WHICH GRASSMANNIANS ARE SPIN/SPIN^c?

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Abstract. The purpose of this note is to determine which (unoriented, oriented, and complex) grassmannians are spin, and which ones are spin^c. In order to achieve this goal, formulae for the first and second Stiefel-Whitney classes of a tensor product are derived. The corresponding non-orientable analogues pin$^+$, pin$^-$, and pin^c are also considered.

Let $\mathrm{Gr}(a, b)$ denote the grassmanian of a-dimensional subspaces of a real b-dimensional vector space, and denote the tautological bundle over it by γ. Recall that $T\,\mathrm{Gr}(a, b) \cong \mathrm{Hom}(\gamma, \gamma^\perp) \cong \gamma^* \otimes \gamma^\perp \cong \gamma \otimes \gamma^\perp$ where $\gamma^\perp$ is the orthogonal complement of $\gamma \subset \varepsilon^b$ with respect to a fixed Riemannian metric on ε^b. As a smooth manifold M is spin if and only if $w_1(M) = 0$ and $w_2(M) = 0$, we need to determine formulae for $w_1(E \otimes F)$ and $w_2(E \otimes F)$.

1. Stiefel-Whitney Classes of a Tensor Product

Lemma. *Let L_1 and L_2 be real line bundles over a paracompact space B. Then $w_1(L_1 \otimes L_2) = w_1(L_1) + w_1(L_2)$.*

Proof. Let $\pi_i : \mathbb{RP}^\infty \times \mathbb{RP}^\infty \to \mathbb{RP}^\infty$ denote projection onto the i^{th} factor and let $\mu : \mathbb{RP}^\infty \times \mathbb{RP}^\infty \to \mathbb{RP}^\infty$ be a classifying map for $\pi_1^*\gamma \otimes \pi_2^*\gamma$. By the Künneth theorem, $\pi_1^* w_1(\gamma)$ and $\pi_2^* w_1(\gamma)$ form a basis for $H^1(\mathbb{RP}^\infty \times \mathbb{RP}^\infty; \mathbb{Z}_2)$, so $w_1(\pi_1^*\gamma \otimes \pi_2^*\gamma) = a\pi_1^* w_1(\gamma) + b\pi_2^* w_1(\gamma)$ for some $a, b \in \mathbb{Z}_2$.

If $\sigma : \mathbb{RP}^\infty \times \mathbb{RP}^\infty \to \mathbb{RP}^\infty \times \mathbb{RP}^\infty$ is the map which interchanges factors, then $\pi_1 \circ \sigma = \pi_2$ and $\pi_2 \circ \sigma = \pi_1$, so $\sigma^*\mu^* w_1(\gamma) = a\pi_2^* w_1(\gamma) + b\pi_1^* w_1(\gamma)$, but $\sigma \circ \mu$ classifies $\pi_2^*\gamma \otimes \pi_1^*\gamma \cong \pi_1^*\gamma \otimes \pi_2^*\gamma$ so $\sigma \circ \mu$ is homotopic to μ. Therefore

$$a\pi_2^* w_1(\gamma) + b\pi_1^* w_1(\gamma) = (\sigma \circ \mu)^* w_1(\gamma) = \mu^* w_1(\gamma) = a\pi_1^* w_1(\gamma) + b\pi_2^* w_1(\gamma),$$

which implies $a = b$. So either $w_1(\pi_1^*\gamma \otimes \pi_2^*\gamma) = \pi_1^* w_1(\gamma) + \pi_2^* w_1(\gamma)$, or $w_1(\pi_1^*\gamma \otimes \pi_2^*\gamma) = 0$.

Now let $f_i : B \to \mathbb{RP}^\infty$ be a classifying map for L_i. Then

$$\begin{aligned}(f_1, f_2)^*(\pi_1^*\gamma \otimes \pi_2^*\gamma) &\cong ((f_1, f_2)^*\pi_1^*\gamma) \otimes ((f_1, f_2)^*\pi_2^*\gamma)\\ &\cong (\pi_1 \circ (f_1, f_2))^*\gamma \otimes (\pi_2 \circ (f_1, f_2)^*)\gamma\\ &\cong f_1^*\gamma \otimes f_2^*\gamma\\ &\cong L_1 \otimes L_2.\end{aligned}$$

