# An Introduction to Operator Algebras

Laurent W. Marcoux

March 30, 2005

# Preface

These notes were designed as lecture notes for a first course in Operator Algebras. The student is assumed to have already taken a first course in Linear Analysis. In particular, they are assumed to already know the Hahn-Banach Theorem, the Open Mapping Theorem, etc. A list of those results which will be used in the sequel is included in the second section of the first chapter.

March 30, 2005

# Contents

CHAPTER 1

# A Brief Review of Banach Space Theory

## 1. Definitions and examples

**1.1. Definition.** *A* ***complex normed linear space*** *is a pair* $(\mathfrak{X}, \|\cdot\|)$ *where* $\mathfrak{X}$ *is a vector space over* $\mathbb{C}$ *and* $\|\cdot\|$ *is a norm on* $\mathfrak{X}$*. That is:*

- $\|x\| \geq 0$ *for all* $x \in \mathfrak{X}$*;*
- $\|x\| = 0$ *if and only if* $x = 0$*;*
- $\|\lambda x\| = |\lambda| \|x\|$ *for all* $\lambda \in \mathbb{C}$, $x \in \mathfrak{X}$
- $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in \mathfrak{X}$*.*

*A* ***Banach space*** *is a complete normed linear space.*

**1.2. Example.** Consider $(\mathbb{C}, \|\cdot\|_p)$, $n \geq 1$, $1 \leq p \leq \infty$. For $\mathbf{x} \in \mathbb{C}$, we define

$$\|\mathbf{x}\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

We can also consider $(\mathbb{C}, \|\cdot\|_\infty)$, for each $n \geq 1$. In this case, given $\mathbf{x} \in \mathbb{C}$, we have $\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$.

**1.3. Remark.** All norms on a finite dimensional Banach space are equivalent. In other words, they generate the same topology.

**1.4. Example.** Consider $(\mathcal{C}(X), \|\cdot\|_\infty)$, where $X$ is a compact Hausdorff space and

$$\mathcal{C}(X) = \{ f : X \to \mathbb{C} : f \text{ is continuous} \}$$

The norm we consider is the **supremum norm**, $\|f\|_\infty = \max_{x \in \mathfrak{X}} |f(x)|$.

**1.5. Example.** Let X be a locally compact Hausdorff space. Then $(\mathcal{C}_0(X), \|\cdot\|_\infty)$ is a Banach space, where

$$\begin{aligned} \mathcal{C}_0(X) &= \{ f \in \mathcal{C}(X) : f \text{ vanishes at } \infty \} \\ &= \{ f \in \mathcal{C}(X) : \forall \varepsilon > 0, \{ x \in X : |f(x)| \geq \varepsilon \} \text{ is compact in } X \}. \end{aligned}$$

As before, the norm here is the supremum norm: $\|f\|_\infty = \sup_{x \in \mathfrak{X}} |f(x)|$.

**1.6. Example.** If $(X, \Omega, \mu)$ is a measure space and $1 \le p \le \infty$, then

$$\begin{aligned} L^p(X, \Omega, \mu) \quad = \quad & \{ f : X \to \mathbb{C} : f \text{ is Lebesgue measurable} \\ & \qquad \text{and } \int_X |f(x)|^p \, d\mu(x) < \infty \} \end{aligned}$$

is a Banach space. The norm here is given by $\|f\|_p = \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}$. With $(X, \Omega, \mu)$ as above, we also define

$$\begin{aligned} L^\infty(X, \Omega, \mu) \quad = \quad & \{ f : X \to \mathbb{C} : f \text{ is Lebesgue measurable and} \\ & \qquad \text{for some } K \ge 0,\ \mu(\{ x \in \mathfrak{X} : f(x) > K \}) = 0 \}. \end{aligned}$$

In this case the norm is $\|f\|_\infty = \inf_{g=f(\text{ } a.e. \text{ } wrt \text{ } \mu)} ( \sup_{x \in \mathfrak{X}} |g(x)| )$.

**1.7. Example.** Let $I$ be a set and let $1 \le p < \infty$. Define $\ell^p(I)$ to be the set of all functions

$$\{ f : I \to \mathbb{C} : \sum_{i \in I} |f(i)|^p < \infty \}$$

and for $f \in \ell^p(I)$, define $\|f\|_p = (\sum_{i \in I} |f(i)|^p)^{1/p}$. Then $\ell^p(I)$ is a Banach space. If $I = \mathbb{N}$, we also write $\ell^p$. Of course,

$$\ell^\infty(I) = \{ f : I \to \mathbb{C} : \sup\{ |f(i)| : i \in I \} < \infty \}$$

and $\|f\|_\infty = \sup\{ |f(i)| : i \in I \}$. A closed subspace of particular interest here is

$$c_0(I) = \{ f \in l^\infty(I) : \text{ for all } \varepsilon > 0, \text{ card } (\{ i \in I : |f(i)| \ge \varepsilon \}) < \infty \}.$$

Again, if $I = \mathbb{N}$, we write only $\ell^\infty$ and $c_0$, respectively.

**1.8. Example.** Consider as a particular case of Example 1.6, $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$, and suppose $\mu$ is normalized Lebesgue measure. Then we can define the so-called **Hardy spaces**

$$H^p(\mathbb{T}, \mu) = \{ f \in L^p(\mathbb{T}, \mu) : \int_0^{2\pi} f(t) \exp^{int} \, dt = 0 \}.$$

These are Banach spaces for each $p \ge 1$, including $p = \infty$.

**1.9. Example.** Let $n \ge 1$ and let

$$\mathcal{C}^{(n)}([0,1]) = \{ f : [0,1] \to \mathbb{C} : f \text{ has } n \text{ continuous derivatives} \}.$$

Define $\|f\| = \max_{0 \le k \le n} \{ \sup\{ |f^{(k)}(x)| : x \in [0,1] \} \}$. Then $\mathcal{C}^{(n)}([0,1])$ is a Banach space.

**1.10. Example.** If $\mathfrak{X}$ is a Banach space and $\mathfrak{Y}$ is a closed subspace of $\mathfrak{X}$, then

- $\mathfrak{Y}$ is a Banach space under the inherited norm, and
- $\mathfrak{X}/\mathfrak{Y}$ is a Banach space — where $\mathfrak{X}/\mathfrak{Y} = \{ x + \mathfrak{Y} : x \in \mathfrak{X} \}$. The norm is the usual quotient norm, namely: $\|x + \mathfrak{Y}\| = \inf_{y \in \mathfrak{Y}} \|x + y\|$.

**1.11. Example.** Examples of Banach spaces can of course be combined. For instance, if $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces, then we can form the so-called $\ell^p$*-direct sum* of $\mathfrak{X}$ and $\mathfrak{Y}$ as follows:

$$\mathfrak{X} \oplus_p \mathfrak{Y} = \{\, (x, y) : x \in \mathfrak{X},\ y \in \mathfrak{Y} \,\},$$

and $\|(x, y)\| = \left( \|x\|_{\mathfrak{X}}^p + \|y\|_{\mathfrak{Y}}^p \right)^{1/p}$.

**1.12. Example.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. Then the set of continuous linear transformations $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ from $\mathfrak{X}$ into $\mathfrak{Y}$ is a Banach space under the *operator norm* $\|\, T \,\| = \sup_{\|x\| \le 1} \|Tx\|$. When $\mathfrak{X} = \mathfrak{Y}$, we also write $\mathcal{B}(\mathfrak{X})$ for $\mathcal{B}(\mathfrak{X}, \mathfrak{X})$.

In particular, $\mathcal{B}(\mathbb{C}^n)$ is isomorphic to the $n \times n$ complex matrices $\mathbb{M}_n$ and forms a Banach space under a variety of norms, including the operator norm from above. On the other hand, as we observed in Remark 1.3, all such norms must be equivalent to the operator norm.

**1.13. Example.** Suppose that $\mathfrak{X}$ is a Banach space. Then $\mathfrak{X}^* = \mathcal{B}(\mathfrak{X}, \mathbb{C})$ is a Banach space, called the *dual space* of $\mathfrak{X}$.

For example, if $\mu$ is $\sigma$-finite measure on the measure space $(X, \Omega)$, $1 \le p < \infty$, and if $q$, $1 < q \le \infty$, is chosen so that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned} [L^p(X, \Omega, \mu)]^* &= L^q(X, \Omega, \mu) \\ [\ell^p]^* &= \ell^q \\ [c_0]^* &= \ell^1. \end{aligned}$$

In general, the first two equalities fail if $p = \infty$.

As a second example, suppose $X$ is a compact, Hausdorff space. Then

$$\begin{aligned} \mathcal{C}(X)^* &= \mathcal{M}(X) \\ &= \{\, \mu : \mu \text{ is a regular Borel measure on } X \,\} \\ &= \{\, f : X \to \mathbb{C} : f \text{ is of bounded variation on } X \,\}. \end{aligned}$$

Note that for $\Phi_\mu \in \mathcal{M}(X)$, the action on $\mathcal{C}(X)$ is given by $\Phi_\mu(f) = \int_X f \, d\mu$.

For our purposes, one of the most important examples of a Banach space will be:

**1.14. Definition.** *A* ***Hilbert space*** $\mathcal{H}$ *is a Banach space whose norm is generated by an* ***inner product*** $\langle \cdot, \cdot \rangle$, *which is a map from* $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$ *satisfying:*

(1) $\langle \lambda x + \beta y, z \rangle = \lambda \langle x, z \rangle + \beta \langle y, z \rangle$
(2) $\langle x, y \rangle = \overline{\langle y, x \rangle}$
(3) $\langle x, x \rangle \ge 0$ *and* $\langle x, x \rangle = 0$ *if and only if* $x = 0$,

*for all* $x, y, z \in \mathcal{H}$*;* $\lambda, \beta \in \mathbb{C}$*. The norm on* $\mathcal{H}$ *is given by* $\|x\| = (\langle x, x\rangle)^{1/2}$*,* $x \in \mathcal{H}$*.*

Each Hilbert space $\mathcal{H}$ is equipped with a **Hilbert space basis**. This is an orthonormal set $\{e_\alpha\}_{\alpha\in\Lambda}$ in $\mathcal{H}$ with the property that any $x \in \mathcal{H}$ can be expressed as $x = \sum_{\alpha\in\Lambda} x_\alpha e_\alpha$ in a unique way. The cardinality of the Hilbert space basis is known as the **dimension** of the space $\mathcal{H}$. It is a standard result that any two Hilbert spaces of the same dimension are isomorphic. Examples of Hilbert spaces are:

- The space $(\mathbb{C}^n, \|\cdot\|_2)$, $n \geq 1$, as defined in Example 1.2.
- The space $L^2(X, \Omega, \mu)$ defined in Example 1.6.
- The space $\ell^2(I)$ defined in Example 1.7.
- The space $H^2(\mathbb{T}, \mu)$ defined in Example 1.8.
- $\mathbb{M}_n$ is a Hilbert space with the inner product $< x, y >= \mathrm{tr}(y^*x)$. Here 'tr' denotes the usual trace functional on $\mathbb{M}_n$, and if $y \in \mathbb{M}_n$, then $y^*$ denotes the conjugate transpose of $y$.

