HOMOTOPY GROUPS OF A WEDGE SUM OF SPHERES

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Abstract. There is a trick for computing the first few homotopy groups of a wedge sum of spheres which uses cellular approximation. But how do you compute the remaining homotopy groups? The answer is given by Hilton’s Theorem. After introducing the trick, I explain Hilton’s theorem and how to implement it to calculate the homotopy groups of a wedge sum of spheres in terms of the homotopy groups of spheres.

Consider the space \( S^{m_1} \vee \cdots \vee S^{m_k} \). As \( A \vee B \) and \( B \vee A \) are homotopy equivalent, we can (and will) assume \( m_1 \leq \cdots \leq m_k \).

If \( m_1 = 0 \), then \( \pi_0(S^{m_1} \vee \cdots \vee S^{m_k}) = \pi_0(S^{m_2} \vee \cdots \vee S^{m_k}) \oplus \pi_0(S^{m_1} \vee \cdots \vee S^{m_k}) \) and for \( n > 0 \), \( \pi_n(S^{m_1} \vee \cdots \vee S^{m_k}) = \pi_n(S^{m_2} \vee \cdots \vee S^{m_k}) \). From now on we will assume that \( m_1 > 0 \) (and hence \( m_i > 0 \) for all \( i \)).

Cellular Approximation

If \( X \) is an \( p \)-dimensional CW complex and \( Y \) is a CW complex, then by cellular approximation \([X, Y] = [X, Y^{(p+1)}]\) where \( Y^{(p+1)} \) denotes the \((p+1)\)-skeleton of \( Y \). If \( X \) and \( Y \) are also pointed, then the same is true of the pointed homotopy classes; in particular, \( \pi_p(Y) = \pi_k(Y^{(p+1)}) \).

Returning to the problem at hand, note that \( S^{m_1} \vee \cdots \vee S^{m_k} \) is a subcomplex of \( S^{m_1} \times \cdots \times S^{m_k} \) - the latter is obtained from the former by attaching cells of dimension at least \( m_1 + m_2 \). However, \( S^{m_1} \vee \cdots \vee S^{m_k} \) need not be the \((m_1 + m_2 - 1)\)-skeleton of \( S^{m_1} \times \cdots \times S^{m_k} \) as it may contain cells of dimension greater than \( m_1 + m_2 - 1 \) (e.g. the 4-skeleton of \( S^2 \times S^3 \times S^5 \) is \( S^2 \vee S^3 \), not \( S^2 \vee S^3 \vee S^5 \)). However, \( S^{m_1} \vee \cdots \vee S^{m_k} \) and \( S^{m_1} \times \cdots \times S^{m_k} \) have the same \((m_1 + m_2 - 1)\)-skeleton, namely \( S^{m_1} \vee \cdots \vee S^{m_a} \) where \( a \) is such that \( m_a \leq m_1 + m_2 - 1 < m_{a+1} \). Therefore, for any \( n < m_1 + m_2 - 1 \) we have

\[
\begin{align*}
\pi_n(S^{m_1} \vee \cdots \vee S^{m_k}) &= \pi_n(S^{m_1} \vee \cdots \vee S^{m_a}) \\
&= \pi_n(S^{m_1} \times \cdots \times S^{m_k}) \\
&= \pi_n(S^{m_1}) \oplus \cdots \oplus \pi_n(S^{m_k}).
\end{align*}
\]

As \( m_1 + m_2 - 1 < m_{a+1} \leq \cdots \leq m_k \), \( \pi_n(S^{m_{a+1}}) = \cdots = \pi_n(S^{m_k}) = 0 \) so we can also express the above result as

\[
\pi_n(S^{m_1} \vee \cdots \vee S^{m_k}) = \pi_n(S^{m_1}) \oplus \cdots \oplus \pi_n(S^{m_a}).
\]

Hilton’s Theorem

Before introducing Hilton’s Theorem, we need to make one further reduction.

If \( m_1 = \cdots = m_b = 1 \) and \( m_{b+1}, \ldots, m_k > 1 \), the Seifert-van Kampen Theorem shows that \( \pi_1(S^{m_1} \vee \cdots \vee S^{m_k}) \cong F_b \), the free group on \( b \) generators. The higher homotopy groups are isomorphic to the higher homotopy groups of the universal cover which is homotopy equivalent to the wedge sum of countably many copies of \( S^{m_{k+1}} \vee \cdots \vee S^{m_k} \). With this in mind, we will assume from now on that \( m_1 > 1 \) (and hence \( m_i > 1 \) for all \( i \)) and set \( m_i = r_i + 1 \); note \( r_i \geq 1 \).