As $w_1(L_1 \otimes L_2) = w_1((f_1, f_2)^*(\pi_1^*\gamma \otimes \pi_2^*\gamma) = (f_1, f_2)^* w_1(\pi_1^*\gamma \otimes \pi_2^*\gamma)$, if $w_1(\pi_1^*\gamma \otimes \pi_2^*\gamma) = 0$, then $w_1(L_1 \otimes L_2) = 0$. This is clearly false, just take L_1 to be non-trivial and L_2 to be trivial. Therefore $w_1(\pi_1^*\gamma \otimes \pi_2^*\gamma) = \pi_1^* w_1(\gamma) + \pi_2^* w_1(\gamma)$ and so

$$\begin{aligned}w_1(L_1 \otimes L_2) &= (f_1, f_2)^* w_1(\pi_1^*\gamma \otimes \pi_2^*\gamma)\\ &= (f_1, f_2)^*(\pi_1^* w_1(\gamma) + \pi_2^* w_1(\gamma))\\ &= (f_1, f_2)^*\pi_1^* w_1(\gamma) + (f_1, f_2)^*\pi_2^* w_1(\gamma)\\ &= (\pi_1 \circ (f_1, f_2))^* w_1(\gamma) + (\pi_2 \circ (f_1, f_2))^* w_1(\gamma)\\ &= f_1^* w_1(\gamma) + f_2^* w_1(\gamma)\end{aligned}$$

$$= w_1(f_1^*\gamma) + w_1(f_2^*\gamma)$$
$$= w_1(L_1) + w_1(L_2).$$

□

With this lemma in hand, we can move on to the general case thanks to the splitting principle.

Theorem. *Let E and F be real vector bundles over a paracompact space B. Let $m = \operatorname{rank} E$ and $n = \operatorname{rank} F$. Then $w(E \otimes F) = p_{m,n}(w_1(E), \dots, w_m(E), w_1(F), \dots, w_n(F))$ where $p_{m,n}$ is the unique polynomial which satisfies*

$$p_{m,n}(\sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_n) = \prod_{i=1}^{m}\prod_{j=1}^{n}(1 + x_i + y_j)$$

where $\sigma_k = \sigma_k(x_1, \dots, x_m)$ and $\tau_k = \tau_k(y_1, \dots, y_n)$ are the k^{th} elementary symmetric polynomials in m and n variables respectively.

Proof. By the splitting principle, there is a paracompact space Y and a map $g : Y \to B$ such that $g^*E \cong \ell_1' \oplus \dots \oplus \ell_m'$ and $g^* : H^*(Y; \mathbb{Z}_2) \to H^*(B; \mathbb{Z}_2)$ is injective. Again by the splitting principle, there is a paracompact space X and a map $f : X \to Y$ such that $f^*g^*F \cong \eta_1 \oplus \dots \oplus \eta_n$, and $f^* : H^*(X; \mathbb{Z}_2) \to H^*(Y; \mathbb{Z}_2)$ is injective. Letting $\ell_i = f^*\ell_i'$, we have $f^*g^*E \cong \ell_1 \oplus \dots \oplus \ell_m$. So

$$f^*g^*(E \otimes F) \cong (f^*g^*E) \otimes (f^*g^*F) \cong (\ell_1 \oplus \dots \oplus \ell_m) \otimes (\eta_1 \oplus \dots \oplus \eta_n) \cong \bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n} \ell_i \otimes \eta_j.$$

Therefore,

$$\begin{aligned} w(f^*g^*(E \otimes F)) &= w\left(\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n} \ell_i \otimes \eta_j\right) \\ &= \prod_{i=1}^{m}\prod_{j=1}^{n} w(\ell_i \otimes \eta_j) \\ &= \prod_{i=1}^{m}\prod_{j=1}^{n}(1 + w_1(\ell_i \otimes \eta_j)) \\ &= \prod_{i=1}^{m}\prod_{j=1}^{n}(1 + w_1(\ell_i) + w_1(\eta_j)) \\ &= \prod_{i=1}^{m}\prod_{j=1}^{n}(1 + x_i + y_j) \end{aligned}$$

where the penultimate equality uses the lemma and $x_i := w_1(\ell_i)$, $y_j := w_1(\eta_j)$.

Denote the final expression above by $q(x_1, \dots, x_m, y_1, \dots, y_n)$. Note that q is a polynomial which is symmetric in the x_i and the y_j separately, so by the fundamental theorem of symmetric polynomials, there is a unique polynomial $p_{m,n}$ such that

$$q(x_1, \dots, x_m, y_1, \dots, y_m) = p_{m,n}(\sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_n).$$

Now note that $\sigma_i(x_1, \dots, x_m) = w_i(\ell_1 \oplus \dots \oplus \ell_m) = w_i(f^*g^*E) = f^*g^*w_i(E)$ and likewise $\tau_j(y_1, \dots, y_n) = f^*g^*w_j(F)$, so

$$\begin{aligned} f^*g^*w(E \otimes F) &= w(f^*g^*(E \otimes F)) \\ &= q(x_1, \dots, x_m, y_1, \dots, y_n) \\ &= p_{m,n}(\sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_n) \end{aligned}$$

$$= p_{m,n}(f^*g^*w_1(E), \dots, f^*g^*w_m(E), f^*g^*w_1(F), \dots, f^*g^*w_n(F))$$
$$= f^*g^*p_{m,n}(w_1(E), \dots, w_m(E), w_1(F), \dots, w_n(F)).$$

By the injectivity of f^* and g^*, we have $w(E \otimes F) = p_{m,n}(w_1(E), \dots, w_m(E), w_1(F), \dots, w_n(F))$. □

The two proofs above constitute a solution to Problem 7-C from [4].