## 2. The main theorems

**2.1. Theorem. [The Hahn-Banach Theorem]** *Suppose* $\mathfrak{X}$ *is a Banach space,* $M \subseteq \mathfrak{X}$ *is a linear manifold and* $f : M \to \mathbb{C}$ *is a continuous linear functional. Then there exists a functional* $F \in \mathfrak{X}^*$ *such that* $\|F\| = \|f\|$ *and* $F|_M = f$*.*

**2.2. Corollary.** *Let* $\mathfrak{X}$ *be a Banach space and suppose* $0 \neq x \in \mathfrak{X}$*. Then there exists* $f \in \mathfrak{X}^*$ *such that* $\|f\| = 1$ *and* $f(x) = \|x\|$*.*

**2.3. Corollary.** *Let* $\mathfrak{X}$ *be a Banach space and suppose* $x \neq y$ *are two vectors in* $\mathfrak{X}$*. Then there exists* $f \in \mathfrak{X}^*$ *such that* $f(x) \neq f(y)$*.*

**2.4. Corollary.** *Let* $\mathfrak{X}$ *be a Banach space,* $M$ *be a closed subspace of* $\mathfrak{X}$ *and* $x$ *be a vector in* $\mathfrak{X}$ *such that* $x \notin M$*. Then there exists* $f \in \mathfrak{X}^*$ *such that* $f \equiv 0$ *on* $M$ *and* $f(x) \neq 0$*.*

**2.5. Theorem. [The Open Mapping Theorem]** *Let* $T : \mathfrak{X} \to \mathfrak{Y}$ *be a continuous linear map of a Banach space* $\mathfrak{X}$ *onto a Banach space* $\mathfrak{Y}$*. Then* $T$ *is open; that is,* $T(V)$ *is open in* $\mathfrak{Y}$ *for all open sets* $V$ *in* $\mathfrak{X}$*.*

**2.6. Corollary. [The Banach Isomorphism Theorem]** *Let* $T : \mathfrak{X} \to \mathfrak{Y}$ *be a continuous, injective linear map of a Banach space* $\mathfrak{X}$ *onto a Banach space* $\mathfrak{Y}$*. Then* $T^{-1}$ *is continuous.*

**2.7. Definition.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces and $M \subseteq \mathfrak{X}$ be a linear manifold. Then a linear map $T : M \to \mathfrak{Y}$ is* ***closed*** *if $x_n \to x$ and $Tx_n \to y$ together imply that $x \in M$ and $Tx = y$. This is equivalent to saying that the graph $\mathfrak{G}(T) = \{ (x, Tx) : x \in M \}$ is a closed subspace of $\mathfrak{X} \oplus \mathfrak{Y}$.*

**2.8. Theorem. [The Closed Graph Theorem]** *If $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces and $T : \mathfrak{X} \to \mathfrak{Y}$ is a closed linear map that is defined everywhere, then $T$ is continuous.*

An alternative formulation reads:

*If $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces, $T : \mathfrak{X} \to \mathfrak{Y}$ is linear, $\{x_n\}_{n=1}^\infty \subseteq \mathfrak{X}$, and if $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} Tx_n = b$ together imply that $b = 0$, then $T$ is continuous.*

**2.9. Theorem. [The Banach-Steinhaus Theorem, also known as the Uniform Boundedness Principle]** *Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces and $\mathcal{F} \subseteq \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. Suppose that for all $x \in \mathfrak{X}$, $K_x := \sup_{T\in\mathcal{F}} \|Tx\| < \infty$. Then $K := \sup_{T\in\mathcal{F}} \|T\| < \infty$.*

**2.10. Corollary.** *Let $\{T_n\}_{n=1}^\infty$ be a sequence of bounded linear operators in $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ such that $Tx = \lim_{n\to\infty} T_n x$ exists for all $x \in \mathfrak{X}$. Then $\sup_{n\geq 1} \|T_n\| < \infty$ and $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$.*

**2.11. Theorem. [The Banach-Alaoglu Theorem]** *Let $\mathfrak{X}$ be a Banach space. Then the unit ball $\mathfrak{X}_1^*$ of $\mathfrak{X}^*$ is compact in the* weak$^*$*-topology.*

**2.12.** Of course, this is but a brief outline of some of the major results and definitions which will be relevant to our study of Operator Algebras. For more information, the reader is encouraged to consult the texts of Dunford and Schwarz [**DS57**], of Bollobás [**Bol90**], and of Pryce [**Pry73**], to name but three.

CHAPTER 2

# Banach Algebras

*For be a man's intellectual superiority what it will, it can never assume the practical, available supremacy over other men, without the aid of some sort of external arts and entrenchments, always, in themselves, more or less paltry and base. This it is, that for ever keeps God's true princes of the Empire from the world's hustings; and leaves the highest honors that this air can give, to those men who become famous more through their infinite inferiority to the choice hidden handful of the Divine inert, than through their undoubted superiority over the dead level of the mass. Such large virtue lurks in these small things when extreme political superstitions invest them, that in some royal instances even to idiot imbecility they have imparted potency.*

**Herman Melville: Moby Dick**

## 1. Basic theory

**1.1. Definition.** *A **Banach algebra** $\mathcal{A}$ is a Banach space together with a norm compatible algebra structure, namely: for all $x, y \in \mathcal{A}$, $\|xy\| \leq \|x\|\,\|y\|$. If $\mathcal{A}$ has a multiplicative unit (denoted by $e$ or $1$), we say that the algebra $\mathcal{A}$ is **unital**. In this case, $\|e\| = \|e^2\| \leq \|e\|^2$, and so $e \neq 0$ implies that $\|e\| \geq 1$. By scaling the norm, we assume that $\|e\| = 1$.*

**1.2. Example.** The set $(\mathcal{C}(X), \|\cdot\|_\infty)$ of continuous functions on a compact Hausdorff space $X$ introduced in Example 1.1.4 becomes a Banach algebra under pointwise multiplication of functions. That is, for $f, g \in (\mathcal{C}(X), \|\cdot\|_\infty)$, we set

$$(fg)(x) = f(x)g(x) \quad \text{for all } x \in X.$$

**1.3. Example.** The set $(\mathcal{C}_0(X), \|\cdot\|_\infty)$ of continuous functions vanishing at infinity on a locally compact Hausdorff space as introduced in Example 1.1.5 also becomes a Banach algebra under pointwise multiplication of functions. Note that this algebra is not unital unless $X$ is compact. In particular, if $X = \mathbb{N}$ with the discrete topology, then $(\mathcal{C}_0(X), \|\cdot\|_\infty) = c_0$ is a Banach algebra under component-wise multiplication.

**1.4. Example.** Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \le 1\}$ and let $\mathbb{D}^o$ be the interior of $\mathbb{D}$. Also, let $\mathcal{A}(\mathbb{D}) = \{f \in \mathcal{C}(\mathbb{D}) : f \text{ is analytic on } \mathbb{D}^o\}$. Then $(\mathcal{A}(\mathbb{D}), \|\cdot\|_\infty)$ is a Banach algebra under pointwise multiplication of functions, called the *disk algebra*.

The map

$$\begin{array}{rccc} \tau : & \mathcal{A}(\mathbb{D}) & \to & \mathcal{C}(\mathbb{T}) \\ & f & \mapsto & f|_{\mathbb{T}} \end{array}$$

is an isometric embedding, by the Maximum Modulus Principle.

It is often useful to identify the disk algebra with its image under this embedding. In this case we have $\mathcal{A}(\mathbb{D}) = \{f \in \mathcal{C}(\mathbb{T}) : f$ can be analytically continued to $\mathbb{D}^o\}$.

In general, given a compact subset $X \subseteq \mathbb{C}$, we let

$$\begin{aligned} \mathcal{A}(X) &= \{f \in \mathcal{C}(X) : f \text{ is analytic on } \operatorname{int}(X)\} \\ \mathcal{R}(X) &= \{f \in \mathcal{C}(X) : f \text{ is a rational function} \\ &\qquad\qquad \text{with poles outside of } X\}^{-\|\ \|} \\ \mathcal{P}(X) &= \{f \in \mathcal{C}(X) : f \text{ is a polynomial}\}^{-\|\ \|} \end{aligned}$$

Each of these is a closed subalgebra of $\mathcal{C}(X)$ under the inherited norm. Clearly $\mathcal{P}(X) \subseteq \mathcal{R}(X) \subseteq \mathcal{A}(X) \subseteq \mathcal{C}(X)$, and it is often an interesting and important problem to decide when the inclusions reduce to equalities. This is the case, for instance, when $X = [0, 1]$.

**1.5. Example.** Let $G$ be a locally compact abelian group, and let $\nu$ denote Haar measure on $G$. Then

$$L^1(G, \nu) = \{f : G \to \mathbb{C} : \int_G |f(x)|\, d\nu(x) < \infty\}.$$

For $f,\ g \in L^1(G, \nu),\ x \in G$, we define the product of $f$ and $g$ via convolution:

$$(f * g)(x) = \int_G f(xy^{-1})\, g(y)\, d\nu(y).$$

We also define $\|f\|_1 = \int_G |f(x)|\, d\nu(x)$.

This is called the *group algebra* of $G$. It is a standard result (see section 4.3.7) that $f * g = g * f$ and that $\|f * g\|_1 \le \|f\|_1\, \|g\|_1$.

Writing $\ell^1(\mathbb{Z}) = L^1(\mathbb{Z}, \nu)$ where $\nu$ represents counting measure, we obtain:

$$\begin{array}{rcl} (f * g)(n) &=& \sum_{k \in \mathbb{Z}} f(n-k) g(k) \\ \|f\|_1 &=& \sum_{k \in \mathbb{Z}} |f(k)|. \end{array}$$

As we shall see in Chapter Four, $\ell^1(\mathbb{Z})$ can be identified with the *Wiener algebra*

$$\mathcal{AC}(\mathbb{T}) = \{f \in \mathcal{C}(\mathbb{T}) : f(e^{i\theta}) = \sum_{n \in \mathbb{Z}} a_n e^{in\theta},\ \ \sum_{n \in \mathbb{Z}} |a_n| < \infty\},$$

where $a_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta}\, d\theta$.

