THE NORMAL BUNDLE OF A SPHERE BUNDLE IS TRIVIAL

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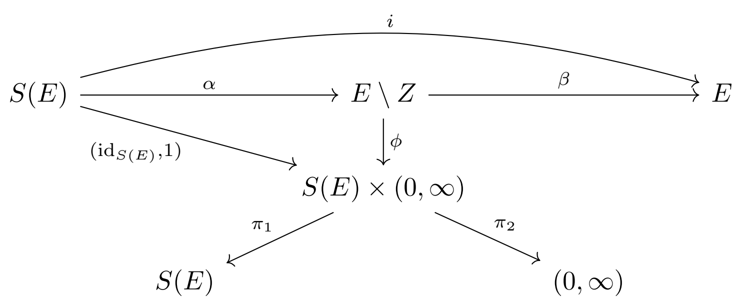
Abstract. Given a Riemannian metric on a smooth vector bundle $E \to M$, one can form its sphere bundle $S(E)$. The purpose of this note is to show that the normal bundle of the inclusion of total spaces $S(E) \to E$ is trivial.

Let $\pi : E \to M$ be a smooth rank k vector bundle and let g be a Riemannian metric on E. The sphere bundle of E with respect to g is $p : S(E) \to M$ where $S(E) = \{e \in E \mid g(e, e) = 1\}$ and $p = \pi|_{S(E)}$. As the name suggests, $p : S(E) \to M$ is a smooth fiber bundle with fiber S^{k-1}.

Let $i : S(E) \to E$ be the inclusion of total spaces. If Z denotes the image of the zero section of $\pi : E \to M$, note that i factors as $\beta \circ \alpha$ where $\alpha : S(E) \to E \setminus Z$ and $\beta : E \setminus Z \to E$ are the natural inclusions. Moreover, there is a diffeomorphism $\phi : E \setminus Z \to S(E) \times (0, \infty)$ given by $e \mapsto (\frac{e}{\|e\|}, \|e\|)$, where $\|e\| = \sqrt{g(e, e)}$. These maps combine into the following commutative diagram



where $1 : S(E) \to (0, \infty)$ denotes the constant function with value 1, and π_1, π_2 are the natural projections. With this in mind, note that

$$\begin{aligned}
i^*TE &\cong (\beta \circ \alpha)^* TE \\
&\cong \alpha^* \beta^* TE \\
&\cong \alpha^* T(E \setminus Z) \\
&\cong \alpha^* \phi^* (T(S(E) \times (0, \infty)) \\
&\cong (\phi \circ \alpha)^* (\pi_1^* T(S(E)) \oplus \pi_2^* T(0, \infty)) \\
&\cong (\mathrm{id}_{S(E)}, 1)^* (\pi_1^* T(S(E)) \oplus \varepsilon^1) \\
&\cong (\mathrm{id}_{S(E)}, 1)^* \pi_1^* T(S(E)) \oplus (\mathrm{id}_{S(E)}, 1)^* \pi_2^* \varepsilon^1 \\
&\cong (\pi_1 \circ (\mathrm{id}_{S(E)}, 1))^* T(S(E)) \oplus (\pi_2 \circ (\mathrm{id}_{S(E)}, 1))^* \varepsilon^1 \\
&\cong \mathrm{id}_{S(E)}^* T(S(E)) \oplus 1^* \varepsilon^1 \\
&\cong T(S(E)) \oplus \varepsilon^1.
\end{aligned}$$

Now let ν be the normal line bundle of the inclusion $i : S(E) \to E$. On $S(E)$ we have a short exact sequence of vector bundles

$$0 \to T(S(E)) \to i^*TE \to \nu \to 0.$$

As the first Stiefel-Whitney class is additive in short exact sequences, we have $w_1(i^*TE) = w_1(T(S(E)))+w_1(\nu)$. On the other hand, by the above isomorphism, $w_1(i^*TE) = w_1(T(S(E)) \oplus \varepsilon^1) = w_1(T(S(E)))$. Therefore $w_1(\nu) = 0$ and hence ν is trivial.