As in the proof, we will use $q(x_1, \dots, x_m, y_1, \dots, y_n)$ to denote the right hand side of the equation in the theorem.

If we can identify the degree k part of $p_{m,n}$, then we can obtain an explicit formula for $w_k(E \otimes F)$ in terms of $w_1(E), \dots, w_k(E), w_1(F), \dots, w_k(F)$. In particular, we need to express the degree k part of q as a polynomial in elementary symmetric polynomials. To achieve our main goal, we only need to do this for $k = 1$ and 2.

The degree one part of q is

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(x_i + y_j) = n\sum_{i=1}^{m} x_i + m\sum_{j=1}^{n} y_j = n\sigma_1(x_1, \dots, x_m) + m\tau_1(y_1, \dots, y_n).$$

Therefore, we have the following:

Proposition. *For vector bundles E, F over a paracompact space with* $\operatorname{rank} E = m$ *and* $\operatorname{rank} F = n$, *we have*

$$w_1(E \otimes F) = nw_1(E) + mw_1(F).$$

Now we need to identify the degree two part of q; this is more difficult. First note that q is the product of mn factors, and any two factors gives rise to four degree two terms, so there should be a total of $4\binom{mn}{2}$ terms in the degree two part of q. There are five distinct types of terms that can appear: x_i^2, y_j^2, $x_i x_{i'}$ with $i \neq i'$, $y_j y_{j'}$ with $j \neq j'$, and $x_i y_j$.

The x_i^2 terms only arise from the subproduct $(1 + x_i + y_1) \dots (1 + x_i + y_n)$, and each choice of two factors gives rise to one such term, so in total there are $\binom{n}{2}$ copies of x_i^2.

The y_j^2 terms only arise from the subproduct $(1 + x_1 + y_j) \dots (1 + x_m + y_j)$, and each choice of two factors gives rise to one such term, so in total there are $\binom{m}{2}$ copies of y_j^2.

The $x_i x_{i'}$ terms with $i \neq i'$ only arise from the subproduct $(1 + x_i + y_1) \dots (1 + x_i + y_n)(1 + x_{i'} + y_1) \dots (1 + x_{i'} + y_n)$, and each choice of a factor from the first n and a factor from the second n gives rise to one such term, so in total there are n^2 copies of $x_i x_{i'}$.

The $y_j y_{j'}$ terms with $j \neq j'$ only arise from the subproduct $(1 + x_1 + y_j) \dots (1 + x_m + y_j)(1 + x_1 + y_{j'}) \dots (1 + x_m + y_{j'})$, and each choice of a factor from the first m and a factor from the second m gives rise to one such term, so in total there are m^2 copies of $y_j y_{j'}$.

Now consider terms of the form $x_i y_j$. They can only arise from products of factors of the form $(1 + x_{i'} + y_{j'})$ where $i = i'$ or $j = j'$. Given one of the $n - 1$ factors of the form $(1 + x_i + y_{j'})$ with $j' \neq j$, there are precisely m factors which contain y_j, namely $(1 + x_1 + y_j), \dots, (1 + x_m + y_j)$, which can combine with $(1 + x_i + y_{j'})$ to produce one $x_i y_j$ term. Likewise, given one of the $m - 1$ factors of the form $(1 + x_{i'} + y_j)$ with $i' \neq i$, there are precisely n factors which contain x_i, namely $(1+x_i+y_1), \dots, (1+x_m+y_j)$, which can combine with $(1+x_{i'}+y_j)$ to produce one $x_i y_j$ term. Finally, the unique factor $(1 + x_i + y_j)$ can combine with $(m - 1) + (n - 1)$ factors to produce one $x_i y_j$ term, namely factors of the form $(1 + x_{i'} + y_{j'})$ where $i = i'$ or $j = j'$, but not both. Note, we have double counted each appearance of $x_i y_j$, so in total there are $\frac{1}{2}[m(n-1)+n(m-1)+(m-1)+(n-1)] = mn-1$ copies of $x_i y_j$.

We should check that we haven't missed any terms. There are m terms of the form x_i^2, n terms of the form y_j^2, $\binom{m}{2}$ terms of the form $x_i x_{i'}$ with $i \neq i'$, $\binom{n}{2}$ terms of the form $y_j y_{j'}$ with $j \neq j'$, and mn terms of the form $x_i y_j$. Therefore, there are a total of

$$\begin{aligned}
&m\binom{n}{2} + n\binom{m}{2} + \binom{m}{2}n^2 + \binom{n}{2}m^2 + mn(mn-1) \\
&= \frac{1}{2}mn(n-1) + \frac{1}{2}mn(m-1) + \frac{1}{2}m^2n(n-1) + \frac{1}{2}mn^2(m-1) + mn(mn-1) \\
&= \frac{1}{2}mn[(n-1) + (m-1) + m(n-1) + n(m-1) + 2(mn-1)] \\
&= \frac{1}{2}mn[n-1+m-1+mn-m+mn-n+2mn-2] \\
&= \frac{1}{2}mn[4mn-4] \\
&= 4\frac{mn(mn-1)}{2} \\
&= 4\binom{mn}{2}
\end{aligned}$$

terms in the degree two part of q as predicted.

