ENLARGEABLE MANIFOLDS

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Abstract. These notes grew out of a talk I gave on enlargeable manifolds in the Student Differential Geometry Seminar at Stony Brook in Spring 2019.

This talk is mostly taken from section 5, chapter IV of [1].

Consider all possible closed surfaces

$$\begin{array}{c|cccc} S^2 & T^2 & \Sigma_2 & \Sigma_3 & \dots \\ \\ \mathbb{RP}^2 & 2\mathbb{RP}^2 & 3\mathbb{RP}^2 & 4\mathbb{RP}^2 & \dots \end{array}$$

By the Gauss-Bonnet Theorem, the only orientable closed surface which can admit metrics of positive scalar curvature is S^2, and it does, e.g. the round metric. As $s_{\pi^*g} = s_g \circ \pi$, the only non-orientable closed surface which can admit metrics of positive scalar curvature is $\mathbb{RP}^2$, and it does, e.g. the round metric descends.

We can ask if something similar happens in higher dimensions. First of all, S^n and $\mathbb{RP}^n$ do admit metrics of positive scalar curvature (in fact constant curvature). What about the non-existence results? That depends on how you generalise to higher dimensions. In the case of the torus, we have an obvious way to generalise to higher dimensions. Along these lines, Geroch asked the following question:

Question. (Geroch) *For $n \geq 3$, can the torus T^n admits a metric of positive scalar curvature?*

Intuitively, manifolds with positive curvature have 'small' fundamental group. In particular, if M has a metric with positive Ricci curvature, Myers' Theorem tells us that $\pi_1(M)$ is finite. Better still, if M is orientable and is even-dimensional, then $\pi_1(M) = 0$. There is no such statement for positive scalar curvature. In particular, $M = N \times S^2$ has positive scalar curvature for any N.

Definition 1. A compact Riemannian n-dimensional manifold is called *enlargeable* if for every $\varepsilon > 0$ there exists an orientable Riemannian covering space which admits an ε-contracting map onto S^n (with its constant sectional curvature 1 metric) which is constant at infinity and is of non-zero degree.

If one can always take a finite covering in the definition of enlargeable, then we say is it *compactly enlargeable.*

Remarks.

- A C^1 map $f : M \to N$ between Riemannian manifolds is *ε-contracting* if $\|df(v)\|_N \leq \varepsilon\|v\|_M$ for all $v \in TM$; i.e. $\sup_{m\in M} \|(df)_m\| \leq \varepsilon$. Note, if f is ε-contracting, then it is ε'-contracting for every $\varepsilon' > \varepsilon$, so we just need to show we can find such maps for ε arbitrarily small. If M is compact, any map is c-contracting for some c.
- A map $f : M \to N$ is said to be *constant at infinity* if it is constant outside a compact set; in particular, if M is compact, every map is constant at infinity. So for compactly enlargeable manifolds, we can drop this condition altogether.

- If $f : M \to N$ is constant at infinity and N is compact, we have $f^* : H^n(N;\mathbb{Z}) \to H^n_c(M;\mathbb{Z})$. Provided M and N are orientable, we have a group homomorphism $\mathbb{Z} \to \mathbb{Z}$ which is necessarily given by multiplication by k for some $k \in \mathbb{Z}$; this is the degree of f. Alternatively, if ω is a volume form on N, the degree of f satisfies $\int_M f^*\omega = k\int_N \omega$; note that the first integral is finite because f is constant at infinity. Yet another definition is a signed count of the preimages of a regular value.

Example 1. Euclidean space $\mathbb{R}^n$, with its usual metric, is enlargeable. As $\mathbb{R}^n$ is simply connected, it has no covering spaces other than itself, so we need to show that for every $\varepsilon > 0$, there are ε-contracting maps $f_\varepsilon : \mathbb{R}^n \to S^n$ which are constant at infinity and have non-zero degree. Let $f : \mathbb{R}^n \to S^n$ be the map which sends rays emanating from the origin to distance π to lines of longitude on the sphere, emanating from the north pole, then send every point of distance at least π to the south pole. This map is c-contracting for some c, is constant at infinity and has degree one. By precomposing with $\mathbb{R}^n \to \mathbb{R}^n$ given by rescaling, we get the family of maps we desire. Note, as ε gets smaller, the balls where the map is non-constant get larger.