**1.6. Example.** The set $L^\infty(X, \Omega, \nu)$ is a Banach algebra under pointwise multiplication.

**1.7. Example.** The set $\mathcal{C}_b(\Omega)$ of continuous, bounded functions on a locally compact space $\Omega$ is a Banach algebra under the supremum norm and pointwise multiplication.

**1.8. Remark.** The above examples are all abelian. The following need not be.

**1.9. Example.** Let $\mathfrak{X}$ be a Banach space. Then the Banach space $\mathcal{B}(\mathfrak{X})$ from Example 1.1.12 is a Banach algebra, using the operator norm and composition of linear maps as our product. To verify this, we need only verify that the operator norm is submultiplicative, that is, that $\|AB\| \leq \|A\|\,\|B\|$ for all operators $A$ and $B$. But

$$\begin{aligned}
\|AB\| &= \sup\{\|ABx\| : \|x\| = 1\} \\
&\leq \sup\{\|A\|\,\|Bx\| : \|x\| = 1\} \\
&\leq \sup\{\|A\|\,\|B\|\,\|x\| : \|x\| = 1\} \\
&= \|A\|\,\|B\|.
\end{aligned}$$

In particular, $\mathbb{M}_n$ can be identified with $\mathcal{B}(\mathbb{C}^n)$ by first fixing an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ for $\mathbb{C}^n$, and then identifying a linear map in $\mathcal{B}(\mathbb{C}^n)$ with its matrix representation with respect to this fixed basis.

It is easy to verify that any closed subalgebra of $\mathcal{B}(\mathfrak{X})$ (or indeed, of any Banach algebra) is itself a Banach algebra.

**1.10. Example.** Let $\mathfrak{X}$ be a Banach space, and let $T \in \mathcal{B}(\mathfrak{X})$. Then

$$\mathrm{Alg}(T) = \{p(T) : p \text{ a polynomial over } \mathbb{C}\}^{-\|\ \|}$$

is a Banach algebra, called the *algebra generated by* $T$. The norm under consideration is the operator norm.

**1.11. Example.** Let $\mathcal{T}_n$ denote the set of upper triangular $n \times n$ matrices in $\mathbb{M}_n$, equipped with the operator norm. Then $\mathcal{T}_n$ is a Banach subalgebra of $\mathbb{M}_n$. After fixing an orthonormal basis for the underlying Hilbert space as in Example 1.9, $\mathcal{T}_n$ can be viewed as a Banach subalgebra of $\mathcal{B}(\mathbb{C}^n)$. In fact, it is the largest subalgebra of $\mathcal{B}(\mathbb{C}^n)$ which leaves each of the subspaces $\mathcal{H}_k = \mathrm{span}\{e_1, e_2, \ldots, e_k\},\ 1 \leq k \leq n$ invariant.

More generally, given a Banach space $\mathfrak{X}$ and a collection $\mathcal{L}$ of closed subspaces $\{L_\alpha\}_{\alpha\in\Lambda}$ of $\mathfrak{X}$, then

$$\mathrm{Alg}(\mathcal{L}) = \{T \in \mathcal{B}(\mathfrak{X}) : TL_\alpha \subseteq L_\alpha \text{ for all } L_\alpha \in \mathcal{L}\}$$

is a Banach algebra. This is closed because if $\lim_{n\to\infty} T_n = T$, then $x \in L_\alpha$ implies $Tx = \lim_{n\to\infty} T_n x \in L_\alpha$ for each $\alpha$.

**1.12. Example.** The space $H^\infty(\mathbb{T}, \mu)$ defined in Example 1.1.8 is a Banach algebra under pointwise multiplication of functions.

**1.13. Example.** Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{K}(\mathcal{H})$ denote the set of compact operators acting on $\mathcal{H}$. (See Chapter 3.) Then $\mathcal{K}(\mathcal{H})$ is a Banach subalgebra of $\mathcal{B}(\mathcal{H})$. In fact, as we shall see, $\mathcal{K}(\mathcal{H})$ is a closed, two-sided ideal of $\mathcal{B}(\mathcal{H})$.

**1.14. Example.** Let $(\mathcal{A}_\alpha, \|\cdot\|_\alpha)_{\alpha\in\Lambda}$ denote a family of Banach algebras indexed by a set $\Lambda$. Then

$$\mathcal{A} = \{(a_\alpha)_{\alpha\in\Lambda} : a_\alpha \in \mathcal{A}_\alpha,\ \alpha \in \Lambda,\ \sup_\alpha \|a_\alpha\| < \infty\}$$

is a Banach algebra when equipped with the norm

$$\|(a_\alpha)_\alpha\| = \sup_\alpha \|a_\alpha\|.$$

**1.15. Example.** Consider the Hilbert space $H^2 = H^2(\mathbb{T}, \mu)$ of Example 1.1.8. Let $P$ denote the orthogonal projection of $L^2(\mathbb{T}, \mu)$ onto $H^2(\mathbb{T}, \mu)$. Define the *Toeplitz algebra* on $H^2$ to be the set of operators in $\mathcal{B}(H^2)$ of the form

$$T_\phi : f \mapsto P(\phi f), \quad \phi \in \mathcal{C}(\mathbb{T}).$$

This is a Banach subalgebra of $\mathcal{B}(H^2)$.

**1.16. Proposition.** *Let $\mathcal{K}$ be a closed ideal in a Banach algebra $\mathcal{A}$. Then the quotient space $\mathcal{A}/\mathcal{K}$ is a Banach algebra with respect to the quotient norm.*

**Proof.** That $\mathcal{A}/\mathcal{K}$ is a Banach space follows from Example 1. 1.10. Let $\pi$ denote the canonical map from $\mathcal{A}$ to $\mathcal{A}/\mathcal{K}$. We must show that

$$\|\pi(x)\pi(y)\| \le \|\pi(x)\|\,\|\pi(y)\|$$

for all $x, y \in \mathcal{A}$.

Suppose $\varepsilon > 0$. By definition of the quotient norm, we can find $m, n \in \mathcal{K}$ such that $\|\pi(x)\| + \varepsilon \ge \|x + m\|$ and $\|\pi(y)\| + \varepsilon \ge \|y + n\|$. Then

$$\begin{aligned}
\|\pi(x)\pi(y)\| &= \|\pi(x+m)\pi(y+n)\| \\
&\le \|\pi((x+m)\,(y+n))\| \\
&\le \|(x+m)\,(y+n)\| \\
&\le \|(x+m)\|\,\|(y+n)\| \\
&\le (\|\pi(x)\| + \varepsilon)\,(\|\pi(y)\| + \varepsilon).
\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain the desired result.

□

Recall from Example 1.13 our claim that the set $\mathcal{K}(\mathcal{H})$ of compact operators is a closed, two-sided ideal of $\mathcal{B}(\mathcal{H})$. Using this and the above Proposition, we obtain the following important example.

**1.17. Example.** Let $\mathcal{H}$ be a Hilbert space. Then the quotient algebra $\mathcal{A}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is a Banach algebra, known as the **Calkin algebra**. The canonical map from $\mathcal{B}(\mathcal{H})$ to $\mathcal{A}(\mathcal{H})$ is denoted by $\pi$.

**1.18. Remark.** In general, if a Banach algebra $\mathcal{A}$ does not have an identity element, it is possible to append one as follows:

Consider the algebra $\mathcal{A}^+ = \mathcal{A} \oplus \mathbb{C}\,1$ with multiplication given by

$$(a, \lambda)\,(b, \nu) = (ab + a\nu + b\lambda, \lambda\nu).$$

We define a norm on $\mathcal{A}^+$ via $\|(a, \lambda 1)\| = \|a\| + |\lambda|$. Then we have

$$\begin{array}{rcl} \|(a,\lambda)\,(b,\nu)\| & = & \|ab + a\nu + b\lambda\| + |\lambda\nu| \\ & \leq & \|a\|\,\|b\| + \|a\|\,|\nu| + \|b\|\,|\lambda| + |\lambda|\,|\nu| \\ & = & (\|a\| + |\lambda|)\,(\|b\| + |\nu|) \\ & = & (\|(a,\lambda)\|)\,(\|(b,\nu)\|) \end{array}$$

It is clear that the embedding of $\mathcal{A}$ into $\mathcal{A}^+$ is linear and isometric, and that $\mathcal{A}$ sits inside of $\mathcal{A}^+$ as a closed ideal. It should be added, however, that this construction is not always natural. The group algebra $L^1(\mathbb{R}, \nu)$ of the real numbers with Lebesgue measure $\nu$ is not unital. On the other hand, the most natural candidate for a multiplicative identity here might be the Dirac delta function (corresponding to a discrete measure with mass one at 0 and zero elsewhere), which clearly does not lie in the algebra. Similarly, $\mathcal{C}_0(\mathbb{R})$ is another much studied non-unital algebra. In this case, there is more than one way to embed this algebra into a unital Banach algebra. For instance, one may want to consider the one-point compactification, or the Stone-Čech compactifications of the reals. Each of these induces an imbedding of $\mathcal{C}_0(\mathbb{R})$ into the corresponding unital Banach algebra of continuous functions on these compactifications.

**1.19. Proposition.** *Every Banach algebra $\mathcal{A}$ embeds isometrically into $\mathcal{B}(\mathfrak{X})$ for some Banach space $\mathfrak{X}$. Here, $\mathcal{A}$ need not have a unit.*

**Proof.** Consider the map

$$\begin{array}{rcl} \mathcal{A} & \to & \mathcal{B}(\mathcal{A}^+) \\ a & \mapsto & L_a \end{array}$$

where $L_a(x, \lambda) = (a, 0)\,(x, \lambda)$ is the *left regular representation* of $\mathcal{A}$.

Then

$$\|L_a\| = \sup_{(x,\lambda)\neq(0,0)} \frac{\|(a,0)\,(x,\lambda)\|}{\|(x,\lambda)\|} \leq \|(a,0)\| = \|a\|$$

and

$$\|L_a\| \geq \|(a,0)\,(0,1)\| = \|a\|,$$

so that $\|L_a\| = \|a\|$. In particular, the map is isometric. That $L_a\,L_b = L_{ab}$ and that $\alpha L_a + \beta L_b = L_{\alpha a + \beta b}$ are easily verified.