So the degree two part of q is

$$\begin{aligned}
&\binom{n}{2}\sum_{i=1}^m x_i^2 + \binom{m}{2}\sum_{j=1}^n y_j^2 + n^2 \sum_{1\leq i<i'\leq m} x_i x_{i'} + m^2 \sum_{1\leq j<j'\leq n} y_j y_{j'} + (mn-1)\sum_{i=1}^n\sum_{j=1}^n x_i y_j \\
&= \binom{n}{2}\left(\sum_{i=1}^m x_i\right)^2 + \binom{m}{2}\left(\sum_{j=1}^n y_j\right)^2 + n^2\sigma_2(x_1,\dots,x_m) + m^2\tau_2(y_1,\dots,y_n) \\
&\quad + (mn-1)\left(\sum_{i=1}^m x_i\right)\left(\sum_{j=1}^n y_j\right) \\
&= \binom{n}{2}\sigma_1(x_1,\dots,x_m)^2 + \binom{m}{2}\tau_1(y_1,\dots,y_n)^2 + n^2\sigma_2(x_1,\dots,x_n) + m^2\tau_2(y_1,\dots,y_n) \\
&\quad + (mn-1)\sigma_1(x_1,\dots,x_m)\tau_1(y_1,\dots,y_n).
\end{aligned}$$

Therefore, we have the following:

Proposition. *For vector bundles E, F over a paracompact space with* $\operatorname{rank} E = m$ *and* $\operatorname{rank} F = n$, *we have*

$$w_2(E\otimes F) = \binom{n}{2}w_1(E)^2 + \binom{m}{2}w_1(F)^2 + n^2w_2(E) + m^2w_2(F) + (mn-1)w_1(E)w_1(F).$$

2. Which Unoriented Grassmannians are spin manifolds?

Write the grassmannian $\operatorname{Gr}(a,b)$ as $\operatorname{Gr}(m,m+n)$ where $m=a$ and $n=b-a$. Then $\gamma^\perp$ has rank n. As $T\operatorname{Gr}(m,m+n) = \gamma\otimes\gamma^\perp$, we have

$$w_1(\operatorname{Gr}(m,m+n)) = nw_1(\gamma) + mw_1(\gamma^\perp).$$

Using the fact that $\gamma\oplus\gamma^\perp \cong \varepsilon^{m+n}$, we see that $w_1(\gamma^\perp) = w_1(\gamma)$ and therefore

$$w_1(\operatorname{Gr}(m,m+n)) = nw_1(\gamma) + mw_1(\gamma^\perp) = nw_1(\gamma) + mw_1(\gamma) = (m+n)w_1(\gamma).$$

Proceeding in a similar way, we have

$$w_2(\mathrm{Gr}(m, m+n)) = \binom{n}{2} w_1(\gamma)^2 + \binom{m}{2} w_1(\gamma^\perp)^2 + n^2 w_2(\gamma) + m^2 w_2(\gamma^\perp) + (mn-1) w_1(\gamma) w_1(\gamma^\perp).$$

Again, as $\gamma \oplus \gamma^\perp \cong \varepsilon^{m+n}$, we see that $w_2(\gamma^\perp) = w_2(\gamma) + w_1(\gamma)w_1(\gamma^\perp) = w_2(\gamma) + w_1(\gamma)^2$, so

$$\begin{aligned}
&w_2(\mathrm{Gr}(m, m+n)) \\
&= \binom{n}{2} w_1(\gamma)^2 + \binom{m}{2} w_1(\gamma^\perp)^2 + n^2 w_2(\gamma) + m^2 w_2(\gamma^\perp) + (mn-1) w_1(\gamma) w_1(\gamma^\perp) \\
&= \binom{n}{2} w_1(\gamma)^2 + \binom{m}{2} w_1(\gamma)^2 + n^2 w_2(\gamma) + m^2 (w_2(\gamma) + w_1(\gamma)^2) + (mn-1) w_1(\gamma) w_1(\gamma) \\
&= \left[\binom{m}{2} + \binom{n}{2} + m^2 + mn - 1\right] w_1(\gamma)^2 + (m^2 + n^2) w_2(\gamma).
\end{aligned}$$