In order to state Hilton’s Theorem, we need to introduce what he calls basic products.
Let \( \alpha_j \) be the positive generator of \( \pi_m(S^m) \) (i.e. the homotopy class of the identity map)\(^1\). We call \( \alpha_1, \ldots, \alpha_k \) basic products of weight one, and we order them as follows: \( \alpha_1 < \cdots < \alpha_k \).

Now assume the basic products of weight less than \( w \) have been defined and ordered. Basic products of weight \( w \) are Whitehead products \([a, b]\) where \( a, b \) are basic products of weights \( u \) and \( v \) respectively, \( u + v = w, a < b \) (in the ordering), and if \( b = [c, d] \) where \( c \) and \( d \) are basic products, then \( c \leq a \). Order the weight \( w \) elements arbitrarily among themselves and greater than all lower weight basic products. It follows that \( u \leq v \).

**Example:** Suppose \( k = 3 \). Then there are three weight one basic products, namely \( \alpha_1, \alpha_2, \alpha_3 \), which are ordered as follows: \( \alpha_1 < \alpha_2 < \alpha_3 \).

The weight two basic products are \([\alpha_1, \alpha_2], [\alpha_1, \alpha_3], [\alpha_2, \alpha_3]\). We choose to extend the order as follows: \( \alpha_1 < \alpha_2 < \alpha_3 < [\alpha_1, \alpha_2] < [\alpha_1, \alpha_3] < [\alpha_2, \alpha_3] \).

The weight three basic products are \([\alpha_1, [\alpha_1, \alpha_2]], [\alpha_1, [\alpha_1, \alpha_3]], [\alpha_1, [\alpha_2, \alpha_3]], [\alpha_2, [\alpha_1, \alpha_3]], [\alpha_2, [\alpha_2, \alpha_3]], [\alpha_3, [\alpha_1, \alpha_3]], [\alpha_3, [\alpha_2, \alpha_3]]\). One possible ordering is

\[
\alpha_1 < \alpha_2 < \alpha_3 < [\alpha_1, \alpha_2] < [\alpha_1, \alpha_3] < [\alpha_2, \alpha_3] < [\alpha_1, \alpha_2] < [\alpha_1, \alpha_2] < [\alpha_2, \alpha_2] < [\alpha_2, \alpha_3] < [\alpha_3, \alpha_1, \alpha_3] < [\alpha_3, \alpha_3, \alpha_3].
\]

Note, the ordering on the weight two basic products played no role in determining the basic products of weight three. However, they do now play a role in determining the basic products of weight four. For example, \([\alpha_1, \alpha_2], [\alpha_1, \alpha_3]\) is a basic product of weight four but \([\alpha_1, \alpha_3], [\alpha_1, \alpha_2]\) is not; this is because we chose an order in which \([\alpha_1, \alpha_2] < [\alpha_1, \alpha_3]\). Had we chosen to extend the order to basic products of weight two in such a way that \([\alpha_1, \alpha_3] < [\alpha_1, \alpha_2]\), then \([\alpha_1, \alpha_3], [\alpha_1, \alpha_2]\) would be a basic product of weight four but \([\alpha_1, \alpha_2], [\alpha_1, \alpha_3]\) would not be. In general, the ordering of the elements of weight \( k \) only affects the basic products of weight \( 2k \) and above.

Any basic product \( p \) of weight \( w \) is a string of symbols \( \alpha_{j_1}, \ldots, \alpha_{j_w} \) suitably bracketed. Let \( w_j \) be the number of occurrences of \( \alpha_j \) in the string representing \( p \). The height of \( p \) is defined to be \( q = \sum_{i=1}^k r_i w_i \).

Let \( \{p_s\} \) be the sequence of basic products written in increasing order, and denote the height of \( p_s \) by \( q_s \). Then Hilton’s Theorem [1] states that there is an isomorphism

\[
\pi_n(S^{m1} \vee \cdots \vee S^{m_k}) \cong \bigoplus_{i=1}^{\infty} \pi_n(S^{q_i+1}).
\]

There were choices involved in the definition of basic products (namely the orderings of the weight \( w \) basic products for \( w \geq 2 \)). It turns out that had we made different choices, the only difference is that the direct summands are reordered. More precisely, by a theorem of Witt [2], the number of basic products involving \( w_j \) copies of \( \alpha_j \), which necessarily have weight \( w = w_1 + \cdots + w_k \), is given by

\[
A(w_1, \ldots, w_k) = \frac{1}{w} \sum_{d|w} \mu(d) \frac{(w/d)!}{(w_1/d)! \cdots (w_k/d)!}
\]

where \( \mu \) denotes the Möbius function defined on the positive integers by

\[
\mu(d) = \begin{cases} 
1 & \text{d is square-free with an even number of prime factors} \\
-1 & \text{d is square-free with an odd number of prime factors} \\
0 & \text{d has a squared prime factor.}
\end{cases}
\]

It follows that for any \( q \), the number of direct summands of the form \( \pi_n(S^{q+i}) \) is independent of the choices made. For the purposes of calculation, it is useful to note that \( A(w_{\sigma(1)}, \ldots, w_{\sigma(k)}) = A(w_1, \ldots, w_k) \) for all \( \sigma \in S_k \).