$\square$

**1.20. Theorem.** *The set $\mathcal{A}^{-1}$ of invertible elements of a unital Banach algebra $\mathcal{A}$ is open in the norm topology.*

**Proof.** If $\|a\| < 1$, then the element $b = \sum_{n=0}^{\infty} a^n$ exists in $\mathcal{A}$ since the defining series is absolutely convergent. As such,

$$\begin{aligned}
(1-a)\,b &= (1-a)\left(\lim_{k\to\infty}\sum_{n=0}^{k} a^n\right) \\
&= \left(\lim_{k\to\infty}\sum_{n=0}^{k} a^n\right) - \left(\lim_{k\to\infty}\sum_{n=1}^{k+1} a^n\right) \\
&= \lim_{k\to\infty} 1 - a^{k+1} \\
&= 1 \\
&= \left(\lim_{k\to\infty}\sum_{n=0}^{k} a^n\right) - \left(\lim_{k\to\infty}\sum_{n=1}^{k+1} a^n\right) \\
&= \left(\lim_{k\to\infty}\sum_{n=0}^{k} a^n\right)(1-a) \\
&= b\,(1-a),
\end{aligned}$$

so that the open ball of radius 1 centred at the identity is contained in the set of invertible elements of $\mathcal{A}$.

Now if $d \in \mathcal{A}^{-1}$ and $\|a\| < \|d^{-1}\|^{-1}$, then $(d-a) = d\,(1-d^{-1}a)$ and $\|d^{-1}a\| < 1$ so that

$$(d-a)^{-1} = (1-d^{-1}a)^{-1}d^{-1}$$

exists. This means that the open ball of radius $\|d^{-1}\|^{-1}$ centred at $d$ is again contained in $\mathcal{A}^{-1}$. Thus $\mathcal{A}^{-1}$ is open.

□

**1.21. Corollary.** *If $\mathcal{A}$ is a unital Banach algebra, then the map $\tau : a \mapsto a^{-1}$ is a homeomorphism of $\mathcal{A}^{-1}$ onto itself. It follows that $\mathcal{A}^{-1}$ is a topological group.*

**Proof.** That multiplication is continuous in $\mathcal{A}^{-1}$ follows from the fact that it is jointly continuous in $\mathcal{A}$. It remains therefore to show that $\tau$ is continuous - as it is clearly a bijection which is equal to its own inverse.

Let us first show that $\tau$ is continuous at 1. If $\|b\| < 1$, then we have just seen that $(1-b)$ is invertible and

$$\begin{aligned} \|1-(1-b)^{-1}\| &= \|1-\sum_{n=0}^{\infty} b^n\| \\ &= \|\sum_{n=1}^{\infty} b^n\| \\ &\leq \sum_{n=1}^{\infty} \|b\|^n \\ &= \|b\|/(1-\|b\|). \end{aligned}$$

Thus as $\|b\| \to 0$ (i.e. as $b \to 0$ and hence $(1-b) \to 1$), we get $(1-b)^{-1} \to 1$, implying that the map $\tau : a \mapsto a^{-1}$ is continuous at 1, as claimed.

If $a \in \mathcal{A}^{-1}$ and $a_n \to a$, then $a_n\, a^{-1} \to a\, a^{-1} = 1$, and also $a^{-1}\, a_n \to a^{-1}\, a = 1$, so that $a_n^{-1} \to a^{-1}$.

$\square$

**1.22. Proposition.** *Let $G$ be a locally connected topological group, and let $G_0$ be the connected component of the identity in $G$. Then $G_0$ is an open and closed normal subgroup of $G$, the cosets of $G_0$ are the components of $G$, and $G/G_0$ is a discrete group.*

**Proof.** A component of a topological space is always closed. If $g \in G$, then $G$ locally connected implies that there exists an open connected neighbourhood $\mathcal{O}_g$ of $g$ which clearly lies in the connected component $C_g$ of $g$. This shows that $C_g$ is open and therefore components of $G$ are both open and closed.

Let $f \in G$. Then the map $L_{f^{-1}} : h \mapsto f^{-1}h$ is a homeomorphism of $G$, and so $f^{-1}G_0$ is open, closed and connected. If, furthermore, $f \in G_0$ and $g \in G_0$, then $f^{-1}G_0$ is a connected set containing 1 and $f^{-1}g$, and therefore $f^{-1}g \in G_0$, implying that $G_0$ is a subgroup of $G$. Since the map $R_f : h \mapsto h\, f$ is also a homeomorphism of $G$, it follows that $f^{-1}\, G_0\, f = L_{f^{-1}}(R_f(G_0))$ is an open, closed and connected subset of $G$ containing 1, so that $f^{-1}\, G_0\, f = G_0$, and therefore $G_0$ is normal.

Since $f^{-1}G_0$ is open, closed and connected for all $f \in G$, the cosets of $G_0$ are precisely the components of $G$. Finally, since the cosets are components, it follows that $G/G_0$ is discrete.

$\square$

**1.23. Definition.** *Let $\mathcal{A}$ be a unital Banach algebra. Let $\mathcal{A}_0^{-1}$ denote the connected component of the identity in $\mathcal{A}^{-1}$. Then the **abstract index group** of $\mathcal{A}$, denoted $\Lambda_{\mathcal{A}}$, is the group $\mathcal{A}^{-1}/\mathcal{A}_0^{-1}$. The **abstract index** is the canonical homomorphism from $\mathcal{A}^{-1}$ to $\Lambda_{\mathcal{A}}$.*

**1.24. Remark.** It follows from Proposition 1.22 that the abstract index group of a Banach algebra $\mathcal{A}$ is well-defined, that $\Lambda_{\mathcal{A}}$ is discrete, and that the components of $\mathcal{A}^{-1}$ are the cosets of $\mathcal{A}_0$ in $\Lambda_{\mathcal{A}}$.

**1.25. Definition.** *Let $\mathcal{A}$ be a Banach algebra and $a \in \mathcal{A}$. If $\mathcal{A}$ is unital, then the* ***spectrum*** *of $a$ relative to $\mathcal{A}$ is the set*

$$\sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible in } \mathcal{A}\}.$$

*If $\mathcal{A}$ is not unital, then $\sigma_{\mathcal{A}}(a)$ is set to be $\sigma_{\mathcal{A}^+}(a) \cup \{0\}$. When the algebra $\mathcal{A}$ is understood, we generally write $\sigma(a)$. The* ***resolvent*** *of $a$ is the set $\rho(a) = \mathbb{C} \backslash \sigma(a)$.*

**1.26. Corollary.** *Let $\mathcal{A}$ be a unital Banach algebra, and let $a \in \mathcal{A}$. Then $\rho(a)$ is open and $\sigma(a)$ is compact.*

**Proof.** Clearly $\rho(a) = \{\lambda \in \mathbb{C} : (a - \lambda 1) \text{ is invertible}\}$ is open, since $\mathcal{A}^{-1}$ is. Indeed, if $a - \lambda_0 1$ is invertible in $\mathcal{A}$, then $\lambda \in \rho(a)$ for all $\lambda \in \mathbb{C}$ such that $|\lambda - \lambda_0| < \|(a - \lambda_0)^{-1}\|^{-1}$. Thus $\sigma(a)$ is closed.

If $|\lambda| > \|a\|$, then $\lambda 1 - a = \lambda\,(1 - \lambda^{-1} a)$ and $\|\lambda^{-1} a\| < 1$, and so $(1 - \lambda^{-1} a)$ is invertible. This implies

$$(\lambda 1 - a)^{-1} = \lambda^{-1}\,(1 - \lambda^{-1} a)^{-1}.$$

Thus $\sigma(a)$ is contained in the disk $D_{\|a\|}(\{0\})$ of radius $\|a\|$ centred at the origin. Since it both closed and bounded, $\sigma(a)$ is compact.

$\square$

**1.27. Definition.** *Let $\mathfrak{X}$ be a Banach space and $U \subseteq \mathbb{C}$ be an open set. Then a function $f : U \to \mathfrak{X}$ is said to be* ***weakly analytic*** *if the map $z \mapsto x^*(f(z))$ is analytic for all $x^* \in \mathfrak{X}^*$.*

**1.28. Theorem.** **[Liouville's Theorem]** *Every bounded, weakly entire function into a Banach space $\mathfrak{X}$ is constant.*

**Proof.** For each linear functional $x^* \in \mathfrak{X}^*$, $x^* \circ f$ is a bounded, entire function into the complex plane. By the complex-valued version of Liouville's Theorem, it must therefore be constant. Now by the Hahn-Banach Theorem, $\mathfrak{X}^*$ separates the points of $\mathfrak{X}$. So if there exist $z_1, z_2 \in \mathbb{C}$ such that $f(z_1) \neq f(z_2)$, then there must exist $x^* \in \mathfrak{X}^*$ such that $x^*(f(z_1)) \neq x^*(f(z_2))$. This contradiction implies that $f$ is constant.

$\square$

**1.29. Definition.** *Let $\mathcal{A}$ be a unital Banach algebra and let $a \in \mathcal{A}$. The map*

$$\begin{array}{rccc} R(\cdot, a) : & \rho(a) & \to & \mathcal{A} \\ & \lambda & \mapsto & (\lambda 1 - a)^{-1} \end{array}$$

*is called the* ***resolvent function*** *of $a$.*

**1.30. Proposition. The Common Denominator Formula.** *Let $a \in \mathcal{A}$, a unital Banach algebra. Then if $\mu, \lambda \in \rho(a)$, we have*

$$R(\lambda, a) - R(\mu, a) = (\mu - \lambda)\, R(\lambda, a)\, R(\mu, a).$$

**Proof.** The proof is transparent if we consider $t \in \mathbb{C}$ and consider the corresponding complex-valued equation:

$$\frac{1}{\lambda - t} - \frac{1}{\mu - t} = \frac{(\mu - t) - (\lambda - t)}{(\lambda - t)\,(\mu - t)} = \frac{(\mu - \lambda)}{(\lambda - t)\,(\mu - t)}.$$

In terms of Banach algebra, we have:

$$\begin{aligned} R(\lambda, a) &= R(\lambda, a)\, R(\mu, a)\, (\mu - a) \\ R(\mu, a) &= R(\mu, a)\, R(\lambda, a)\, (\lambda - a). \end{aligned}$$

Noting that $R(\lambda, a)$ and $R(\mu, a)$ clearly commute, we obtain the desired equation by simply subtracting the second equation from the first.

□

We shall return to this formula when establishing the holomorphic functional calculus in the next section.

**1.31. Proposition.** *If $a \in \mathcal{A}$, a unital Banach algebra, then $R(\cdot, a)$ is analytic on $\rho(a)$.*

**Proof.** Let $\lambda_0 \in \rho(a)$. Then

$$\begin{aligned} \lim_{\lambda \to \lambda_0} \frac{R(\lambda, a) - R(\lambda_0, a)}{\lambda - \lambda_0} &= \lim_{\lambda \to \lambda_0} \frac{(\lambda_0 - \lambda)\, R(\lambda, a)\, R(\lambda_0, a)}{\lambda - \lambda_0} \\ &= -R(\lambda_0, a)^2 \end{aligned}$$

since inversion is continuous on $\rho(a)$. Thus the limit of the Newton quotient exists, and so $R(\cdot, a)$ is analytic.

