A MANIFOLD OF DIMENSION 4 WITH EVEN INTERSECTION FORM AND SIGNATURE -8

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What are the symmetric unimodular bilinear forms that can be the intersection form of a closed 4-manifold? It follows from the theorem of Rochlin [1] that the signature of a closed, simply connected manifold of dimension four, whose intersection form is even, is divisible by 16. In this note, we present an example which shows that the simply connected hypothesis is essential in the above statement.

The theorem of Rochlin says that the signature of a manifold of dimension 4 which is closed and almost parallelisable is divisible by 16. On the other hand, an orientable manifold M of dimension 4 is almost parallelisable if and only if its second Stiefel-Whitney class $w_2(M) \in H^2(M;\mathbb{Z}_2)$ is zero. This condition can also be expressed in terms of the mod 2 intersection form as follows: Let $u_2 = u_2(M) \in H^2(M;\mathbb{Z}_2)$, be the second Wu class defined by the formula $\langle u_2 \cdot x, [M]\rangle = \langle \mathrm{Sq}^2(x), [M]\rangle$ for all $x \in H^{n-2}(M;\mathbb{Z}_2)$. We recall the formula of Wu, $w = \mathrm{Sq}(u)$, where $w = 1 + w_1 + w_2 + \dots$ and $u = 1 + u_1 + u_2 + \dots$ are respectively the total classes of Stiefel-Whitney and of Wu. For the remainder, M now denotes a closed orientable manifold of dimension 4. Then $w_1(M) = 0$ and the formula of Wu gives $w_2(M) = u_2(M)$. Thus we have the formula $w_2 \cdot x = \mathrm{Sq}^2(x) = x^2$ for all $x \in H^2(M;\mathbb{Z}_2)$. So for M closed, orientable of dimension 4, we have $w_2 = 0$ if and only if $x^2 = 0$ for all $x \in H^2(M;\mathbb{Z}_2)$.

Let $T^2 = T^2(M;\mathbb{Z})$ be the torsion subgroup of $H^2 = H^2(M;\mathbb{Z})$ and let $\rho : H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z}_2)$ be reduction mod 2. The subspaces $\rho(T^2)$ and $\rho(H^2)$ are mutually orthogonal (for the product mod 2). In fact, for dimension reasons (cf. [2]) each is the orthogonal complement of the other. It follows that:

(i) $w_2(M) \in \rho(H^2)$

(ii) $w_2(M) \in \rho(T^2) \Leftrightarrow$ the intersection form on H^2/T^2 is even.

Thus for M^4 simply connected ($T^2 = 0$) the conditions

(a) M has even intersection form ($w_2 \in \rho(T^2)$)

(b) M is almost parallelisable ($w_2 = 0$)

are equivalent.

Thus we find the statement at the beginning as a corollary of the theorem of Rochlin. If M is simply connected of dimension 4 and possesses an even intersection form, then its signature is divisible by 16. But in general, condition (a) above is weaker than (b) as can be seen in the following example:

Let $\tilde{M} = S^2 \times S^2$ and $M = \tilde{M}/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\tilde{M}$ by $(x, y) \to (-x, -y)$. We have $\operatorname{rank} H^2(M) + 2 = \chi(M) = \frac{1}{2}\chi(\tilde{M}) = 2$ so $\operatorname{rank} H^2(M) = 0$, that is to say $H^2(M) = T^2(M)$. On the other hand, from the diagonal embedding $S^2 \hookrightarrow S^2 \times S^2$ we obtain by passing to quotients an embedding $\mathbb{RP}^2 \overset{i}{\hookrightarrow} M$ with self-intersection 1. If $x \in H^2(M;\mathbb{Z}_2)$ is the Poincaré dual of $i_*[\mathbb{RP}^2] \in H_2(M;\mathbb{Z}_2)$, we have $w_2 \cdot x = x^2 = 1$ and so $w_2 \neq 0$.

The manifold above has signature zero, since $H^2/T^2 = 0$. Nevertheless, this construction gave us hope that there could be a manifold with even intersection form and signature $\equiv 8 \bmod 16$. Here is such an example: Let $\tilde{M}$ be the hypersurface of degree 4 of $\mathbb{CP}^3$ given by the equation $z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$. We define an involution on $\mathbb{CP}^3$ by $(z_0, z_1, z_2, z_3) \mapsto (\overline{z_1}, -\overline{z_0}, \overline{z_3}, -\overline{z_2})$. It is easy to verify that it is an involution without fixed points which leaves $\tilde{M}$ invariant. By taking quotients we obtain an embedding $M = \tilde{M}/\mathbb{Z}_2 \xrightarrow{i} Q = \mathbb{CP}^3/\mathbb{Z}_2$. It is easy to verify that M is orientable with normal bundle $\nu(i)$ non-orientable.

