. can be obtained from $\mathbb{CP}^2$ by a series of blowups and blowdowns). The only minimal rational surfaces are $\mathbb{CP}^2$ and the Hirzebruch surfaces $\Sigma_n = \mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O})$ for $n = 0$ or $n \geq 2$ which are all $\mathbb{CP}^1$-bundles over $\mathbb{CP}^1$.

A *ruled surface* is a $\mathbb{CP}^1$-bundle over a Riemann surface C of positive genus. It is necessarily of the form $\mathbb{P}(E) \to C$ where $E \to C$ is a rank 2 holomorphic vector bundle.

A *class* VII *surface* is a surface with Kodaira dimension $-\infty$ and b_1 odd. Examples include Hopf surfaces and Inoue surfaces.

Tori are quotients of $\mathbb{C}^2$ by a lattice Λ (i.e. a free abelian subgroup of maximal rank, $\Lambda \cong \mathbb{Z}^4$). A *hyperelliptic surface* is a surface which is finitely covered by a product of two elliptic curves.

A *K3 surface* is a surface with $b_1 = 0$ and trivial canonical bundle. An example of a K3 surface is the Fermat quartic $X = \{[x_0, x_1, x_2, x_3] \in \mathbb{CP}^3 \mid x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}$. An *Enriques surface* is a surface X with $b_1(X) = 0$ and a non-trivial canonical bundle whose square is trivial, i.e. K_X^2 is trivial, but K_X is not. Every Enriques surface is double covered by a K3 surface.

A *primary Kodaira surface* is a complex surface with $b_1 = 3$ and is a holomorphic fiber bundle of elliptic curves over an elliptic curve. For example, let $L \to \mathbb{C}/\Lambda$ be a non-trivial holomorphic line bundle, then $(L \setminus Z)/\mathbb{Z}$ is a primary Kodaira surface; here Z denotes the image of the zero section and $\mathbb{Z}$ acts by rescaling, $\ell \mapsto 2\ell$; $b_1(X) = 3$ follows from the Gysin sequence. A *secondary Kodaira surface* is a surface which is finitely covered by a primary Kodaira surface.

An *elliptic surface* is a surface X which admits a map $p : X \to C$ where C is a Riemann surface such that almost all fibers are elliptic curves. An elliptic surface is called *properly elliptic* if it has Kodaira dimension 1.

A surface is called *general type* if it has Kodaira dimension 2. For a generic homogeneous polynomial p on $\mathbb{C}^4$ of degree $d \geq 5$, the hypersurface $X = \{[x_0, x_1, x_2, x_3] \in \mathbb{CP}^3 \mid p(x_0, x_1, x_2, x_3) = 0\}$ is a general type surface.

Let X be a complex manifold and let J denote the associated almost complex structure. A Riemannian metric g is called *hermitian* if $g(JV, JW) = g(V, W)$ for all V and W. There is an associated non-degenerate two-form ω given by $\omega(V, W) = g(JV, W)$. If ω is closed, g is called a Kähler metric.

On a compact Kähler manifold, we have the Hodge decomposition

$$H^k_{\text{dR}}(X) \otimes_{\mathbb{R}} \mathbb{C} \cong \bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(X),$$

so $b_k(X) = \sum_{p+q=k} h^{p,q}(X)$. As $H^{q,p}_{\bar{\partial}}(X) \cong \overline{H^{p,q}_{\bar{\partial}}(X)}$, we have $h^{p,q}(X) = h^{q,p}(X)$ and hence the odd Betti numbers of a compact Kähler manifold are even. In particular, if X is a compact Kähler surface, then $b_1(X)$ and $b_3(X)$ are even; by Poincaré duality, we have $b_1(X) = b_3(X)$.

Kodaira conjectured that if X is a connected compact complex surface with $b_1(X)$ even, then X admits a Kähler metric. By checking the cases in the classification, a proof was eventually achieved with the final cases of K3 surface done by Siu. A proof of Kodaira's conjecture which does not rely on the classification was later given by Buchdahl and Lamari independently.

Theorem 3.2. *A connected compact complex surface X admits a Kähler metric if and only if $b_1(X)$ is even.*

In higher dimensions, there are connected compact complex manifolds have odd Betti numbers even, but do not admit Kähler metrics, e.g. Calabi-Eckmann manifolds diffeomorphic to $S^{2k+1} \times S^{2k+1}$, $k \geq 0$.