As $\binom{d}{2} = \frac{1}{2}d(d-1)$, its parity is determined by $d \bmod 4$. More precisely, $\binom{d}{2}$ is even if $d \equiv 0, 1 \bmod 4$ and odd if $d \equiv 2, 3 \bmod 4$. So the parity of the first two terms is determined by the values of m and n modulo 4, while the parity of remaining terms is determined by the values of m and n modulo 2. So we see that

$$w_2(\mathrm{Gr}(m, m+n)) = \begin{cases} 0 & (m, n) \equiv (0,2), (1,3), (2,0), (3,1) \bmod 4 \\ w_2(\gamma) & (m, n) \equiv (0,3), (1,0), (2,1), (3,2) \bmod 4 \\ w_1(\gamma)^2 & (m, n) \equiv (0,0), (1,1), (2,2), (3,3) \bmod 4 \\ w_2(\gamma) + w_1(\gamma)^2 & (m, n) \equiv (0,1), (1,2), (2,3), (3,0) \bmod 4. \end{cases}$$

Note that the difference $m - n$ is constant in each row, so we can more succinctly express the above as

$$w_2(\mathrm{Gr}(m, m+n)) = \begin{cases} 0 & m - n \equiv 2 \bmod 4 \\ w_2(\gamma) & m - n \equiv 1 \bmod 4 \\ w_1(\gamma)^2 & m - n \equiv 0 \bmod 4 \\ w_2(\gamma) + w_1(\gamma)^2 & m - n \equiv 3 \bmod 4. \end{cases}$$

Upon first glance, the above description seems to contradict the fact that $\mathrm{Gr}(m, m+n)$ and $\mathrm{Gr}(n, m+n)$ are diffeomorphic, at least in the case where $m - n$ is odd. Why does interchanging m and n give a different expression for w_2? In order to understand this disparity, denote the tautological bundles over $\mathrm{Gr}(m, m+n)$ and $\mathrm{Gr}(n, m+n)$ by γ_m and γ_n respectively.

Recall that there is a diffeomorphism $\varphi : \mathrm{Gr}(m, m+n) \to \mathrm{Gr}(n, m+n)$ given by $P \mapsto P^\perp$; note, this requires an inner product on the ambient vector space. It follows that $\varphi^*\gamma_n \cong \gamma_m^\perp$. So, if $m - n \equiv 3 \bmod 4$, we have $w_2(\mathrm{Gr}(m, m+n)) = w_2(\gamma_m) + w_1(\gamma_m)^2 \in H^2(\mathrm{Gr}(m, m+n); \mathbb{Z}_2)$ and $w_2(\mathrm{Gr}(n, m+n)) = w_2(\gamma_n) \in H^2(\mathrm{Gr}(n, m+n); \mathbb{Z}_2)$. The cohomology rings are not equal, so we cannot compare these two elements, but the diffeomorphism φ gives rise to an isomorphism between them, namely φ^*. Under this isomorphism,

$$\varphi^* w_2(\gamma_n) = w_2(\varphi^*\gamma_n) = w_2(\gamma_m^\perp) = w_2(\gamma_m) + w_1(\gamma_m)^2.$$

The case $m - n \equiv 1 \bmod 4$ is similar.

Now that we have expressions for $w_1(\mathrm{Gr}(m, m+n))$ and $w_2(\mathrm{Gr}(m, m+n))$, we can finally determine for which m and n the manifold $\mathrm{Gr}(m, m+n)$ is spin. We can also ask about the non-orientable anologues of spin, namely pin$^+$ and pin$^-$. The obstruction to a smooth manifold M admitting a pin$^+$ structure is $w_2(M)$, and the obstruction to admitting a pin$^-$ structure is $w_2(M) + w_1(M)^2$.

Recall that $H^*(\mathrm{Gr}(m, m+n); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1(\gamma), \dots, w_m(\gamma)]/(\bar{w}_{n+1}, \dots, \bar{w}_{m+n})$ where $\bar{w}_i$ is the degree i component of the formal inverse of $1 + w_1(\gamma) + \dots + w_m(\gamma)$ in $\mathbb{Z}_2[w_1(\gamma), \dots, w_m(\gamma)]$. It follows that if $m, n \geq 2$, then $w_1(\gamma)$, $w_2(\gamma)$, $w_1(\gamma)^2$, and $w_2(\gamma) + w_1(\gamma)^2$ are all non-zero. If $m = 1$ or $n = 1$, then the grassmannian is a projective space, in which case it is easy to check whether $w_1(\gamma)$, $w_2(\gamma)$, $w_1(\gamma)^2$, and $w_2(\gamma) + w_1(\gamma)^2$ are non-zero or not.

Theorem. *The grassmannian* $\mathrm{Gr}(m, m+n)$ *is:*

- *orientable if and only if* $m+n$ *is even.*
- *spin if and only if* $m - n \equiv 2 \bmod 4$*, or* $m = n = 1$*, i.e.* $\mathrm{Gr}(1,2) = \mathbb{RP}^1 = S^1$.
- *pin*$^+$ *if and only if it is spin or is a projective space of dimension* $4k$.
- *pin*$^-$ *if and only if it is spin or is a projective space of dimension* $4k+2$.