\(^1\)Note that Hilton uses the notation \( \iota_j \) instead of \( \alpha_j \).
We can calculate $\pi_n(S^{m_1} \vee \cdots \vee S^{m_k})$ as follows: for each $q$, find the sum of all $A(w_1, \ldots, w_k)$ for which $\sum_{i=1}^k r_i w_i = q$; call it $c_{q+1}$ (this is the number of direct summands of the form $\pi_n(S^{q+1})$). Therefore we have

$$\pi_n(S^{m_1} \vee \cdots \vee S^{m_k}) \cong \bigoplus_{q=1}^{\infty} \pi_n(S^{q+1})^{\otimes c_{q+1}} = \bigoplus_{q=2}^{\infty} \pi_n(S^q)^{c_q}.$$ 

Furthermore, $\pi_n(S^q) = 0$ for $q > n$, so we only need to consider $q \leq n$ and hence

$$\pi_n(S^{m_1} \vee \cdots \vee S^{m_k}) \cong \bigoplus_{q=2}^n \pi_n(S^q)^{c_q}.$$ 

If $n < m_1 + m_2 - 1$, this agrees with the expression we found earlier. To see this, note that for $2 \leq q < m_1 + m_2 - 1$, any solution of the equation $(m_1 - 1)w_1 + \cdots + (m_k - 1)w_k = q - 1$ must be of the form $w_i = 1$ for some $i$ with $m_i = q$ and zero for all other $m_j$. This solution corresponds to a unique basic product of weight one, namely $\alpha_i$. So we see that $c_q$ is equal to the number of spheres in the wedge product of dimension $q$ and hence recover the previous result.

**Example:** Suppose we want to calculate the homotopy groups of $S^3 \vee S^4 \vee S^5 = S^{2+1} \vee S^{3+1} \vee S^{4+1}$. For $n < 3 + 4 - 1 = 6$ we have $\pi_n(S^3 \vee S^4 \vee S^5) = \pi_n(S^3) \oplus \pi_n(S^4) \oplus \pi_n(S^5)$.

For $n = 6$ we want to find the solutions of the equation $2w_1 + 3w_2 + 4w_3 = 5$. The only solution is $(1, 1, 0)$ and the only basic product involving $\alpha_1$ once and $\alpha_2$ once is $\{\alpha_1, \alpha_2\}$, so $c_6 = A(1, 1, 0) = 1$ and therefore

$$\pi_6(S^3 \vee S^4 \vee S^5) \cong \pi_6(S^3) \oplus \pi_6(S^4) \oplus \pi_6(S^5) \oplus \pi_6(S^6).$$

For $n = 7$ the equation of interest is $2w_1 + 3w_2 + 4w_3 = 6$. The solutions are $(3, 0, 0)$, $(0, 2, 0)$, and $(1, 0, 1)$. There are no basic products involving $\alpha_1$ three times but no $\alpha_2$ and $\alpha_3$. Let’s check with the formula

$$A(3, 0, 0) = \frac{1}{3} \sum_{d \mid w_i} \mu(d)(3/d)! = \frac{1}{3} \sum_{d \mid w_i} \mu(d) = \frac{1}{3} (\mu(1) + \mu(3)) = \frac{1}{3} (1 - 1) = 0$$

Likewise, there are no basic products with $\alpha_2$ twice, but no $\alpha_1$ or $\alpha_3$ (note, $[\alpha_2, \alpha_2]$ is not a basic product), so $A(0, 2, 0) = 0$. Finally, there is only one basic product with $(w_1, w_2, w_3) = (1, 0, 1)$, namely $[\alpha_1, \alpha_3]$. Therefore, $c_7 = A(3, 0, 0) + A(0, 2, 0) + A(1, 0, 1) = 1$, so

