THE HIRZEBRUCH χ_y GENUS AND A THEOREM OF HIRZEBRUCH ON ALMOST COMPLEX MANIFOLDS

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ABSTRACT. The purpose of this note is to give an introduction to the Hirzebruch χ_y genus and to give a proof of a theorem of Hirzebruch which states that on a closed almost complex manifold M of dimension $4m$ we have $\chi(M) \equiv (-1)^m\sigma(M) \bmod 4$.

Let (M, g) be an $2n$-dimensional closed Riemannian manifold. Given a spin^c structure, one can form the complex spin^c bundles $\mathbb{S}^+_{\mathbb{C}}$ and $\mathbb{S}^-_{\mathbb{C}}$. Then there is a spin^c Dirac operator $\partial\!\!\!/^c : \Gamma(\mathbb{S}^+_{\mathbb{C}}) \to \Gamma(\mathbb{S}^-_{\mathbb{C}})$ which has index

$$\operatorname{ind}(\partial\!\!\!/^c) = \int_M \exp(c_1(L)/2)\hat{A}(TM)$$

where L is the complex line bundle associated to the spin^c structure; see Theorem D.15 of [5].

If $E \to M$ is a hermitian vector bundle, then there is a twisted spin^c Dirac operator $\partial\!\!\!/^c_E : \Gamma(\mathbb{S}^+_{\mathbb{C}} \otimes E) \to \Gamma(\mathbb{S}^-_{\mathbb{C}} \otimes E)$ which has index

$$\operatorname{ind}(\partial\!\!\!/^c_E) = \int_M \exp(c_1(L)/2)\operatorname{ch}(E)\hat{A}(TM).$$

I don't know a reference for this precise statement (if you do, please let me know), but the fact that this quantity is an integer is Theorem 26.1.1 of [3].

Suppose now that M admits an almost complex structure and g is hermitian. Then there is a canonical spin^c structure which has associated line bundle $L = \det_{\mathbb{C}}(TM)$, so $c_1(L) = c_1(M)$; see Example D.6 of [5]. Using the fact that $\exp(c_1(M)/2)\hat{A}(TM) = \operatorname{Td}(TM)$, the index becomes

$$\operatorname{ind}(\partial\!\!\!/^c_E) = \int_M \operatorname{ch}(E)\operatorname{Td}(TM). \tag{1}$$

In addition, the complex spin^c bundles take the form $\mathbb{S}^+_{\mathbb{C}} \cong \bigwedge^{0,\text{even}} M$ and $\mathbb{S}^-_{\mathbb{C}} = \bigwedge^{0,\text{odd}} M$; see corollary 3.4.6 of [6]. If $E = \bigwedge^{p,0} M$, then we have a twisted spin^c Dirac operator $\partial\!\!\!/^c_{\bigwedge^{p,0} M} : \Gamma(\bigwedge^{p,\text{even}} M) \to \Gamma(\bigwedge^{p,\text{odd}} M)$; for notational convenience, we will instead write $\partial\!\!\!/^c_p$ for this operator. We define $\chi^p(M) := \operatorname{ind}(\partial\!\!\!/^c_p)$; if $p = 0$, this is just the Todd genus. The Hirzebruch χ_y genus is defined to be

$$\chi_y(M) := \sum_{p=0}^{n} \chi^p(M)y^p.$$

INTEGRABLE CASE

Suppose now that J is integrable, in which case $n = \dim_{\mathbb{C}} M$. Then, modulo order zero terms, we have $\partial\!\!\!/^c = \sqrt{2}(\bar\partial + \bar\partial^*)$; see Proposition 8 of [1]. In addition, if E is holomorphic, then modulo order zero terms $\partial\!\!\!/^c_E = \sqrt{2}(\bar\partial_E + \bar\partial^*_E)$ and (1) becomes the statement of the Hirzebruch-Riemann-Roch theorem.