3. Which oriented grassmannians are spin?

Let $\mathrm{Gr}^+(a,b)$ denote the grassmanian of oriented a-dimensional subspaces of a real b-dimensional vector space, and denote the tautological bundle over it by γ_+. Similar to the unoriented case, we have $T\,\mathrm{Gr}^+(a,b) \cong \gamma_+ \otimes \gamma_+^\perp$ where $\gamma_+^\perp$ is the orthogonal complement of $\gamma_+ \subset \varepsilon^b$ with respect to a fixed Riemannian metric on ε^b.

There is a double covering $\pi : \mathrm{Gr}^+(a,b) \to \mathrm{Gr}(a,b)$ which forgets the orientation of the subspace. It follows that $\pi^*\gamma \cong \gamma_+$, and hence $w_i(\gamma_+) = w_i(\pi^*\gamma) = \pi^* w_i(\gamma)$. The Gysin sequence associated to π is

$$\cdots \to H^{i-1}(\mathrm{Gr}(a,b);\mathbb{Z}_2) \xrightarrow{w_1\cup} H^i(\mathrm{Gr}(a,b);\mathbb{Z}_2) \xrightarrow{\pi^*} H^i(\mathrm{Gr}^+(a,b);\mathbb{Z}_2) \xrightarrow{\pi_*} H^i(\mathrm{Gr}(a,b);\mathbb{Z}_2) \to \dots$$

where $w_1 \in H^1(\mathrm{Gr}(a,b);\mathbb{Z}_2) = \{0, w_1(\gamma)\}$ is the first Stiefel-Whitney class of the real line bundle associated to π; as π is not the trivial double cover, we have $w_1 = w_1(\gamma)$.

By the exactness of the Gysin sequence, the class $w_i(\gamma_+)$ is zero if and only if $w_i(\gamma) = w_1(\gamma) \cup \alpha$ for some α, i.e. $w_i(\gamma)$ is in the ideal generated by $w_1(\gamma)$. In particular, $w_1(\gamma_+) = 0$, and hence $w_1(\mathrm{Gr}^+(m, m+n)) = 0$.

It now follows from the computation of $w_2(\mathrm{Gr}(m, m+n))$ in the previous section that

$$w_2(\mathrm{Gr}^+(m, m+n)) = \begin{cases} 0 & m - n \equiv 0 \bmod 2 \\ w_2(\gamma_+) & m - n \equiv 1 \bmod 2. \end{cases}$$

As $H^*(\mathrm{Gr}(m, m+n);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1(\gamma), \dots, w_m(\gamma)]/(\overline{w}_{n+1}, \dots, \overline{w}_{m+n})$, if $m, n \geq 2$, then $w_2(\gamma)$ is not in the ideal generated by $w_1(\gamma)$ and hence $w_2(\gamma^+) \neq 0$. If $m = 1$ or $n = 1$, then the orientable grassmannian is a sphere and hence $w_2(\mathrm{Gr}^+(m, m+n)) = 0$.

Theorem. *The grassmannian* $\mathrm{Gr}^+(m, m+n)$ *is always orientable. Moreover, the obstructions to spin, pin*$^+$*, and pin*$^-$ *structures coincide and they vanish if and only if* $m - n$ *is even,* $m = 1$*, or* $n = 1$.

This agrees with Theorem 8 of [1].

4. Which Complex Grassmannians are spin?

Let $\mathrm{Gr}^{\mathbb{C}}(a,b)$ denote the grassmanian of complex a-dimensional subspaces of a complex b-dimensional vector space, and denote the tautological bundle over it by $\gamma_{\mathbb{C}}$. Similar to the previous cases, we have $T\,\mathrm{Gr}^{\mathbb{C}}(a,b) \cong \overline{\gamma_{\mathbb{C}}} \otimes \gamma_{\mathbb{C}}^\perp$ as complex vector bundles, where $\gamma_{\mathbb{C}}^\perp$ is the orthogonal complement of $\gamma_{\mathbb{C}} \subset \varepsilon_{\mathbb{C}}^b$ with respect to some fixed hermitian metric on $\varepsilon_{\mathbb{C}}^b$.

As $\mathrm{Gr}^{\mathbb{C}}(m, m+n)$ is a complex manifold, it is orientable, i.e. $w_1(\mathrm{Gr}^{\mathbb{C}}(m, m+n)) = 0$. Instead of using the formula for $w_2(E \otimes F)$, we have a shortcut in the complex case: we can use the Chern character to compute $c_1(\mathrm{Gr}^{\mathbb{C}}(m, m+n))$ and hence $w_2(\mathrm{Gr}^{\mathbb{C}}(m, m+n))$.