4. LeBrun's Theorem

Reference: LeBrun - Kodaira Dimension and the Yamabe Problem

Recall that for Riemann surfaces, we have calculated both their Kodaira dimensions and their Yamabe invariants.

genus	0	1	≥ 2
$\kappa(X)$	$-\infty$	0	1
$Y(X)$	> 0	0	< 0

Surprisingly, there is an analogue of this relationship in complex dimension two.

Theorem 4.1 (LeBrun '99)**.** *Let X be a connected compact complex surface which admits a Kähler metric (equivalently, with $b_1(X)$ even). Then the Yamabe invariant of X satisifies*

$\kappa(X)$	$-\infty$	0	1	2
$Y(X)$	> 0	0	0	< 0

Moreover, if $\kappa(X) = 2$ and X has minimal model X_0, then $Y(X) = Y(X_0) = -4\pi\sqrt{2c_1(X_0)^2} = -4\pi\sqrt{4\chi(X_0) + 6\sigma(X_0)}$.

The proof consists of several parts:

(1) If $\kappa(X) = -\infty$, let X_0 be a minimal model of X. Then by the Kodaira-Enriques classification, X_0 is diffeomorphic to $\mathbb{CP}^2$, $S^2 \times S^2$, $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, or an S^2-bundle over a surface of genus $g > 0$. One can construct positive scalar curvature metrics on these manifolds, so they have positive Yamabe invariant. Now note that X is diffeomorphic to $X_0\#k\overline{\mathbb{CP}^2}$ for some $k > 0$. The connected sum of manifolds which admit psc metrics also admits psc metrics in dimension at least three by a surgery result of Gromov and Lawson. Therefore X admits a psc metric and hence $Y(X) > 0$.

(2) Kähler surfaces are symplectic, so they have a non-trivial Seiberg-Witten invariant by a result of Taubes. If $b^+(X) > 1$, then it follows from a Weitzenbock formula that X does not admit psc metrics (note that $b^+(X) = 1$ if $\kappa(X) = -\infty$). If $b^+(X) = 1$ and $\kappa(X) \geq 0$, then X also fails to admit psc metrics. This was shown by Friedman and Morgan via a non-trivial application of Seiberg-Witten theory. So if $\kappa(X) \geq 0$, we have $Y(X) \leq 0$.

(3) Every surface with $\kappa(X) = 0, 1$ is deformation equivalent to, and hence diffeomorphic to, an elliptic surface. LeBrun shows such manifolds collapse with bounded scalar curvature. It follows that $Y(X) \geq 0$, and hence $Y(X) = 0$.

(4) Using an alternative characterisation of the Yamabe invariant[1] and a careful analysis of the aforementioned Weitzenbock formula, LeBrun shows the final statement regarding the case $\kappa(X) = 2$. In particular, $Y(X) < 0$ for such surfaces.

In the same paper, LeBrun shows that if $Y(X) = 0$ is realised by a metric g (necessarily Ricci-flat), then X is minimal and $\kappa(X) = 0$. Such surfaces admit Ricci-flat Kähler metrics by Yau's solution of the Calabi conjecture.

In the Yamabe positive case, it's much harder to compute $Y(X)$. The only known example is $Y(\mathbb{CP}^2) = 12\sqrt{2}\pi$ due to Gursky and LeBrun (after an earlier proof by LeBrun); note that $Y(S^4) = 8\sqrt{6}\pi$. The proof uses a perturbed version of the conformal Laplacian and gives rise to an upper bound on the Yamabe invariant of any positive definite four-manifold. It follows from LeBrun's theorem that if X is a Kähler surface with $\kappa(X) \geq 0$, then the Yamabe invariant is unchanged under blowup; in particular, the Yamabe invariant of X is equal to the Yamabe invariant of a minimal model. It is not known if this is still true for Kähler surfaces with $\kappa(X) = -\infty$.

How can we generalise this theorem? There are two restrictions we could try to remove: dimension two and the existence of a Kähler metric.