In particular, $\bigwedge^{p,0} M$ is holomorphic and $\partial\!\!\!/_p^c : \Gamma(\bigwedge^{p,\text{even}} M) \to \Gamma(\bigwedge^{p,\text{odd}} M)$ is just $\sqrt{2}(\bar\partial + \bar\partial^*)$ to highest order. If $\mathcal{H}^{p,q}_{\bar\partial}(M)$ denotes the $\bar\partial$-harmonic (p,q)-forms on M, then

$$\begin{aligned}
\operatorname{ind}(\partial\!\!\!/_p^c) &= \operatorname{ind}\sqrt{2}(\bar\partial + \bar\partial^*)\\
&= \dim\left(\bigoplus_{q\text{ even}} \mathcal{H}^{p,q}_{\bar\partial}(M)\right) - \dim\left(\bigoplus_{q\text{ odd}} \mathcal{H}^{p,q}_{\bar\partial}(M)\right)\\
&= \sum_{q\text{ even}} \dim \mathcal{H}^{p,q}_{\bar\partial}(M) - \sum_{q\text{ odd}} \dim \mathcal{H}^{p,q}_{\bar\partial}(M)\\
&= \sum_{q\text{ even}} h^{p,q}(M) - \sum_{q\text{ odd}} h^{p,q}(M)\\
&= \sum_{q=0}^{n} (-1)^q h^{p,q}(M)\\
&= \chi(M, \Omega^p).
\end{aligned}$$

Using the penultimate expression above, we have

$$\chi_y(M) = \sum_{p=0}^{n}\sum_{q=0}^{n} (-1)^q h^{p,q}(M) y^p.$$

We now list some properties of $\chi_y(M)$ in the integrable case.

Property 1. $\chi_y(M) = (-y)^n \chi_{y^{-1}}(M)$.

Proof. As

$$(-y)^n \chi_{y^{-1}}(M) = (-y)^n \sum_{p=0}^{n} \chi^p(M) y^{-p} = \sum_{p=0}^{n} (-1)^n \chi^p(M) y^{n-p},$$

this property is equivalent to $\chi^p(M) = (-1)^n \chi^{n-p}(M)$.

By Serre duality we have $h^{p,q}(M) = h^{n-p,n-q}(M)$, so

$$\begin{aligned}
\chi^p(M) &= \sum_{q=0}^{n} (-1)^q h^{p,q}(M)\\
&= \sum_{q=0}^{n} (-1)^q h^{n-p,n-q}(M)\\
&= (-1)^n \sum_{q=0}^{n} (-1)^{n-q} h^{n-p,n-q}(M)\\
&= (-1)^n \chi^{n-p}(M).
\end{aligned}$$

□

Property 2. *If M admits a Kähler metric, then $\chi_{-1}(M) = \chi(M)$.*

Proof. Note that in the Kähler case

$$\chi_{-1}(M) = \sum_{p=0}^{n}\sum_{q=0}^{n} (-1)^{p+q} h^{p,q}(M) = \sum_{k=0}^{2n} (-1)^k \sum_{p+q=k} h^{p,q}(M) = \sum_{k=0}^{2n} (-1)^k b_k(M) = \chi(M).$$

□

Property 3. *Suppose that n is even and M admits a Kähler metric. Then $\chi_1(M) = \sigma(M)$.*

Proof. Using the fact that $h^{p,q}(M) = h^{q,p}(M)$, we have

$$\chi_1(M) = \sum_{p=0}^{n}\sum_{q=0}^{n}(-1)^q h^{p,q}(M) = \sum_{p=0}^{n}\sum_{q=0}^{n}(-1)^q h^{q,p}(M) = \sum_{p=0}^{n}\sum_{q=0}^{n}(-1)^p h^{p,q}(M).$$

It follows from the Hard Lefschetz Theorem that the final expression is equal to $\sigma(M)$; see Corollary 3.3.18 of [4]. □

As we will see in the next section, all three of these properties hold in general.

Non-Integrable Case

Suppose now that J is not integrable.

In order to establish the properties mentioned in the previous section, we need the following expression for $\chi_y(M)$.