$$\pi_7(S^3 \vee S^4 \vee S^5) \cong \pi_7(S^3) \oplus \pi_7(S^4) \oplus \pi_7(S^5) \oplus \pi_7(S^6) \oplus \pi_7(S^7).$$
Here is a table of the relevant information for the next few values of $n$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>solutions of $2w_1 + 3w_2 + 4w_3 = n - 1$</th>
<th>$A$</th>
<th>$c_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>$(2, 1, 0)$, $(0, 1, 1)$</td>
<td>$A(2, 1, 0) = 1$</td>
<td>$1 + 1 = 2$</td>
</tr>
<tr>
<td>9</td>
<td>$(4, 0, 0)$, $(2, 0, 1)$, $(1, 2, 0)$, $(0, 0, 2)$</td>
<td>$A(4, 0, 0) = 0$</td>
<td>$0 + 1 + 1 + 0 = 2$</td>
</tr>
<tr>
<td>10</td>
<td>$(3, 1, 0)$, $(1, 1, 1)$, $(0, 3, 0)$</td>
<td>$A(3, 1, 0) = 1$</td>
<td>$1 + 2 + 0 = 3$</td>
</tr>
<tr>
<td>11</td>
<td>$(5, 0, 0)$, $(3, 0, 1)$, $(2, 2, 0)$, $(1, 0, 2)$, $(0, 2, 1)$</td>
<td>$A(5, 0, 0) = 0$</td>
<td>$0 + 1 + 1 + 1 + 1 = 4$</td>
</tr>
<tr>
<td>12</td>
<td>$(4, 1, 0)$, $(2, 1, 1)$, $(1, 2, 0)$, $(0, 1, 2)$</td>
<td>$A(4, 1, 0) = 1$</td>
<td>$1 + 3 + 1 + 1 = 6$</td>
</tr>
</tbody>
</table>

So, using our knowledge of the homotopy groups of spheres, we see for example that

$$
\pi_{12}(S^3 \vee S^4 \vee S^5) \\
\cong \pi_{12}(S^3) \oplus \pi_{12}(S^4) \oplus \pi_{12}(S^5) \oplus \pi_{12}(S^6) \oplus \pi_{12}(S^7) \oplus \pi_{12}(S^8)^2 \oplus \pi_{12}(S^9)^2 \oplus \pi_{12}(S^{10})^3 \\
\oplus \pi_{12}(S^{11})^4 \oplus \pi_{12}(S^{12})^6 \\
\cong \mathbb{Z}_2^3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{30} \oplus \mathbb{Z}_2 \oplus 0 \oplus 0^2 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_2^3 \oplus \mathbb{Z}_2^4 \oplus \mathbb{Z}^6 \\
\cong \mathbb{Z}^6 \oplus \mathbb{Z}_{30} \oplus \mathbb{Z}_2^3 \oplus \mathbb{Z}_2^4 \oplus \mathbb{Z}_4^2 \\
\cong \mathbb{Z}^6 \oplus (\mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2) \oplus (\mathbb{Z}_5 \oplus \mathbb{Z}_3)^2 \oplus \mathbb{Z}_2^{11} \\
\cong \mathbb{Z}^6 \oplus \mathbb{Z}_5^2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_2^{12}
$$

In this example, every sphere of dimension greater than $m_1 + m_2 - 1$ occurs at least once. This is not always the case. For example, consider $S^3 \vee S^5$. As the expression $2w_1 + 4w_2$ is never odd, $c_q = 0$ for $q$ even, i.e. only homotopy groups of odd-dimensional spheres appear as direct summands. More generally, if the greatest common divisor of $r_1, \ldots, r_k$ is $r$, then $c_q = 0$ if $r \nmid q$. Even if the relevant equation has solutions for a given $q$, there may not be any corresponding basic products. For example, $S^3 \vee S^4$. The equation $2w_1 + 3w_2 = 4$ has a unique solution, namely $(2, 0)$, but there are no basic products with two $\alpha_1$ and no $\alpha_2$, so $\pi_n(S^5)$ does not appear as a direct summand of $\pi_n(S^3 \vee S^4)$.

Here is some pseudocode for calculating the values of $c_q$

- Enter $n$.
- Enter $m_1, \ldots, m_k$.
- Reorder $m_i$ in increasing order to get $m_i'$
- Set $r_i = m_i' - 1$.
- For $q = 2, \ldots, \min(r_1 + r_2, n)$, set $c_q$ to be the number of elements of $[r_i]$ equal to $q - 1$.
- For $q = r_1 + r_2 + 1, \ldots, n$
  - Calculate non-negative integer solutions of $[r_i]^T w = q - 1$
  - For each solution $w$, calculate (using Witt's Theorem) $A(w)$. 


\[ \text{Set } c_q = \text{sum of } A(w) \]

- Output \([c_q]\)

\textbf{References}