Example: Let $m \geq 3$, and let X be a smooth degree $m + 3$ hypersurface in $\mathbb{CP}^{m+1}$. Then

- $\kappa(X) = m$ (a smooth degree d hypersurface of $\mathbb{CP}^n$ has $\kappa(X) = -\infty$ if $d < n + 1$, $\kappa(X) = 0$ if $d = n + 1$, and $\kappa(X) = m$ if $d > n + 1$)
- $\dim_{\mathbb{C}} X = m$, so $\dim_{\mathbb{R}} X = 2m \geq 6$,
- $\pi_1(X) = 0$
- X is non-spin.

Using their surgery result, Gromov and Lawson proved that any closed, simply connected, non-spin manifold of dimension at least 5 admits psc metrics, so $Y(X) > 0$. On the other hand, X has maximal Kodaira dimension - this does not fit the pattern we see in complex dimensions one and two.

One reason why such problems arise in higher dimensions is that the Kodaira dimension is not a diffeomorphism invariant. That is, there are complex manifolds X_1 and X_2 which are diffeomorphic but $\kappa(X_1) \neq \kappa(X_2)$. Many examples can be constructed using the s-cobordism theorem, see Răsdeaconu '06. The diffeomorphism invariance of the Kodaira dimension for Kähler surfaces is a theorem proved using Seiberg-Witten theory (even the plurigenera are diffeomorphism invariants).

Given that we cannot remove the dimension restriction, we instead focus on the Yamabe invariant of non-Kähler surfaces.

[1] $\displaystyle\inf_{g\in\mathrm{Riem}(M)} \int_M |s_g|^{\frac{n}{2}}\, d\mu_g = \begin{cases} 0 & \text{if } Y(M) \geq 0 \\ |Y(M)|^{\frac{n}{2}} & \text{if } Y(M) < 0 \end{cases}.$

5. The Yamabe Invariant of Non-Kähler Surfaces

References: Albanese - The Yamabe invariants of Inoue surfaces, Kodaira surfaces, and their blowups, Albanese & LeBrun - Kodaira Dimension and the Yamabe Problem, II.

We will proceed by considering each value of the Kodaira dimension separately.

5.1. **Class VII.** Hopf constructed the first examples of compact complex manifolds which do not admit Kähler metrics. Namely, $X = (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$ where the $\mathbb{Z}$-action is generated by $z \mapsto 2z$. As X is diffeomorphic to $S^1 \times S^{2n-1}$, it does not admit a Kähler metric because $b_1(X) = 1$ is odd. We call a compact complex manifold a *Hopf surface* if its universal cover is $\mathbb{C}^n \setminus \{0\}$. We say X is *primary* if $\pi_1(X) \cong \mathbb{Z}$ and *secondary* otherwise. Every secondary Hopf manifold is finitely covered by a primary Hopf manifold.

In complex dimension 2, every primary Hopf surface is diffeomorphic to $S^1 \times S^3$ and hence has positive Yamabe invariant. This doesn't automatically imply that secondary Hopf surfaces have positive Yamabe invariant though: if $M' \to M$ is a finite covering and M' admits a psc metric, then M does not necessarily admit a psc metric[2]. The diffeomorphism types of secondary Hopf surfaces were classified by Kato. Some are diffeomorphic to $S^1 \times (S^3/H)$ for a finite free group action H, and these have positive Yamabe invariant. There are also some which are diffeomorphic to the mapping torus of S^3/H by a diffeomorphism of order two or three. In an upcoming paper, I show these also have positive Yamabe invariant.

By the surgery result of Gromov and Lawson, the blowups of Hopf surfaces also have positive Yamabe invariant. Moreover, there is upper bound on the Yamabe invariant in this case as they are definite and hence the result of Gursky and LeBrun applies.

Gursky and LeBrun showed that $Y((S^1 \times S^3)\#\overline{\mathbb{CP}^2}) \leq 12\pi\sqrt{2}$; note, this is the same as $Y(\mathbb{CP}^2)$ which is calculated in the same paper. On the other hand, a result of Schoen shows that $Y(S^1 \times S^3) = Y(S^4) = 8\sqrt{6}\pi$. In particular, the Yamabe invariant does change under blowups for class VII surfaces, which is not the case for Kähler surfaces with $\kappa(X) \geq 0$.