Theorem. *Let x_i be the Chern roots of TM. Then*

$$\chi_y(M) = \int_M \prod_{i=1}^{n} \frac{x_i(1 + ye^{-x_i})}{1 - e^{-x_i}}.$$

Proof. By the splitting principle, we can suppose that $TM = \ell_1 \oplus \cdots \oplus \ell_n$, and hence $T^*M = \ell_1^* \oplus \cdots \oplus \ell_n^*$, without any loss of generality. Defining $x_i = c_1(\ell_i)$, we have $-x_i = c_1(\ell_i^*)$. Note that

$$\bigwedge^{p,0} M = \bigwedge^{p} T^*M = \bigwedge^{p}(\ell_1^* \oplus \cdots \oplus \ell_n^*) = s_p(\ell_1^*, \ldots, \ell_n^*)$$

where s_p is the p^{th} elementary symmetric polynomial (addition and multiplication correspond to direct sum and tensor product respectively). Therefore

$$\operatorname{ch}\left(\bigwedge^{p,0} M\right) = \operatorname{ch}(s_p(\ell_1^*, \ldots, \ell_n^*)) = s_p(\operatorname{ch}(\ell_1^*), \ldots, \operatorname{ch}(\ell_n^*)) = s_p(e^{-x_1}, \ldots, e^{-x_n}).$$

So we have

$$\begin{aligned}
\chi_y(M) &= \sum_{p=0}^{n} \chi^p(M) y^p \\
&= \sum_{p=0}^{n} \operatorname{ind}(\partial\!\!\!/_p^c) y^p \\
&= \sum_{p=0}^{n} \left(\int_M \operatorname{ch}\left(\bigwedge^{p,0} M\right) \operatorname{Td}(M)\right) y^p \\
&= \int_M \left(\sum_{p=0}^{n} \operatorname{ch}\left(\bigwedge^{p,0} M\right) y^p\right) \operatorname{Td}(M) \\
&= \int_M \left(\sum_{p=0}^{n} s_p(e^{-x_1}, \ldots, e^{-x_n}) y^p\right) \prod_{i=1}^{n} \frac{x_i}{1 - e^{-x_i}} \\
&= \int_M \prod_{i=1}^{n}(1 + e^{-x_i} y) \prod_{i=1}^{n} \frac{x_i}{1 - e^{-x_i}} \\
&= \int_M \prod_{i=1}^{n} \frac{x_i(1 + e^{-x_i} y)}{1 - e^{-x_i}}.
\end{aligned}$$

□

Setting $y = -1$, we now see that property 2 holds in the non-integrable case:

$$\chi_{-1}(M) = \int_M \prod_{i=1}^n \frac{x_i(1-e^{-x_i})}{1-e^{-x_i}} = \int_M \prod_{i=1}^n x_i = \int_M c_n(M) = \int_M e(M) = \chi(M).$$

When J is integrable, property 2 gives us the following result (which also follows from the existence of the Frölicher spectral sequence).

Corollary. *Let M be an n-dimensional compact complex manifold. The Euler characteristic of M is given by*

$$\chi(M) = \sum_{p=0}^n \sum_{q=0}^n (-1)^{p+q} h^{p,q}(M).$$

For the other two properties, we need the following lemma. Thanks to Professor Ping Li for pointing this out to me.

Lemma. *Let t be a parameter. Then*

$$\chi_y(M) = \int_M \prod_{i=1}^n \frac{x_i(1+ye^{-tx_i})}{1-e^{-tx_i}}.$$

Proof. The key is to note that $\chi_y(M)$ only depends on the degree $2n$ part of the integrand. As $\deg x_i = 2$, if we replace x_i by tx_i, then we have

$$\int_M \prod_{i=1}^n \frac{tx_i(1+ye^{-tx_i})}{1-e^{-tx_i}} = t^n \int_M \prod_{i=1}^n \frac{x_i(1+ye^{-x_i})}{1-e^{-x_i}} = t^n \chi_y(M).$$