The Chern character is extremely useful as it satisfies $\mathrm{ch}(E\otimes F) = \mathrm{ch}(E)\,\mathrm{ch}(F)$. As $\mathrm{ch}(E) = \mathrm{rank}(E)+c_1(E) + \dots$ this immediately implies

$$c_1(E\otimes F) = \mathrm{rank}(F)c_1(E) + \mathrm{rank}(E)c_1(F).$$

In particular,

$$c_1(\mathrm{Gr}^{\mathbb{C}}(m, m+n)) = c_1(\overline{\gamma_{\mathbb{C}}}\otimes\gamma_{\mathbb{C}}^{\perp}) = nc_1(\overline{\gamma_{\mathbb{C}}}) + mc_1(\gamma_{\mathbb{C}}^{\perp}).$$

As $\gamma_{\mathbb{C}}\oplus\gamma_{\mathbb{C}}^{\perp}\cong\varepsilon_{\mathbb{C}}^{m+n}$, we see that $c_1(\gamma_{\mathbb{C}}^{\perp}) = -c_1(\gamma^{\mathbb{C}})$, while for the other term we use the fact that $c_i(\overline{E}) = (-1)^i c_i(E)$, so we conclude that

$$c_1(\mathrm{Gr}^{+}(m, m+n)) = nc_1(\overline{\gamma_{\mathbb{C}}}) + mc_1(\gamma_{\mathbb{C}}^{\perp}) = -nc_1(\gamma_{\mathbb{C}}) - mc_1(\gamma_{\mathbb{C}}) = -(m+n)c_1(\gamma_{\mathbb{C}}).$$

As $H^*(\mathrm{Gr}^{\mathbb{C}}(m, m+n);\mathbb{Z})\cong\mathbb{Z}[c_1(\gamma_{\mathbb{C}}),\dots,c_m(\gamma_{\mathbb{C}})]/(\overline{c}_{m+1},\dots,\overline{c}_{m+n})$ where $\overline{c}_i$ are defined in analogy with the previous cases, we see that $c_1(\gamma_{\mathbb{C}})$ is non-zero and is not divisible by 2. Therefore $w_2(\mathrm{Gr}^{\mathbb{C}}(m, m+n)) = (m+n)w_2(\gamma_{\mathbb{C}})$; as $c_1(\gamma_{\mathbb{C}})$ is not divisible by 2, we see that $w_2(\gamma_{\mathbb{C}})\neq 0$. Therefore, we arrive at the following result.

Theorem. *The grassmannian $\mathrm{Gr}^{\mathbb{C}}(m, m+n)$ is always orientable. Moreover, the obstructions to spin, pin$^+$, and pin$^-$ structures coincide and they vanish if and only if $m+n$ is even.*

5. Which grassmannians are spin^c?

Recall that a smooth manifold M is spin^c if and only if $w_1(M) = 0$ and $w_2(M)$ has an integral lift. More generally, a smooth manifold M is pin^c if and only if $w_2(M)$ has an integral lift, so an orientable smooth manifold is pin^c if and only if it is spin^c. The obstruction to lifting $w_2(M)$ to an integral class is the integral Stiefel-Whitney class $W_3(M) = \beta(w_2(M))\in H^3(M;\mathbb{Z})$; note, as $w_2(M)$ is 2-torsion, so is $W_3(M)$.

On an almost complex manifold M, the first Chern class $c_1(M)$ is an integral lift of $w_2(M)$, so M is spin^c; better still, almost complex manifolds have a canonical spin^c structure (see Example D.6 of [3]). Therefore, all complex grassmannians are spin^c (and hence pin^c).

Turning our attention to oriented grassmannians, first note that if $m = 1$ or $n = 1$, then $\mathrm{Gr}^+(m, m+n)$ is a sphere which is spin and hence spin^c. For $m, n > 1$, the oriented grassmannian $\mathrm{Gr}^+(m, m+n)$ is simply connected, so

$$W_3(\mathrm{Gr}^+(m, m+n))\in H^3(\mathrm{Gr}^+(m, m+n);\mathbb{Z})_{\mathrm{tors}}\cong H_2(\mathrm{Gr}^+(m, m+n);\mathbb{Z})_{\mathrm{tors}}\cong\pi_2(\mathrm{Gr}^+(m, m+n))_{\mathrm{tors}}.$$

To determine $\pi_2(\mathrm{Gr}^+(m, m+n))$, recall that $\mathrm{Gr}^+(m, m+n)$ is diffeomorphic to the homogeneous space $SO(m+n)/(SO(m)\times SO(n))$, so there is a fibre bundle $SO(m)\times SO(n)\to SO(m+n)\to\mathrm{Gr}^+(m, m+n)$. From the associated long exact sequence in homotopy, we deduce

$$\pi_2(\mathrm{Gr}^+(m, m+n))\cong\begin{cases}0 & m = 1, n\neq 2, \text{ or } m\neq 2, n = 1\\ \mathbb{Z} & m = 2 \text{ or } n = 2, \text{ but not both}\\ \mathbb{Z}\oplus\mathbb{Z} & m = n = 2\\ \mathbb{Z}_2 & m, n\geq 3\end{cases}$$