Example 2. The same idea shows that the standard flat torus is compactly enlargeable. As the required balls get larger (i.e. as ε gets smaller), need to pass to a higher covering space, but only need it to be finite.

In some sense, enlargeability is about being able to pass to higher and higher covering spaces to 'enlarge' these balls. Note, a manifold has finite fundamental group, which is the case if it has positive Ricci curvature for example, then we cannot do this.

The collection of enlargeable manifolds is quite extensive.

Theorem. *For compact manifolds, we have the following.*

1. *Enlargeability is independent of the Riemannian metric.*
2. *Enlargeability only depends on the homotopy type of the manifold.*
3. *The product of enlargeable manifolds is enlargeable.*
4. *The connected sum of any manifold with an enlargeable manifold is again enlargeable.*
5. *Any manifold which admits a map of non-zero degree onto an enlargeable manifold is itself enlargeable.*
6. *Any manifold which admits a metric of non-positive sectional curvature is enlargeable.*

Proof. (Sketch) Note that $5 \Rightarrow 2 \Rightarrow 1$ and $5 \Rightarrow 4$, so we only need to establish statements 3, 5, and 6.

3. Suppose M^m and N^n are enlargeable. Fix a degree one map $\phi : S^m \times S^n \to S^{m+n}$ which is constant on $(S^m \times \{q\}) \cup (\{p\} \times S^n)$. By compactness, $c = \sup \|d\phi\| < \infty$. Given $\frac{\varepsilon}{c}$-contracting maps $f : M' \to S^m$ and $g : N' \to S^n$, then $\phi \circ (f \times g) : M' \times N' \to S^{m+n}$ is an ε-contracting map; moreover $\deg(\phi \circ (f \times g)) = \deg(f)\deg(g)$. By post-composing with an isometries if necessary, we can assume f is constant $= p$ at infinity, and g is constant $= q$ at infinity, and hence $\phi \circ (f \times g)$ will be constant at infinity.

Even though M and N are compact, M' and N' might be non-compact, hence the need to illustrate the map $\phi \circ (f \times g)$ is constant at infinity. If the manifolds were compactly enlargeable, we can drop the constant at infinity condition.

5. Let $F : X \to Y$ be of non-zero degree. As X is compact, F is c-contracting for some c. There is a finite orientable covering $p_Y : Y' \to Y$ such that there is an $\frac{\varepsilon}{c}$-contracting map $f : Y' \to S^n$ which is constant outside a compact set $K \subset Y'$. Pulling back the covering p_Y by F, we obtain the following commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{F'} & Y' \\ \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow{F} & Y \end{array}$$

As covering maps are local isometries, F' is also a c-contracting map, so $f \circ F' : X' \to S^n$ is ε-contracting. The map F' is proper, and so $f \circ F'$ is constant outside the compact set $(F')^{-1}(K)$. Finally, $\deg(f \circ F') = \deg(f)\deg(F) \neq 0$.

6. Let X be a compact manifold equipped with a metric of non-positive sectional curvature. By the Cartan-Hadamard theorem, for any $x \in \widetilde{X}$, the universal cover of X, the exponential map at x is a diffeomorphism and its inverse $\exp_x^{-1} : \widetilde{X} \to T_x\widetilde{X}$ is distance-decreasing (i.e. 1-contracting) where $\widetilde{X}$ is equipped with the pullback metric and $T_x\widetilde{X} \cong \mathbb{R}^n$ is equipped with the Euclidean metric. Now just compose with the maps described in Example 1. □

The relationship between enlargeable manifolds and positive scalar curvature is given by the following theorem.

Theorem. (Gromov–Lawson) *An enlargeable spin manifold cannot carry a metric of positive scalar curvature. Moreover, any metric with non-negative scalar curvature must be flat.*

The spin condition can be weakened. We only need the covering space which admits the desired ε-contracting map to be a spin manifold.

This theorem immediately gives some interesting conclusions:

- As tori are enlargeable and spin, we obtain a negative answer to Geroch's question.
- Compact manifolds with non-positive sectional curvature don't admit positive scalar curvature metrics; note that $\mathbb{R}^n$, the domain of the ε-contracting maps, is spin. This can be seen as a generalisation of the situation for closed surfaces.
- (2) above shows that enlargeability only depends on the homotopy type. By Wu's theorem (see Theorem 11.14 of [2]), so does the spin condition. So, for example, any smooth manifold homotopy equivalent to the standard torus (in particular, exotic tori) is enlargeable and spin, and therefore does not admit a positive scalar curvature metric. In fact, such manifolds do not admit non-negative scalar curvature metrics as such a metric is flat which is impossible: a closed flat manifold is uniquely determined up to diffeomorphism by its fundamental group (Bieberbach Theorem).