There is another construction of class VII surfaces due to Inoue. They are quotients of $\mathbb{C} \times \mathbb{H}$ by a group of affine transformations. There are four families: S^+_M, S^-_M, $S^+_{N,p,q,r,t}$, and $S^-_{N,p,q,rt}$. The first two are diffeomorphic to the mapping torus of T^3, and the final two are diffeomorphic to the mapping torus of a non-trivial circle bundle over T^2; in both cases, the diffeomorphism is induced by a linear map on $\mathbb{R}^3$.

Theorem 5.1 (A. '21)**.** *Inoue surfaces and their blowups have Yamabe invariant zero.*

Corollary 5.2. *The Kähler hypothesis of LeBrun's theorem is necessary.*

Recall that LeBrun showed that elliptic surfaces collapse with bounded scalar curvature, and hence have non-negative Yamabe invariant. Inoue surfaces do not contain any complex curves; in particular, they are not elliptic. However, one can still show they collapse with bounded scalar curvature as they have a $\mathcal{T}$-structure.

A $\mathcal{T}$-structure on a closed smooth manifold is a finite open covering $\{U_1, \dots, U_N\}$ and a non-trivial torus action on each U_i such that each intersection $U_{i_1} \cap \dots \cap U_{i_k}$ is invariant under the torus actions on $U_{i_1}, \dots, U_{i_k}$, and the torus actions commute.

Example: Let M be the mapping torus of a diffeomorphism $T^n \to T^n$ induced by a linear diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$. If $p : M \to S^1$ is the projection, then let $U_1 = p^{-1}(S^1 \setminus \{1\})$ and $U_2 = p^{-1}(S^1 \setminus \{-1\})$. Note that U_1 and U_2 are both diffeomorphic to $(0,1) \times T^n$ so they admit

[2]An example attributed to Bérard Bergery: $M' = S^2 \times S^7$ and $M = (S^2 \times \mathbb{RP}^7)\#\Sigma$ where $\Sigma \in \Theta_9$ has $\alpha(\Sigma) \neq 0$. Note that $\Theta_9 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, so $\Sigma\#\Sigma = S^9$.

effective torus actions acting by translations. Moreover, the intersection $U_1 \cap U_2$ is invariant under the torus actions, and as f is linear, they commute and hence M has a $\mathcal{T}$-structure.

In particular, Inoue surfaces in the families S^+_M and S^-_M have $\mathcal{T}$-structures. Paternain and Petean showed that the other two families also admit $\mathcal{T}$-structures, so all Inoue surfaces have non-negative Yamabe invariant. A theorem of Kobayashi implies that this remains true after blowing up: if $\dim M_1 = \dim M_2 \geq 3$ and $Y(M_1), Y(M_2) \geq 0$, then $Y(M_1 \# M_2) \geq 0$.

All that remains is to show that such surfaces do not admit psc metrics. We'll come back to this.

Even though the non-Kähler analogue of LeBrun's theorem is not true, we can still hope to determine the sign of the Yamabe invariant for non-Kähler surfaces. What class VII surfaces remain?

Theorem 5.3 (Bogomolov, Li-Yau-Zheng, Teleman)**.** *If X is a class VII surface with $b_2(X) = 0$, then X is a Hopf surface or an Inoue surface.*

A *global spherical shell* in a complex surface X is an open subset U biholomorphic to a neighbourhood of S^3 in $\mathbb{C}^2 \setminus \{0\}$ such that $X \setminus U$ is connected. This notion was introduced by Kato. He showed that if X admits a global spherical shell, then X is a degeneration of blownup primary Hopf surfaces. That is, there is a holomorphic submersion $\pi : \mathcal{X} \to \mathbb{D}$ such that $\mathcal{X}_0 := \pi^{-1}(0)$ is biholomorphic to X, but $\mathcal{X}_t$ is a primary Hopf surface blownup at $b_2(X)$ points for all $t \neq 0$. It follows that X is diffeomorphic to $(S^1 \times S^3) \# b_2(X)\overline{\mathbb{CP}^2}$.

Conjecture 5.4 (Nakamura '89)**.** Let X be a class VII surface with $b_2(X) > 0$. Then X has a global spherical shell.

If the conjecture is true, then all such surfaces have positive Yamabe invariant.