As $W_3(\mathrm{Gr}^+(m, m+n))$ is 2-torsion, it could only be non-zero when $m, n\geq 3$ in which case it would be the unique non-zero element of $H^3(\mathrm{Gr}^+(m, m+n);\mathbb{Z})\cong\mathbb{Z}_2$. It follows that $W_3(\mathrm{Gr}^+(m, m+n))$ is non-zero if and only if its reduction modulo 2 is. In general, we have $W_3(M) = \beta(w_2(M))\equiv\mathrm{Sq}^1(w_2(M))\bmod 2$. Recall, if $m-n$ is even, $w_2(\mathrm{Gr}^+(m, m+n)) = 0$, and if $m-n$ is odd, then $w_2(\mathrm{Gr}^+(m, m+n)) = w_2(\gamma^+)$. Now note that $\mathrm{Sq}^1(w_2(\gamma^+)) = w_1(\gamma^+)w_2(\gamma^+) + w_3(\gamma^+) = w_3(\gamma^+)$. As $H^*(\mathrm{Gr}(m, m+n);\mathbb{Z}_2)\cong\mathbb{Z}_2[w_1(\gamma),\dots,w_m(\gamma)]/(\overline{w}_{n+1},\dots,\overline{w}_{m+n})$, the class $w_3(\gamma)$ is not in the ideal generated by $w_1(\gamma)$ for $m, n\geq 3$. As we did in section 3, we deduce from the Gysin sequence that $w_3(\gamma^+) = \pi^*w_3(\gamma)$ is non-zero. Therefore $\mathrm{Gr}^+(m, m+n)$ is not spin^c (and hence not pin^c) when $m-n$ is odd and $m, n\geq 3$.

To determine which unoriented grassmannians are pin^c, note that if $\mathrm{Gr}(m, m+n)$ is pin^c, then so is $\mathrm{Gr}^+(m, m+n)$, which by the above implies that $m-n$ is even, $m \leq 2$, or $n \leq 2$. When $m-n$ is even, $\mathrm{Gr}(m, m+n)$ is orientable and $w_2(\mathrm{Gr}(m, m+n)) = 0$ or $w_1(\gamma)^2$. Note that $w_1(\gamma)^2 = \mathrm{Sq}^1(w_1(\gamma))$ which has $W_2(\gamma) = \beta(w_1(\gamma))$ as an integral lift, so $\mathrm{Gr}(m, m+n)$ is pin^c when $m-n$ is even. If $m = 1$ or $n = 1$, then $\mathrm{Gr}(m, m+n)$ is a projective space. As

$$H^3(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 3 \\ 0 & n \neq 3 \end{cases}$$

we see that $W_3(\mathbb{RP}^n) = 0$ and hence projective spaces are pin^c. Finally, suppose $m = 2$ and $n > 1$ is odd. Then $w_2(\mathrm{Gr}(2, 2+n))$ is either $w_2(\gamma)$ or $w_2(\gamma) + w_1(\gamma)^2$; in both cases, we see that $W_3(\mathrm{Gr}(2, 2+n)) = W_3(\gamma)$. Now note that $W_3(\gamma) = \beta(w_2(\gamma)) \equiv \mathrm{Sq}^1(w_2(\gamma)) \bmod 2$, and $\mathrm{Sq}^1(w_2(\gamma)) = w_1(\gamma)w_2(\gamma) + w_3(\gamma) = w_1(\gamma)w_2(\gamma)$ as γ has rank 2. Given that $H^*(\mathrm{Gr}(2, 2+n); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1(\gamma), w_2(\gamma)]/(\overline{w}_{n+1}, \overline{w}_{n+2})$ and $n+1 > 3$, we see that $w_1(\gamma)w_2(\gamma) \neq 0$ and hence $W_3(\mathrm{Gr}(2, 2+n)) = W_3(\gamma) \neq 0$. The case $n = 2$ and $m > 1$ odd is completely analogous.

In summary, we have the following:

Theorem.

- *The complex grassmannians* $\mathrm{Gr}^{\mathbb{C}}(m, m+n)$ *are all pin^c/spin^c.*
- *The oriented grassmannians* $\mathrm{Gr}^+(m, m+n)$ *are pin^c/spin^c if and only if $m-n$ is even, $m \leq 2$, or $n \leq 2$.*
- *The unoriented grassmannians* $\mathrm{Gr}(m, m+n)$ *are pin^c if and only if $m-n$ is even, $m = 1$, or $n = 1$. In particular, they are spin^c if and only if $m-n$ is even.*

In particular, for $k > 1$, the oriented grassmannians $\mathrm{Gr}^+(2, 2k+1)$ are spin^c but not spin. On the other hand, for $m, n \neq 2$, we see that $\mathrm{Gr}^+(m, m+n)$ is spin^c if and only if it is spin. More generally, a simply connected manifold M with $\pi_2(M)$ finite is spin^c if and only if it is spin, see page 50 of [2].

From the above, we discover an example of a manifold which is not pin^c but admits a finite cover which is. Namely, the manifold $\mathrm{Gr}(2, 2k+1)$ is not pin^c for $k \geq 2$, but its double cover $\mathrm{Gr}^+(2, 2k+1)$ is pin^c.

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