□

**1.32. Corollary. [Gelfand]** *If $a \in \mathcal{A}$, a Banach algebra, then $\sigma(a)$ is non-empty.*

**Proof.** We may assume that $\mathcal{A}$ is unital, for otherwise $0 \in \sigma(a)$ and we are done. Similarly, if $a = 0$, then $0 \in \sigma(a)$. If $\rho(a) = \mathbb{C}$, then clearly $R(\cdot, a)$ is entire. Now for $|\lambda| > \|a\|$, we have

$$\begin{aligned} (\lambda - a)^{-1} &= \big(\lambda(1 - \lambda^{-1} a)\big)^{-1} \\ &= \lambda^{-1} \sum_{n=0}^{\infty} (\lambda^{-1} a)^n \\ &= \sum_{n=0}^{\infty} \lambda^{-n-1} a^n \end{aligned}$$

so that if $|\lambda| \geq 2\|a\|$, then

$$\|(\lambda - a)^{-1}\| \leq \sum_{n=0}^{\infty} \frac{\|a\|^n}{(2\|a\|)^{n+1}} \leq \frac{1}{\|a\|}.$$

That is, $\|R(\lambda, a)\| \leq \|a\|^{-1}$ for all $\lambda \geq 2\|a\|$.

Clearly there exists $M < \infty$ such that

$$\max_{|\lambda| \leq 2\|a\|} \|R(\lambda, a)\| \leq M$$

since $R(\cdot, a)$ is a continuous function on this compact set. The conclusion is that $R(\cdot, a)$ is a bounded, entire function. By Theorem 1.28, the resolvent function must be constant. This obvious contradiction implies that $\sigma(a)$ is non-empty.

□

Recall that a ***division algebra*** is an algebra in which each non-zero element is invertible.

**1.33. Theorem. [Gelfand-Mazur]** *If $\mathcal{A}$ is a Banach algebra and a division algebra, then there is a unique isometric isomorphism of $\mathcal{A}$ onto $\mathbb{C}$.*

**Proof.** If $b \in \mathcal{A}$, then $\sigma(b)$ is non-empty by Corollary 1.32. Let $\beta \in \sigma(b)$. Then $\beta 1 - b$ is not invertible, and since $\mathcal{A}$ is a division algebra, we conclude that $\beta 1 = b$; that is to say, that $\sigma(b)$ is a singleton.

Given $a \in \mathcal{A}$, $\sigma(a)$ is a singleton, say $\{\lambda_a\}$. The complex-valued map $\phi : a \mapsto \lambda_a$ is an algebra isomorphism. Moreover, $\|a\| = \|\lambda_a 1\| = |\lambda_a| = \|\phi(a)\|$, so the map is isometric as well.

If $\phi_0 : \mathcal{A} \to \mathbb{C}$ were another such map, then $\phi_0(a) \in \sigma(a)$, implying that $\phi_0(a) = \phi(a)$.

□

**1.34. Definition.** *Let $a \in \mathcal{A}$, a Banach algebra. The* ***spectral radius*** *of $a$ is*

$$\text{spr}(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

**1.35. Lemma. The Spectral Mapping Theorem - polynomial version.** *Let $a \in \mathcal{A}$, a unital Banach algebra, and suppose $p \in \mathbb{C}[z]$ is a polynomial. Then*

$$\sigma(p(a)) = p(\sigma(a)) := \{p(\lambda) : \lambda \in \sigma(a)\}.$$

**Proof.** Let $\alpha \in \mathbb{C}$. Then for some $\gamma \in \mathbb{C}$,

$$p(z) - \alpha = \gamma\,(z - \beta_1)\,(z - \beta_2) \cdots (z - \beta_n)$$

and so

$$p(a) - \alpha = \gamma\,(a - \beta_1)\,(a - \beta_2) \cdots (a - \beta_n).$$

Thus (as all of the terms $(a - \beta_i)$ commute),

$$\begin{aligned} \alpha \in \sigma(p(a)) &\iff \beta_i \in \sigma(a) \text{ for some } 1 \leq i \leq n \\ &\iff p(z) - \alpha = 0 \text{ for some } z \in \sigma(a) \\ &\iff \alpha \in p(\sigma(a)). \end{aligned}$$

□

**1.36. Theorem. [Beurling : The Spectral Radius Formula]** *If $a \in \mathcal{A}$, a Banach algebra, then*

$$\operatorname{spr}(a) = \lim_{n \to \infty} \|a^n\|^{1/n}.$$

**Proof.** First observe that if $\mathcal{A}$ is not unital, then we can always embed it isometrically into a unital Banach algebra $\mathcal{A}^+$. Since both the left and right hand sides of the above equation remain unchanged when $a$ is considered as an element of $\mathcal{A}^+$, we may (and do) assume that $\mathcal{A}$ is already unital.

Now $\sigma(a^n) = (\sigma(a))^n$, and so $\operatorname{spr}(a^n) = (\operatorname{spr}(a))^n$. Moreover, for all $b \in \mathcal{A}$, the proof of Corollary 1.26 shows that $\operatorname{spr}(b) \leq \|b\|$. Thus

$$\operatorname{spr}(a) = (\operatorname{spr}(a^n))^{1/n} \leq \|a^n\|^{1/n} \text{ for all } n \geq 1.$$

This tells us that $\operatorname{spr}(a) \leq \inf_{n \geq 1} \|a^n\|^{1/n}$.

On the other hand, $R(\cdot, a)$ is analytic on $\rho(a)$ and hence is analytic on $\{\lambda \in \mathbb{C} : |\lambda| > \operatorname{spr}(a)\}$. Furthermore, if $|\lambda| > \|a\|$, then

$$\begin{aligned} R(\lambda, a) &= (\lambda - a)^{-1} \\ &= \lambda^{-1}(1 - \lambda^{-1}a)^{-1} \\ &= \sum_{n=0}^{\infty} a^n / \lambda^{n+1}. \end{aligned}$$

Let $\phi \in \mathcal{A}^*$. Then $\phi \circ R(\cdot, a)$ is an analytic, complex-valued function,

$$[\phi \circ R(\cdot, a)](\lambda) = \sum_{n=0}^{\infty} \phi(a^n)/\lambda^{n+1}$$

and this Laurent expansion is still valid for $\{\lambda \in \mathbb{C} : |\lambda| > \|a\|\}$, since the series for $R(\cdot, a)$ is absolutely convergent on this set, and applying $\phi$ introduces at most a factor of $\|\phi\|$ to the absolutely convergent sum. Since $[\phi \circ R(\cdot, a)]$ is analytic on $\{\lambda \in \mathbb{C} : |\lambda| > \operatorname{spr}(a)\}$, the complex-valued series converges on this larger set.

From this it follows that the sequence $\{\phi(a^n)/\lambda^{n+1}\}_{n=1}^{\infty}$ converges to 0 as $n$ tends to infinity for all $\phi \in \mathcal{A}^*$, so therefore is bounded for all $\phi \in \mathcal{A}^*$. It is now a consequence of the Uniform Boundedness Principle that $\{a^n/\lambda^{n+1}\}_{n=1}^{\infty}$ is bounded in norm, say by $M_\lambda > 0$, *for each* $\lambda \in \mathbb{C}$ satisfying $|\lambda| > \operatorname{spr}(a)$. That is:

$$\|a^n\| \leq M_\lambda |\lambda^{n+1}|$$

for all $|\lambda| > \operatorname{spr}(a)$. But then, for all $|\lambda| > \operatorname{spr}(a)$,

$$\limsup_{n\geq 1} \|a^n\|^{1/n} \leq \limsup_{n\geq 1} M_\lambda^{1/n} |\lambda^{n+1/n}| = |\lambda|.$$

Combining this estimate with the above yields $\operatorname{spr}(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

$\square$

*Never kiss a gift horse on the mouth.*

## 2. The functional calculus

**2.1. Integration in a Banach space.** Let $\alpha \le \beta \in \mathbb{R}$ and let $\mathfrak{X}$ be a Banach space. An **$\mathfrak{X}$-valued step function** $f$ is a function on $[\alpha, \beta]$ for which there exists a partition $P = \{\alpha = \alpha_0 < \alpha_1 < \ldots < \alpha_n = \beta\}$ of $[\alpha, \beta]$ so that

$$f(t) = c_k, \qquad \alpha_{k-1} < t \le \alpha_k,\ 1 \le k \le n \tag{1}$$

for some $c_k \in \mathfrak{X}$, $1 \le k \le n$, and $f(\alpha_0) = c_1$. Given an $\mathfrak{X}$-valued step function $f$, a partition $P$ satisfying (1) will be referred to as an **admissible partition** for $f$.

Denote by $S = S([\alpha, \beta], \mathfrak{X})$ the linear manifold of $\mathfrak{X}$-valued step functions in the Banach space $L^\infty([\alpha, \beta], \mathfrak{X})$. For each $f \in S$, define

$$\int_\alpha^\beta f = \sum_{k=1}^n (\alpha_k - \alpha_{k-1}) c_k$$

whenever $P = \{\alpha = \alpha_0 < \alpha_1 < \ldots < \alpha_n = \beta\}$ is an admissible partition for $f$. We remark that this sum is easily seen to be independent of the choice of admissible partitions, and so $\int_\alpha^\beta f$ is well-defined. Moreover, $\| \int_\alpha^\beta f \| \le (\beta - \alpha) \|f\|_\infty$. It follows that the map

$$\begin{array}{rccc} \Phi: & S & \to & \mathfrak{X} \\ & f & \mapsto & \int_\alpha^\beta f \end{array}$$

is continuous.

We may therefore extend $\Phi$ to the closure $\overline{S}$ in $L^\infty([\alpha, \beta], \mathfrak{X})$ and continue to write $\int_\alpha^\beta f$ or $\int_\alpha^\beta f(t)\, dt$ for $f \in \overline{S}$. Clearly we still have

$$\| \int_\alpha^\beta f \| \le (\beta - \alpha) \|f\|_\infty$$

for all $f \in \overline{S}$.

If $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ for some Banach space $\mathfrak{Y}$, then it is easy to check that $T \circ f \in \overline{S([\alpha, \beta], \mathfrak{Y})}$ for all $f \in \overline{S}$, and

$$T\left( \int_\alpha^\beta f \right) = \int_\alpha^\beta T \circ f.$$

**2.2. Proposition.** *Let $f \in \mathcal{C}([\alpha, \beta], \mathfrak{X})$ and let $\varepsilon > 0$. Then $f \in \overline{S}$ and there exists $\delta > 0$ such that for every partition $P = \{\alpha = \alpha_0 < \alpha_1 < \ldots < \alpha_n = \beta\}$ of $[\alpha, \beta]$ such that $\|P\| := \max_{1 \le k \le n} (\alpha_k - \alpha_{k-1}) < \delta$, and for all $t_1, t_2, \ldots, t_n$ satisfying $\alpha_{k-1} \le t_k \le \alpha_k$, $1 \le k \le n$, the following statements hold:*

1. *there exists $g \in S([\alpha, \beta], \mathfrak{X})$ with $g(t) = f(t_k)$, $(\alpha_{k-1} \le t < \alpha_k,\ 1 \le k \le n)$ and $\|f - g\| \le \varepsilon$.*
2. $\| \int_\alpha^\beta f - \sum_{k=1}^n (\alpha_k - \alpha_{k-1}) f(t_k) \| \le (\beta - \alpha)\varepsilon$.