5.2. $\kappa(X) = 0$**.** In complex dimension two, $K_X \cong \mathcal{O}_X$ implies symplectic: let α be a nowhere-zero holomorphic two-form, then $\omega = \mathrm{Re}(\alpha) = \frac{1}{2}(\alpha + \overline{\alpha})$ is a symplectic form. In higher dimensions, triviality of the canonical bundle does not imply symplectic. For example, there are so-called non-Kähler Calabi-Yau manifolds which are diffeomorphic to $k(S^3 \times S^3)$, $k \geq 2$, which have trivial canonical bundle.

Primary Kodaira surfaces have trivial canonical bundle and are therefore symplectic and have $b^+(X) = 2h^{2,0}(X) = 2h^0(X, K_X) = 2h^0(X, \mathcal{O}_X) = 2 > 1$. There is a notion of symplectic blowup, and the diffeomorphism type coincides with the complex blowup. Therefore blowups of a primary Kodaira surface (viewed as a complex manifold) are symplectic with $b^+ > 1$, so they do not admit psc metrics by Seiberg-Witten theory. As secondary Kodaira surfaces and their blowups are finitely covered by primary Kodaira surfaces and their blowups, they cannot admit psc metrics either. $Y(X) \leq 0$. On the other hand, they are all elliptic surfaces so argument from LeBrun's theorem shows $Y(X) \geq 0$.

5.3. $\kappa(X) = 1$**.**

Theorem 5.5 (A. & LeBrun '21)**.** *Non-Kähler properly elliptic surfaces and their blowups have Yamabe invariant zero.*

LeBrun previously showed that elliptic surfaces collapse with bounded scalar curvature, and hence have non-negative Yamabe invariant. So all that remains is to establish that non-Kähler properly elliptic surfaces do not admit psc metrics. In the Kähler case, we used the existence of a symplectic form to utilise Seiberg-Witten theory. However, non-Kähler properly elliptic surfaces and their blowups are never symplectic (Biquard '98).

Theorem 5.6. *Let N be a compact oriented connected 3-manifold, and let X be a mapping torus of N. Let P be any connected smooth compact oriented 4-manifold, and let $M = X\#P$. Then $Y(N) \leq 0 \Rightarrow Y(M) \leq 0$, i.e. if N does not admit a psc metric, then neither does M.*

A closed orientable three-manifold admits a psc metric if and only if it contains no aspherical factors in its prime decomposition. In particular, T^3 and non-trivial circle bundles over T^2 do not admit psc metrics. The theorem therefore implies that Inoue surfaces and their blowups do not admit psc metrics.

The proof of the theorem uses the Schoen-Yau minimal hypersurface method. Choose a metric g on M. The Poincaré dual of the pullback of an orientation form on S^1 can be represented by a hypersurface Σ which is stable minimal with respect to g. The proof constructs a map $\Sigma \to N$ of non-zero degree. If N does not admit a psc metric, it follows that Σ doesn't either. If g were psc, then $g|_\Sigma$ would be conformal to a psc metric, so we see that g can't be psc.

Let X be a non-Kähler elliptic surface. Unlike in the Kähler case, non-Kähler elliptic surfaces have no singular fibers, only multiple fibers. By passing to a finite cover X', we obtain a principal elliptic bundle over a curve C. This gives rise to two circle bundles over C. By passing to another cover X'', we can take one of them to be trivial, and hence $X'' = S^1 \times N$ where N is a non-trivial circle bundle over C. The Kodaira dimension of C is equal to the Kodaira dimension of X (and hence X''), so if $\kappa(X) \geq 0$, then N is aspherical and hence does not admit psc metrics. Therefore X'' does not admit psc metrics, so neither does X. Taking P to be $k\overline{\mathbb{CP}^2}$, we see the claim also holds for blowups of non-Kähler elliptic surfaces with $\kappa(X) \geq 0$.

All Kodaira surfaces are elliptic, so this gives another proof that such surfaces and their blowups do not admit psc metrics.

Theorem 5.7. *If the global spherical shell conjecture is true, then Inoue surfaces and their blowups are the only counterexamples to the non-Kähler analogue of LeBrun's Theorem.*

What about realising the Yamabe invariant? Conjecturally, if it is realised, it must be realised by an Einstein metric. Using the Kodaira-Enriques classification and the Hitchin-Thorpe inequality, it is not too hard to show that non-Kähler surfaces do not admit Einstein metrics.

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