  Note that, in general, the question of whether a manifold admits positive scalar curvature metrics does not depend only on the homotopy type, or even the homeomorphism type. For example, $Bl_p(K3)$, the blowup of a $K3$ surface at a point p, and $3\mathbb{CP}^2\#20\overline{\mathbb{CP}^2}$ are homeomorphic but not diffeomorphic – the latter admits positive scalar curvature metrics, while the former doesn't.

Proof. (Sketch) Only deal with the compactly enlargeable case. The general case requires the Relative Index Theorem.

Suppose X is a compactly enlargeable manifold with a positive scalar curvature metric g, and let $s_0 = \inf_{x \in X} s_g(x)$; by compactness, $s_0 > 0$. By replacing X with $X \times S^1$ if necessary, we can assume $\dim X = 2n$.

Let $E_0 \to S^{2n}$ be a rank n complex vector bundle with $c_n(E_0) \neq 0$, then

$$\mathrm{ch}(E_0) = n + \frac{1}{(n-1)!}c_n(E_0) \in H^0(S^{2n};\mathbb{Q}) \oplus H^{2n}(S^{2n};\mathbb{Q}).$$

In fact, we can replace $\mathbb{Q}$ by $\mathbb{Z}$.

Fix $\varepsilon > 0$, and let $p : X' \to X$ be a finite spin covering which admits am ε-contracting map $f : X' \to S^{2n}$ of non-zero degree, and set $E = f^*E_0$. As X' is spin of even dimension, we can form the complex spinor bundle $\mathbb{S}_{\mathbb{C}} \cong \mathbb{S}_{\mathbb{C}}^+ \oplus \mathbb{S}_{\mathbb{C}}^-$. Recall that we have the Dirac operator $\partial\!\!\!/ : \Gamma(\mathbb{S}_{\mathbb{C}}) \to \Gamma(\mathbb{S}_{\mathbb{C}})$, but we also have the twisted Dirac operator $\partial\!\!\!/_E : \Gamma(\mathbb{S}_{\mathbb{C}} \otimes E) \to \Gamma(\mathbb{S}_{\mathbb{C}} \otimes E)$. The Bochner formula for $\partial\!\!\!/_E$ is

$$\partial\!\!\!/_E^2 = \nabla^*\nabla + \frac{s}{4} + \mathfrak{R}_E$$

where $\mathfrak{R}_E$ depends only on R_E, the curvature of E. It follows that $\|\mathfrak{R}_E\| \leq k_n\|R_E\|$ for some constant k_n. Using the fact that f is ε-contracting, one can show that $\|R_E\| \leq \varepsilon^2\|R_{E_0}\|$. So, we have

$$\begin{aligned}\|\partial\!\!\!/_E(\sigma)\|^2 &= \langle \partial\!\!\!/_E^2(\sigma), \sigma\rangle \\ &= \langle \nabla^*\nabla\sigma, \sigma\rangle + \langle \frac{s}{4}\sigma, \sigma\rangle + \langle \mathfrak{R}_E(\sigma), \sigma\rangle \\ &\geq \|\nabla\sigma\|^2 + \frac{s_0}{4}\|\sigma\|^2 - \|\mathfrak{R}_E\|\|\sigma\|^2 \\ &\geq \|\nabla\sigma\|^2 + \frac{s_0}{4}\|\sigma\|^2 - k_n\varepsilon^2\|\sigma\|^2.\end{aligned}$$