**Proof.** Since $f$ is continuous on the compact set $[\alpha, \beta]$, it is uniformly continuous there, and so we can choose $\delta > 0$ such that $|a - b| < \delta$ implies that $\|f(a) - f(b)\| < \varepsilon$.

Let $P$ be any partition of $[\alpha, \beta]$ with $\|P\| < \delta$, and choose $\{t_k\}_{k=1}^n$ such that $\alpha_{k-1} \leq t_k < \alpha_k$, $1 \leq k \leq n$. Let $g(\alpha_0) = f(t_1)$, and for $1 \leq k \leq n$, let $g(t) = f(t_k)$, $\alpha_{k-1} < t \leq \alpha_k$.

(1) Now

$$\begin{aligned} \|f - g\|_\infty &= \sup_{t \in [\alpha, \beta]} \|f(t) - g(t)\| \\ &= \max_{1 \leq k \leq n} \sup_{t \in (\alpha_{k-1}, \alpha_k]} \|f(t) - g(t)\| \\ &= \max_{1 \leq k \leq n} \sup_{t \in (\alpha_{k-1}, \alpha_k]} \|f(t) - f(t_k)\| \\ &< \varepsilon. \end{aligned}$$

(2) Secondly,

$$\begin{aligned} \|\int_\alpha^\beta f - \sum_{k=1}^n (\alpha_k - \alpha_{k-1}) f(t_k)\| &= \|\int_\alpha^\beta f - \int_\alpha^\beta g\| \\ &= \|\int_\alpha^\beta f - g\| \\ &\leq \int_\alpha^\beta \|f - g\|_\infty \\ &\leq (\beta - \alpha)\varepsilon. \end{aligned}$$

□

We remark in passing that a minor adaptation of the above proof shows that piecewise continuous functions also lie in $\overline{S}$.

**2.3.** With $\alpha \leq \beta \in \mathbb{R}$ as above, we define a **curve** in $\mathfrak{X}$ to be a continuous function $\tau : [\alpha, \beta] \to \mathfrak{X}$. The interval $[\alpha, \beta]$ is referred to as the **parameter interval** of the curve, and we denote the image of $\tau$ in $\mathfrak{X}$ by $\tau^*$. The point $\tau(\alpha)$ is then called the **initial point** of the curve, while $\tau(\beta)$ is referred to as the **final point**.

A **contour** in $\mathfrak{X}$ is a piecewise continuously differentiable curve. That is, there exists a partition $P = \{\alpha = \alpha_0 < \alpha_1 < \ldots < \alpha_n = \beta\}$ of $[\alpha, \beta]$ such that $\tau|_{[\alpha_{i-1}, \alpha_i]}$ is continuously differentiable, $1 \leq i \leq n$. If $\tau(\alpha) = \tau(\beta)$, we say that the contour $\tau$ is **closed**.

Suppose that $\tau$ is a contour in $\mathbb{C}$, and that $f : \tau^* \to \mathfrak{X}$ is a continuous function. We can then think of $\tau$ as a parametrization of $\tau^*$. We shall define the **integral of $f$ over $\tau$** as

$$\int_{\tau^*} f(z)\, dz = \int_\alpha^\beta f(\tau(x))\tau'(x)\, dx. \tag{2}$$

Note that the integral on the right hand side exists by the comment following Proposition 2.2.

Suppose next that $\gamma : [\alpha_1, \beta_1] \to [\alpha, \beta]$ is a continuously differentiable bijection with $\gamma(\alpha_1) = \alpha$ and $\gamma(\beta_1) = \beta$. Let $\tau_1 = \tau \circ \gamma$. Then

$$\begin{aligned}\int_{\tau_1^*} f(z)\,dz &= \int_{\alpha_1}^{\beta_1} f(\tau_1(x))\tau_1'(x)\,dx \\ &= \int_{\alpha_1}^{\beta_1} f(\tau(\gamma(x))\tau'(\gamma(x))\gamma'(x)\,dx \\ &= \int_{\alpha}^{\beta} f(\tau(y))\tau'(y)\,dy \\ &= \int_{\tau^*} f(z)\,dz,\end{aligned}$$

and so the integral is seen to be independent of the parametrization of the contour. Any two such contours $\tau_1$ and $\tau_2$ for which

$$\int_{\tau_1^*} f(z)\,dz = \int_{\tau_2^*} f(z)\,dz$$

for all continuous functions $f \in \mathcal{C}(\tau_1^* = \tau_2^*)$ will be considered **equivalent**.

The notion of equivalence of contours allows us to manipulate vector-valued integrals in the standard way. For instance, suppose that the final point of $\tau_1$ equals the initial point of $\tau_2$, and suppose $f \in \mathcal{C}(\tau_1^* \cup \tau_2^*)$. We can "concatenate" the two contours into one longer contour $\tau$ satisfying

$$\int_{\tau^*} f(z)\,dz = \int_{\tau_1^*} f(z)\,dz + \int_{\tau_2^*} f(z)\,dz.$$

Moreover, equation (2) shows that

$$\begin{aligned}\| \int_{\tau^*} f(z)\,dz\| &= \| \int_{\alpha}^{\beta} f(\tau(x))\tau'(x)\,dx\| \\ &\leq \|f\|_\infty \, \| \int_{\alpha}^{\beta} \tau'(x)\,dx\| \\ &= \|f\|_\infty \, \|\tau^*\|,\end{aligned}$$

where $\|f\|_\infty = \max\{\|f(x)\| : x \in \tau^*\}$, while $\|\tau^*\| = \| \int_\alpha^\beta \tau'(x)\,dx\|$ is (by definition) the **length** of $\tau^*$. Note that this length is finite as $\tau'$ is piecewise continuous.

Finally, observe that as before, if $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ for some Banach space $\mathfrak{Y}$, then

$$T\left(\int_{\tau^*} f(z)\,dz\right) = \int_{\tau^*} (T \circ f)(z)\,dz.$$

**2.4.** Our present goal is to make sense of expressions of the form $f(a)$, where $a \in \mathcal{A}$, a Banach algebra, and $f$ is a function. An important question in this regard is to find the largest set of functions for which $f(a)$ makes sense. Clearly if $p(z) = \sum_{k=0}^{n} c_k z^k$ is polynomial over the complex numbers, then

$$p(a) = \sum_{k=0}^{n} c_k \, a^k$$

can be defined in any unital Banach algebra which contains $a$. (If we also stipulate that $c_0 = 0$, then $p(a)$ makes sense even if the algebra is not unital.)

Suppose now that the algebra $\mathcal{A}$ is unital, that $p$ and $q$ are polynomials over $\mathbb{C}$, and that $0 \notin q(\sigma(a))$. Then $q(z) = \beta(\Pi_{k=1}^{m}(z-\lambda_k))$, where $\lambda_k \notin \sigma(a)$ for $1 \le k \le m$, so we can define $r(z) = p(z)/q(z)$ as an analytic function on some neighbourhood of $\sigma(a)$ and

$$r(a) = p(a)\beta^{-1} \, (\Pi_{k=1}^{m}(a - \lambda_k)^{-1}).$$

The question remains: can we do better than rational functions? For general Banach algebras $\mathcal{A}$ and arbitrary elements $a \in \mathcal{A}$, we are now in a position to develop an *analytic functional calculus*: that is, we shall make sense of $f(a)$ whenever $f$ is a function which is analytic on some neighbourhood of $\sigma(a)$.

This is definitely not the only possible functional calculus that exists. For example, later we shall see that if $\mathcal{A}$ is a $C^*$-algebra and $a \in \mathcal{A}$ is *normal*, then one can develop a continuous functional calculus. As another example, if $T \in \mathcal{B}(\mathcal{H})$ is a contraction, then an $H^\infty$ functional calculus is possible.

Recall from Complex Analysis the following:

**2.5. Definition.** *If $\Gamma$ is a finite system of closed contours in $\mathbb{C}$ and $\lambda \notin \Gamma$, then the* ***index*** *or* ***winding number*** *of $\Gamma$ with respect to $\lambda$ is*

$$\mathrm{Ind}_\Gamma(\lambda) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{(z-\lambda)} \, dz,$$

*and represents the number of times that $\Gamma$ wraps around $\lambda$ in the positive (i.e. counterclockwise) direction.*

**2.6. Theorem. [Cauchy's Theorem]** *Let $f$ be analytic on an open set $U \subseteq \mathbb{C}$, and let $z_0 \in U$. Let $\Gamma$ be a finite system of closed contours in $U$ such that $z_0 \notin \Gamma$, $\mathrm{Ind}_\Gamma(z_0) = 1$, and $\{z \in \mathbb{C} : \mathrm{Ind}_\Gamma(z) \neq 0\} \subseteq U$. Then*

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z-z_0)} \, dz.$$

*Furthermore,*

$$\int_\Gamma f(z) \, dz = 0.$$

**2.7. Remarks.**

- We shall say that a complex valued function $f$ is ***analytic*** on a compact subset $K$ of $\mathbb{C}$ if $f$ is analytic on some open subset $U$ of $\mathbb{C}$ which contains $K$.
- Let $U \subseteq \mathbb{C}$ be open and $K \subseteq U$ be compact. Then there exists a finite system of contours $\Gamma \subseteq U$ such that
  (1) $\mathrm{Ind}_\Gamma(\lambda) \subseteq \{0, 1\}$;
  (2) $\mathrm{Ind}_\Gamma(\lambda) = 1$ for all $\lambda \in K$;
  (3) $\{z \in \mathbb{C} : \mathrm{Ind}_\Gamma(z) \neq 0\} \subseteq U$.
  The existence of such a system $\Gamma$ is a relatively standard result from Complex Analysis, and follows from the Jordan Curve Theorem. A proof can be found in [**Con78**], although (to quote Conway himself [**Con85**]), "some details are missing".

  In practice, the idea is to cover $K$ by open disks of sufficiently small radius so as to ensure that their closures still lie in $U$. Since $K$ is presumed to be compact, there will exist a finite subcover $V$ by these disks. Modulo some technicalities, the boundary of $V$ will then yield the desired system $\Gamma$ of contours.