Dividing through by t^n, we arrive at the result. □

With this lemma in hand, we have

$$\begin{aligned}
\chi_y(M) &= \int_M \prod_{i=1}^n \frac{x_i(1+ye^{-x_i})}{1-e^{-x_i}} \\
&= \int_M \prod_{i=1}^n \frac{x_i(1+ye^{x_i})}{1-e^{x_i}} \\
&= \int_M \prod_{i=1}^n (-y) \frac{x_i(y^{-1}+e^{x_i})}{e^{x_i}-1} \\
&= (-y)^n \int_M \prod_{i=1}^n \frac{x_i(e^{x_i}+y^{-1})}{e^{x_i}-1} \\
&= (-y)^n \int_M \prod_{i=1}^n \frac{x_i(1+y^{-1}e^{-x_i})}{1-e^{-x_i}} \\
&= (-y)^n \chi_{y^{-1}}(M)
\end{aligned}$$

where the second equality uses the lemma with $t = -1$. So we see that property 1 holds.

For property 3, suppose $n = 2m$. Setting $y = 1$ gives

$$\chi_1(M) = \int_M \prod_{i=1}^n \frac{x_i(1+e^{-x_i})}{1-e^{-x_i}} = \int_M \prod_{i=1}^n \frac{x_i(1+e^{2x_i})}{1-e^{2x_i}} = \int_M \prod_{i=1}^n \frac{x_i}{\tanh(x_i)}$$

where the lemma was used with $t = -2$ in the second equality. Recall that the power series which generates the L genus (in terms of Pontryagin classes) is

$$Q(x) = \frac{\sqrt{x}}{\tanh(\sqrt{x})}.$$

By lemma 1.3.1 of [3], the corresponding power series which generates the L genus (in terms of Chern classes) is

$$\widetilde{Q}(x) = \frac{x}{\tanh(x)}.$$

So the above computation shows that

$$\chi_1(M) = \int_M \prod_{i=1}^{n} \frac{x_i}{\tanh(x_i)} = \int_M L_m(p_1, \ldots, p_m) = \sigma(M).$$

In the integrable case, we immediately obtain the following corollary.

Corollary. *Let M be an n-dimensional compact complex manifold with n even. The signature of M is given by*

$$\sigma(M) = \sum_{p=0}^{n} \sum_{q=0}^{n} (-1)^q h^{p,q}(M).$$

Note, in the proof of property 3 in the Kähler case, the exponent of -1 is p, not q. Because $h^{p,q}(M) = h^{q,p}(M)$, it doesn't make any difference. However, in the non-Kähler case, having exponent p does not compute the signature as can easily be checked for a Hopf surface.

If M has odd dimension, it follows from Serre duality that $\chi_1(M) = 0$.

A Theorem of Hirzebruch

The three properties of $\chi_y(M)$ give rise to the following result of Hirzebruch.

Theorem. *Suppose M is a closed $4m$-dimensional manifold which admits an almost complex structure, then $\chi(M) \equiv (-1)^m \sigma(M) \bmod 4$.*

Proof. We separate the proof into two cases based on the parity of m.

If $m = 2k$ is even, then $n = 2m = 4k$ so

$$\begin{aligned}
\chi(M) &= \chi_{-1}(M) \\
&= \sum_{p=0}^{4k} (-1)^p \chi^p(M) \\
&= \sum_{p=0}^{4k} \chi^p(M) - 2 \sum_{p=0}^{2k-1} \chi^{2p+1}(M) \\
&= \chi_1(M) - 2 \left[\sum_{p=0}^{k-1} \chi^{2p+1}(M) + \sum_{p=k}^{2k-1} \chi^{2p+1}(M) \right] \\
&= \sigma(M) - 2 \left[\sum_{p=0}^{k-1} \chi^{2p+1}(M) + \sum_{p=k}^{2k-1} (-1)^{4k} \chi^{4k-(2p+1)}(M) \right] \\
&= \sigma(M) - 2 \left[\sum_{p=0}^{k-1} \chi^{2p+1}(M) + \sum_{p=k}^{2k-1} \chi^{2(2k-1-p)+1}(M) \right]
\end{aligned}$$

$$= \sigma(M) - 2\left[\sum_{p=0}^{k-1}\chi^{2p+1}(M) + \sum_{p=0}^{k-1}\chi^{2p+1}(M)\right]$$
$$= \sigma(M) - 4\sum_{p=0}^{k-1}\chi^{2p+1}(M).$$