We will verify that the intersection form of M is even, signature$(M) = -8$, $\operatorname{rank} H^2(M;\mathbb{Z}) = 10$. According to the classification of unimodular symmetric forms (cf [4]), we obtain

Proposition. *Any even unimodular symmetric bilinear form with* $|\text{signature}| \le \frac{4}{5}\,\text{rank}$ *is the intersection form of a closed manifold of dimension 4.*

The manifold $\tilde{M}$ has the following properties (cf. [3]). $\tilde{M}$ is simply connected, $\operatorname{rank} H^2(\tilde{M};\mathbb{Z}) = 22$, signature$(\tilde{M}) = -16$. We have $-16 = \text{signature}(\tilde{M}) = \langle p_1(\tilde{M})/3, [\tilde{M}]\rangle = \langle p_1(M)/3, 2[M]\rangle = 2\,\text{signature}(M)$, therefore signature$(M) = -8$. On the other hand $\operatorname{rank} H^2(M)+2 = \chi(M) = \frac{1}{2}\chi(\tilde{M}) = \frac{1}{2}(\operatorname{rank} H^2(\tilde{M}) + 2) = 12$ and therefore $\operatorname{rank} H^2(M) = 10$.

It remains to be seen that the intersection form of M is even. From what precedes, it suffices to see that $w_2(M) \in \rho(T^2(M;\mathbb{Z}))$. From the fibre equation $TM + \nu(i) = i^*TQ$ induced by the inclusion $M \xrightarrow{i} Q$, we deduce that $w_2(M) + w_2(\nu(i)) = i^*w_2(Q)$. We will show that $H^2(Q;\mathbb{Z}_2) = \rho(T^2(Q;\mathbb{Z}))$ and $w_2(\nu(i)) = 0$. It follows that $w_2(M) = i^*w_2(Q) \in i^*\rho(T^2(Q;\mathbb{Z})) \subset \rho(T^2(M;\mathbb{Z}))$.

Let $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^3$ be given by $z_2 = z_3 = 0$. By taking quotients, we obtain $\mathbb{RP}^2 = \mathbb{CP}^1/\mathbb{Z}_2 \overset{j}{\hookrightarrow} Q$. Recall that for a two-sheeted covering $\tilde{X} \to X$, we have the short exact sequences of chain complexes: $0 \to C(X)\otimes\mathbb{Z}_2 \to C(\tilde{X})\otimes\mathbb{Z}_2 \to C(X)\otimes\mathbb{Z}_2 \to 0$. So we obtain the diagram

$$\begin{array}{ccccccc}
H_2(\mathbb{CP}^1;\mathbb{Z}_2) & \xrightarrow{0} & H_2(\mathbb{RP}^2;\mathbb{Z}_2) & \longrightarrow & H_1(\mathbb{RP}^2;\mathbb{Z}_2) & \longrightarrow & H_1(\mathbb{CP}^1;\mathbb{Z}_2) = 0 \\
\downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow \\
H_2(\mathbb{CP}^3;\mathbb{Z}_2) & \longrightarrow & H_2(Q;\mathbb{Z}_2) & \longrightarrow & H_1(Q;\mathbb{Z}_2) & \longrightarrow & H_1(\mathbb{CP}^3;\mathbb{Z}_2) = 0
\end{array}$$

It follows that $H_2(\mathbb{RP}^2;\mathbb{Z}_2) \to H_2(Q;\mathbb{Z}_2)$ is an isomorphism. By duality, $H^2(Q;\mathbb{Z}_2) \to H^2(\mathbb{RP}^2;\mathbb{Z}_2)$ is an isomorphism. The commutative diagram

$$\begin{array}{ccccc}
\operatorname{Ext}(H_1(Q;\mathbb{Z}),\mathbb{Z}) & \longrightarrow & H^2(Q;\mathbb{Z}) & \xrightarrow{\rho} & H^2(Q;\mathbb{Z}_2) \\
\downarrow\cong & & \downarrow & & \downarrow\cong \\
\operatorname{Ext}(H_1(\mathbb{RP}^2;\mathbb{Z}),\mathbb{Z}) & \xrightarrow{\cong} & H^2(\mathbb{RP}^2;\mathbb{Z}) & \underset{\cong}{\xrightarrow{\rho}} & H^2(\mathbb{RP}^2;\mathbb{Z}_2)
\end{array}$$

establishes that $\rho(T^2(Q;\mathbb{Z})) = H^2(Q;\mathbb{Z}_2)$.

As $j_*[\mathbb{RP}^2]$ generates $H_2(Q;\mathbb{Z}_2)$ and the intersection of $\mathbb{RP}^2$ and M is zero ($\tilde{M}$ is of degree 4), it follows that $i_*[M] = 0$ in $H_4(Q;\mathbb{Z}_2)$. The Euler class of the normal bundle $\nu(i)$ is therefore also trivial, which shows that $w_2(\nu(i)) = 0$.

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References

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