  In fact, with a bit more work, one can even assume that $\Gamma$ consists of a finite system of *infinitely differentiable* curves [**Con85**], Proposition 4.4.

**2.8. The Riesz-Dunford Functional Calculus.** Let $a \in \mathcal{A}$, a unital Banach algebra, and fix $U$ be an open subset of $\mathbb{C}$ such that $\sigma(a) \subseteq U$. Set

$$\mathcal{F}(a) = \{f : U \to \mathbb{C} \, : f \text{ is analytic}\}.$$

Choose a system $\Gamma \subseteq U$ of closed contours such that

(1) $\mathrm{Ind}_\Gamma(\lambda) = 1$ for all $\lambda \in \sigma(a)$;
(2) $\{z \in \mathbb{C} : \mathrm{Ind}_\Gamma(z) \neq 0\} \subseteq U$.

We define

$$f(a) = \frac{1}{2\pi i} \int_\Gamma f(z) \, (z - a)^{-1} \, dz.$$

The first question we must ask is whether or not this definition makes sense. As stated, the definition appears to depend upon the choice of the system $\Gamma$ and of $U$. The following Theorem addresses this issue.

**2.9. Theorem. [The Riesz-Dunford Functional Calculus, 1]** *With the above setting, $f(a)$ is well-defined (i.e. independent of the choice of curves $\Gamma$), and for $f, g \in \mathcal{F}(a)$, $h \in \mathcal{F}(f(a))$, and $\lambda \in \mathbb{C}$,*

(i) $(f + g)(a) = f(a) + g(a)$;
(ii) $(\lambda f)(a) = \lambda(f(a))$;
(iii) $(fg)(a) = f(a)\, g(a)$;

**Proof.** First note that if $U_1$ and $U_2$ are two open sets containing $\sigma(a)$, then so is $U := U_1 \cap U_2$. If $\Gamma_1$ (resp. $\Gamma_2$) is an eligible system of contours in $U_1$ (resp. $U_2$), then it suffices to show that the integral along each of $\Gamma_1$ and $\Gamma_2$ agrees with the integral along an eligible system of contours $\Gamma$ contained in $U$. By symmetry, it suffices to show that the integral along $\Gamma_1$ agrees with the integral along $\Gamma$. Since $U \subseteq U_1$, this implies that the problem reduces to the case where $\Gamma_1$ and $\Gamma_2$ sit inside the same open set $U$.

Let $\Gamma_1$ and $\Gamma_2$ be two eligible systems of contours. We must show that

$$b = \frac{1}{2\pi i}\int_{\Gamma_1} f(z)(z-a)^{-1}\,dz - \frac{1}{2\pi i}\int_{\Gamma_2} f(z)(z-a)^{-1}\,dz = 0.$$

By the Corollary to the Hahn-Banach Theorem [Cor. 1. 2.2], it suffices to show that $x^*(b) = 0$ for all $x^* \in \mathcal{A}^*$. Now

$$x^*(b) = \frac{1}{2\pi i}\int_{\Gamma_1 - \Gamma_2} f(z)\, x^*(z-a)^{-1}\,dz.$$

Also, $f(z)$ is analytic on $U$, $R(z,a) = (z-a)^{-1}$ is analytic on $\rho(a) \supseteq \Gamma_1, \Gamma_2$ and so $x^*((z-a)^{-1})$ is analytic on $\rho(a)$ for all $x^* \in \mathcal{A}^*$. So the integrand is analytic on the open set $U \cap \rho(a)$. To apply Cauchy's Theorem above, we need only verify the index conditions.

If $\lambda \notin U$, then we have $\mathrm{Ind}_{\Gamma_1}(\lambda) = \mathrm{Ind}_{\Gamma_2}(\lambda) = 0$, and therefore

$$\mathrm{Ind}_{\Gamma_1-\Gamma_2}(\lambda) = \mathrm{Ind}_{\Gamma_1}(\lambda) - \mathrm{Ind}_{\Gamma_2}(\lambda) = 0.$$

If $\lambda \in \sigma(a)$, then $\mathrm{Ind}_{\Gamma_1}(\lambda) = \mathrm{Ind}_{\Gamma_2}(\lambda) = 1$, therefore $\mathrm{Ind}_{\Gamma_1-\Gamma_2}(\lambda) = 0$.

Thus $\{z \in \mathbb{C} : \mathrm{Ind}_{\Gamma_1-\Gamma_2}(\lambda) \neq 0\} \subseteq U \cap \rho(a)$ and so Cauchy's Theorem applies, namely:

$$x^*(b) = \frac{1}{2\pi i}\int_{\Gamma_1 - \Gamma_2} f(z)\, x^*(z-a)^{-1}\,dz = 0 \textit{ for all } x^* \in \mathcal{A}^*.$$

Thus $b = 0$ and so $f(a)$ is indeed well-defined.

(i) $(f+g)(a) = f(a) + g(a)$:
This follows for the linearity of the integral, and is left as an exercise.

(ii) $(\lambda f)(a) = \lambda(f(a))$:
Again, this follows from the linearity of the integral.

(iii) $(fg)(a) = f(a)\,g(a)$:
Now $f$ and $g$ are both analytic on some open set $U \supseteq \sigma(a)$. Choose two systems of contours $\Gamma_1$ and $\Gamma_2$ such that
   (a) $\{z \in \mathbb{C} : \mathrm{Ind}_{\Gamma_i}(z) \neq 0\} \subseteq U$, $i = 1, 2$;
   (b) $\mathrm{Ind}_{\Gamma_i}(z) = 1$ for all $z \in \sigma(a)$, $i = 1, 2$;
   (c) $\mathrm{Ind}_{\Gamma_1}(z) = 1$ for all $z \in \Gamma_2$.

To get part (c), we choose $\Gamma_2$ first and then choose $\Gamma_1$ to lie "outside" of $\Gamma_2$. Then

$$
\begin{aligned}
f(a)\,g(a) &= \frac{1}{2\pi i}\int_{\Gamma_1} f(z)(z-a)^{-1}\,dz \;\; \frac{1}{2\pi i}\int_{\Gamma_2} g(w)(w-a)^{-1}\,dw \\
&= (\frac{1}{2\pi i})^2 \int_{\Gamma_1}\int_{\Gamma_2} f(z)g(w)(z-a)^{-1}(w-a)^{-1}\,dw\,dz \\
&= (\frac{1}{2\pi i})^2 \int_{\Gamma_1}\int_{\Gamma_2} f(z)g(w)(\frac{1}{w-z})[(z-a)^{-1}-(w-a)^{-1}]\,dz\,dw \\
(3) \qquad &= (\frac{1}{2\pi i})^2 \int_{\Gamma_1} f(z)(z-a)^{-1}\int_{\Gamma_2} g(w)(w-z)^{-1}\,dw\,dz - \\
&\quad (\frac{1}{2\pi i})^2 \int_{\Gamma_2} g(w)(w-a)^{-1}\int_{\Gamma_1} f(z)(w-z)^{-1}\,dz\,dw \\
&= \frac{1}{2\pi i}\int_{\Gamma_2} g(w)(w-a)^{-1}f(w)\,dw \\
&= (fg)(a).
\end{aligned}
$$

where the first integral in equation (3) is zero since $z$ lies "outside" of $\Gamma_2$ and $g$ is analytic.

$\square$

**2.10. Remark.** Let $a$ be an element of a unital Banach algebra $\mathcal{A}$ and let $U$ be an open set in the complex plane such that $\sigma(a) \subseteq U$. Let

$$H(U) = \{f : U \to \mathbb{C} : \ f \text{ is analytic}\}.$$

From (i), (ii) and (iii) above, we conclude that the map:

$$
\begin{array}{rccc}
\Phi : & H(U) & \to & \mathcal{A} \\
& f & \mapsto & f(a)
\end{array}
$$

is an algebra homomorphism. Moreover, for all $a \in \mathcal{A}$ and $f, g \in H(U)$, we have $f(a)g(a) = g(a)f(a)$ since $f(z)g(z) = g(z)f(z)$.

**2.11. Proposition.** *Suppose $\mathcal{A}$ is a unital Banach algebra and that $a \in \mathcal{A}$. Let $U \subseteq \mathbb{C}$ be an open set containing $\sigma(a)$, and let $\{f_n\}_{n=0}^{\infty}$ be a sequence of analytic functions on $U$ converging uniformly to $f$ on compact subsets of $U$. Then $f$ is also analytic on $U$ and*

$$\lim_{n\to\infty} \|f_n(a) - f(a)\| = 0.$$

**Proof.** Choose an open set $V$ with boundary $\Gamma$ consisting of a finite number of piecewise smooth curves such that $\sigma(a) \subseteq V \subseteq \overline{V} \subseteq U$.

Since $\{f_n\}_{n=0}^{\infty}$ converges uniformly on compact subsets of $U$, $f$ is analytic on $U$. Thus $f \in H(U)$ and $\{f_n\}_{n=0}^{\infty}$ converges uniformly to $f$ on $\Gamma$. It follows that

$$\begin{aligned}\|f_n(a) - f(a)\| &= \|(1/2\pi i)\int_\Gamma [f_n(z) - f(z)](z-a)^{-1}\,dz\| \\ &\leq (1/2\pi)K\,\|\Gamma\|\,\|f_n - f\|_\Gamma,\end{aligned}$$

where $K = \sup\{\|(z-a)^{-1}\| : z \in \Gamma\}$, $\|\Gamma\|$ represents the arclength of the contour, and $\|f_n - f\|_\Gamma = \sup\{|f_n(z) - f(z)| : z \in \Gamma\}$. Since both $K$ and $\|\Gamma\|$ are fixed and $\|f_n - f\|_\Gamma$ tends to zero as $n \to \infty$, we obtain the desired conclusion.

□

**2.12. Theorem. [The Riesz-Dunford Functional Calculus, 2]** *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. If $f(z) = \sum_{n=0}^\infty c_n z^n$ converges to a function analytic in a neighbourhood of $\sigma(a)$, then $f(a) = \sum_{n=0}^\infty c_n a^n$.*

**Proof.** Suppose $f(z) = \sum_{n=0}^\infty c_n z^n$ converges in $D_R(\{0\}) \supseteq \sigma(a)$. Then consider the curve $\Gamma = \{re^{i\theta} : 0 \leq \theta \leq 2\pi\}$ for some $r$, $\mathrm{spr}(a) < r \leq R$ and consider

$$\begin{aligned}f(a) &= \frac{1}{2\pi i}\int_\Gamma \left(\sum_{n=0}^\infty c_n z^n\right)(z-a)^{-1}\,dz \\ &= \frac{1}{2\pi i}\sum_{n=0}^\infty c_n \int_\Gamma z^n\,(z-a)^{-1}\,dz \qquad (4)\\ &= \sum_{n=0}^\infty c_n z^n(a),\end{aligned}$$

where $z(a) = \frac{1}{2\pi i}\int_\Gamma z(z-a)^{-1}\,dz$ is the identity function evaluated at $a$. Note that (4) uses the uniform convergence of the series on $\Gamma$.