So if $\frac{s_0}{4} - k_n\varepsilon^2 > 0$, that is $\varepsilon < \sqrt{s_0/4k_n}$, then for $\sigma \neq 0$ we see that $\partial\!\!\!/_E(\sigma) \neq 0$. It follows that $\partial\!\!\!/_E^+ : \Gamma(\mathbb{S}_{\mathbb{C}}^+ \otimes E) \to \Gamma(\mathbb{S}_{\mathbb{C}}^- \otimes E)$ and $\partial\!\!\!/_E^- : \Gamma(\mathbb{S}_{\mathbb{C}}^- \otimes E)$ have trivial kernel, so

$$\begin{aligned}\mathrm{ind}\,\partial\!\!\!/_E^+ &= \dim\ker\partial\!\!\!/_E^+ - \dim\mathrm{coker}\,\partial\!\!\!/_E^+ \\ &= 0 - \dim\ker(\partial\!\!\!/_E^+)^* \\ &= -\dim\ker\partial\!\!\!/_E^- \\ &= 0.\end{aligned}$$

On the other hand, we can compute the index via the Atiyah-Singer Index Theorem:

$$\begin{aligned}\mathrm{ind}\,\partial\!\!\!/_E^+ &= \int_X ch(E)\hat{A}(TX') \\ &= \int_{X'} \left(n + \frac{1}{(n-1)!}c_n(E)\right)(1 + \hat{A}_{2n}(TX')) \\ &= \int_{X'} n\hat{A}_{2n}(TX') + \frac{1}{(n-1)!}c_n(E) \\ &= \frac{1}{(n-1)!}\int_{X'} c_n(E) \\ &= \frac{1}{(n-1)!}\int_{X'} c_n(f^*E_0) \\ &= \frac{1}{(n-1)!}\int_{X'} f^*c_n(E_0) \\ &= \frac{\deg f}{(n-1)!}\int_{S^{2n}} c_n(E_0)\end{aligned}$$

which is non-zero because $c_n(E_0) \neq 0$. The fourth equality holds because either n is odd, in which case $\hat{A}_{2n}(TX') = 0$, or n is even, in which case $\hat{A}(X') = \int_{X'} \hat{A}_{2n}(TX')$ is zero by Lichnerowicz's Theorem (X' is a compact spin manifold which admits a positive scalar curvature metric).

This is a contradiction, so X cannot admit a metric of positive scalar curvature. □

Another corollary of the theorem of Gromov and Lawson is that $M\#T^n$ does not admit positive scalar curvature metrics for any closed spin manifold M. This is particularly interesting in light of the following:

Theorem. *Suppose for all closed n-dimensional manifolds M, the manifolds $M\#T^n$ have no metric with positive scalar curvature. Then the ADM mass is nonnegative for all asymptotically flat manifolds with non-negative scalar curvature*

The case for M not necessarily spin was settled by Schoen and Yau in their recent paper [3] which generalises their hypersurface technique, settling the Positive Mass Conjecture. Prior to that, their technique was limited to dimension $n \leq 7$ due to the singularities that begin to occur in minimal hypersurfaces of higher dimensional manifolds.

As mentioned before, one could see the statement that a closed manifold of non-positive sectional curvature cannot admit positive scalar curvature metrics as the correct generalisation of the fact that closed surfaces other than S^2 and $\mathbb{RP}^2$ do not admit positive scalar curvature metrics. But those surfaces can be characterised in another way.

A topological space X is called *aspherical* if $\pi_n(X) = 0$ for all $n \geq 2$; if X has a universal cover $\widetilde{X}$, which is the case if X is a manifold or even a CW complex, then it is equivalent to say that $\widetilde{X}$ is contractible. By the Cartan-Hadamard Theorem, a closed manifold which admits a metric of non-positive sectional curvature is aspherical, but not every closed aspherical manifold admits a metric of non-positive sectional curvature, for example the Heisenberg manifold $H(3,\mathbb{R})/H(3,\mathbb{Z})$; see Theorem 2.1 of [4]. So one could instead ask the following question:

Question. *Do all aspherical manifolds fail to admit positive scalar curvature metrics?*

The answer is yes in dimensions two and three, but the question is open in all other dimensions. By the discussion above, if there is a counterexample, it fails to admit a metric of non-positive sectional curvature. More generally, one could ask if a compact orientable manifold which admits a map to aspherical manifold with non-zero degree could admit a positive scalar curvature metric. Schoen and Yau's recent result answers this question when the aspherical manifold is a torus.

Remarks.

- The notion of enlargeability has been generalised to $\hat{A}$-enlargeability and weak enlargeability. The latter can be used to rule out the existence of complete metrics of positive scalar curvature on non-compact manifolds.
- In 2018, Cecchini & Schick posted a preprint which claims to remove the spin hypothesis.

References

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