Therefore $\chi(M) \equiv \sigma(M) \bmod 4$.

If $m = 2k+1$ is odd, then $n = 2m = 4k+2$ so

$$\begin{aligned}
\chi(M) &= \chi_{-1}(M) \\
&= \sum_{p=0}^{4k+2}(-1)^p\chi^p(M) \\
&= -\sum_{p=0}^{4k+2}\chi^p(M) + 2\sum_{p=0}^{2k+1}\chi^{2p}(M) \\
&= -\chi_1(M) + 2\left[\sum_{p=0}^{k}\chi^{2p}(M) + \sum_{p=k+1}^{2k+1}\chi^{2p}(M)\right] \\
&= -\sigma(M) + 2\left[\sum_{p=0}^{k}\chi^{2p}(M) + \sum_{p=k+1}^{2k+1}(-1)^{4k+2}\chi^{4k+2-2p}(M)\right] \\
&= -\sigma(M) + 2\left[\sum_{p=0}^{k}\chi^{2p}(M) + \sum_{p=k+1}^{2k+1}\chi^{2(2k+1-p)}(M)\right] \\
&= -\sigma(M) + 2\left[\sum_{p=0}^{k}\chi^{2p}(M) + \sum_{p=0}^{k}\chi^{2p}(M)\right] \\
&= -\sigma(M) + 4\sum_{p=0}^{k}\chi^{2p}(M).
\end{aligned}$$

Therefore $\chi(M) \equiv -\sigma(M) \bmod 4$. □

We end with an application of this theorem.

Let $M_k = k\mathbb{CP}^{2m}$ denote the connected sum of $k \geq 0$ copies of $\mathbb{CP}^{2m}$. Note that $\chi(M_k) = (2m-1)k + 2 = 2mk - k + 2$ and $\sigma(M_k) = k$. For k even, we have $\chi(M_k) \equiv -k + 2 \bmod 4$ while $(-1)^m\sigma(M_k) \equiv (-1)^m k \bmod 4$. Therefore, the manifolds M_k with k even do not admit almost complex structures. On the other hand, for k odd, we have $\chi(M_k) \equiv 2m - k + 2 \bmod 4$ and $(-1)^m\sigma(M_k) \equiv (-1)^m k \bmod 4$. By splitting into cases (either m even and m odd, or $k \equiv 1 \bmod 4$ and $k \equiv 3 \bmod 4$), one can verify that $\chi(M_k) \equiv (-1)^m\sigma(M_k) \bmod 4$ for k odd; that is, we can't use the above theorem to rule out the existence of almost complex structures on these manifolds. In fact, these manifolds do admit almost complex structures, see [2].

On the other hand, the manifolds $N_k = k\mathbb{CP}^{2m+1}$ admit almost complex structures (even integrable ones) for all $k > 0$. To see this, note that the map $\mathbb{CP}^{2m+1} \to \mathbb{CP}^{2m+1}$ given by $[z_0, \dots, z_{2m+1}] \mapsto [\overline{z_0}, \dots, \overline{z_{2m+1}}]$ is an orientation-reserving diffeomorphism, and hence $\mathbb{CP}^{2m+1}$ and $\overline{\mathbb{CP}^{2m+1}}$ are diffeomorphic as oriented manifolds. Therefore, we see that N_k is diffeomorphic, as an oriented manifold, to the blowup of $\mathbb{CP}^{2m+1}$ at $k-1$ points.

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