But

$$\begin{aligned}z(a) &= \frac{1}{2\pi i}\int_\Gamma z(z-a)^{-1}\,dz \\ &= \frac{1}{2\pi i}\int_\Gamma \sum_{n=0}^\infty z^{-n} a^n\,dz \\ &= \sum_{n=0}^\infty a^n \frac{1}{2\pi i}\int_\Gamma z^{-n}\,dz \qquad (5)\\ &= \sum_{n=0}^\infty a^n(\delta_{n1}) \qquad (6)\\ &= a.\end{aligned}$$

Here (5) uses the uniform convergence of the series when $|z| = r > \mathrm{spr}(a)$, and (6) uses the Residue Theorem. We can now apply induction on part

(iii) of Theorem 2.9 to get $z^n(a) = (z(a))^n = a^n$, and so

$$f(a) = \sum_{n=0}^{\infty} c_n a^n,$$

as desired.

□

**2.13. Corollary. [Dunford: The Spectral Mapping Theorem]** *Let $a \in \mathcal{A}$, a unital Banach algebra and suppose that $f$ is analytic on $\sigma(a)$. Then*

$$\sigma(f(a)) = f(\sigma(a)).$$

**Proof.** If $\lambda \notin f(\sigma(a))$, then $g(z) = (\lambda - f(z))^{-1}$ is analytic on $\sigma(a)$. From the functional calculus,

$$\begin{aligned} g(a)\,(\lambda - f(a)) &= (g(z)\,(\lambda - f(z)))\,(a) \\ &= 1(a) \\ &= 1 \\ &= (\lambda - f(a))\,g(a), \end{aligned}$$

since everything commutes. Thus $\lambda \notin \sigma(f(a))$.

If $\lambda \in f(\sigma(a))$, then $\lambda - f(z)$ has a zero on $\sigma(a)$, say at $z_0$. As such,

$$\lambda - f(z) = (z_0 - z)g(z)$$

for some function $g$ which is analytic on $\sigma(a)$. Via the functional calculus, we obtain

$$\lambda - f(a) = (z_0 - a)g(a),$$

and since $(z_0 - a)$ is not invertible and $(z_0 - a)$ commutes with $g(a)$, we conclude that $\lambda - f(a)$ is not invertible either. Thus $\lambda \in \sigma(f(a))$.

Combining the two results, $f(\sigma(a)) = \sigma(f(a))$.

□

**2.14. Theorem. [The Riesz-Dunford Functional Calculus, 3]** *Suppose that $\mathcal{A}$ is a unital Banach algebra, and that $g$ is a complex-valued function which is analytic on $\sigma(a)$ while $f$ is a complex-valued function which is analytic on $g(\sigma(a))$. Then $(f \circ g)(a) = f(g(a))$.*

**Proof.** Let $V$ be an open neighbourhood of $g(\sigma(a))$ upon which $f$ is analytic and consider $U = g^{-1}(V)$, an open neighbourhood of $\sigma(a)$. Let $\Gamma_1$ be a system of closed contours in $U$ such that

(a) $\mathrm{Ind}_{\Gamma_1}(\lambda) = 1$ for all $\lambda \in \sigma(a)$;
(b) $\mathrm{Ind}_{\Gamma_1}(\lambda) \neq 0$ implies that $\lambda \in U$.

Let $\Gamma_2$ be a system of closed contours in $V$ such that

(A) $\mathrm{Ind}_{\Gamma_2}(\beta) = 1$ for all $\beta \in g(\sigma(a))$;
(B) $\mathrm{Ind}_{\Gamma_2}(\beta) = 1$ for all $\beta \in g(\Gamma_1)$;

(C) $\mathrm{Ind}_{\Gamma_1}(\beta) \neq 0$ implies that $\beta \in V$.

(One can view $\Gamma_2$ as lying "outside" of $g(\Gamma_1)$ in $V$.)

Then

$$\begin{aligned}
(f \circ g)(a) &= \frac{1}{2\pi i}\int_{\Gamma_1} (f \circ g)(z)(z-a)^{-1}\,dz \\
&= \frac{1}{2\pi i}\int_{\Gamma_1} f(g(z))(z-a)^{-1}\,dz \\
&= \frac{1}{2\pi i}\int_{\Gamma_1} \frac{1}{2\pi i}\int_{\Gamma_2} f(w)(w-g(z))^{-1}\,dw\,(z-a)^{-1}\,dz \\
&= \frac{1}{2\pi i}\int_{\Gamma_2} \frac{1}{2\pi i}\int_{\Gamma_1} (w-g(z))^{-1}(z-a)^{-1}\,dz\,dw \\
&= \frac{1}{2\pi i}\int_{\Gamma_2} f(w)(w-g(a))^{-1}\,dw \\
&= f(g(a)).
\end{aligned}$$

□

**2.15. Corollary. [The Riesz Decomposition Theorem]** *Let $a \in \mathcal{A}$, a unital Banach algebra, and suppose that $\Delta$ is a non-trivial, relatively closed and open subset of $\sigma(a)$.*

(i) *There exists a non-trivial idempotent $E(\Delta)$ in $\mathcal{A}$ which commutes with $a$;*

(ii) *If $\mathcal{A} \subseteq \mathcal{B}(\mathfrak{X})$ for some Banach space $\mathfrak{X}$, then $E(\Delta)\mathfrak{X}$ and $(I - E(\Delta))\mathfrak{X}$ are complementary subspaces invariant under $a$.*

(iii) *Let $a_\Delta = a|_{E(\Delta)\mathfrak{X}}$. Then $\sigma(a_\Delta) = \Delta$. Moreover, for any function $f$ which is analytic on $\sigma(a)$, we have $f(a_\Delta) = f(a)|_{E(\Delta)\mathfrak{X}}$.*

**Proof.**

(i) Consider an analytic function $g$ such that $g \equiv 1$ on $\Delta$ and $g \equiv 0$ on $\sigma(a) \setminus \Delta$. Let $E(\Delta) = g(a)$. Then $g^2 = g$ and so $E^2(\Delta) = g^2(a) = g(a) = E(\Delta)$ is an idempotent. Note that $E(\Delta) \neq 0$ since $1 \in \sigma(g(a)) = g(\sigma(a))$. Similarly, $E(\Delta) \neq I$ as $0 \in \sigma(g(a)) = g(\sigma(a))$.

Since $zg(z) = g(z)z$, $E(\Delta)$ commutes with $a$.

(ii) Let $\mathfrak{X}$ be a Banach space and assume that $\mathcal{A} \subseteq \mathcal{B}(\mathfrak{X})$. Then $E(\Delta)\mathfrak{X} = \ker(I - E(\Delta))$ is closed, as is $(I - E(\Delta))\mathfrak{X} = \ker E(\Delta)$. Clearly

$$\mathfrak{X} = E(\Delta)\mathfrak{X} + (I - E(\Delta))\mathfrak{X}.$$

Moreover, if $y \in E(\Delta)\mathfrak{X} \cap (I - E(\Delta))\mathfrak{X}$, then

$$y = E(\Delta)y = E(\Delta)\,(I - E(\Delta))y = 0.$$

Thus $E(\Delta)\mathfrak{X}$ and $(I - E(\Delta))\mathfrak{X}$ are complementary. Finally, let $x \in E(\Delta)\mathfrak{X}$. Then $ax = aE(\Delta)x = E(\Delta)ax \in E(\Delta)\mathfrak{X}$. Therefore $E(\Delta)\mathfrak{X}$ is invariant under $a$, as is $(I - E(\Delta))\mathfrak{X}$.

(iii) First we show that $\sigma(a_\Delta) \subseteq \Delta$.

If $\lambda \notin \Delta$, let

$$h(z) = \begin{cases} (\lambda - z)^{-1} & \text{for } z \text{ in a neighbourhood of } \Delta \\ 0 & \text{for } z \text{ in a neighbourhood of } \sigma(a) \setminus \Delta. \end{cases}$$

Then $h(z)(\lambda - z) = g(z)$. Thus $h(a)(\lambda - a) = g(a) = E(\Delta)$. Now $h(a)$ leaves $E(\Delta)\mathfrak{X}$ and $(I - E(\Delta))\mathfrak{X}$ invariant (since $h(a)$ commutes with $g(a)$). If $R_\lambda := h(a)|_{E(\Delta)\mathfrak{X}}$, then

$$R_\lambda(\lambda - a_\Delta) = (\lambda - a_\Delta)R_\lambda = I_{E(\Delta)\mathfrak{X}},$$

so that $\lambda \in \rho(a_\Delta)$, i.e. $\sigma(a_\Delta) \subseteq \Delta$.

Suppose now that $\lambda \in \Delta \cap \rho(a_\Delta)$, so that for some $b \in \mathcal{B}(E(\Delta)\mathfrak{X})$, we have

$$b\,(\lambda - a_\Delta) = (\lambda - a_\Delta)\,b = I_{E(\Delta)\mathfrak{X}}.$$

Let

$$k(z) = \begin{cases} (\lambda - z)^{-1} & \text{for } z \text{ in a neighbourhood of } \sigma(a) \setminus \Delta \\ 0 & \text{for } z \text{ in a neighbourhood of } \Delta. \end{cases}$$

Then

$$k(a)(\lambda - a) = (\lambda - a)k(a) = I - E(\Delta).$$

Define $r = k(a) + bE(\Delta)$. Then

$$\begin{aligned} r\,(\lambda - a) &= k(a)\,(\lambda - a) + b\,E(\Delta)\,(\lambda - a) \\ &= (I - E(\Delta)) + b\,(\lambda - a_\Delta)\,E(\Delta) \\ &= (I - E(\Delta)) + E(\Delta) \\ &= I. \end{aligned}$$

Similarly, $(\lambda - a)\,r = I$, and so $\lambda \in \rho(a)$, a contradiction. We conclude that $\sigma(a_\Delta) = \Delta$.

Finally, suppose that $f$ is analytic on $\sigma(a)$. Then for an eligible system $\Gamma$ of contours we obtain

$$\begin{aligned} f(a_\Delta) &= \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - a_\Delta)^{-1}\,d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - a)^{-1}|_{E(\Delta)\mathfrak{X}}\,d\lambda \\ &= \frac{1}{2\pi i}\left(\int_\Gamma f(\lambda)(\lambda - a)^{-1}\,d\lambda\right)|_{E(\Delta)\mathfrak{X}} \\ &= f(a)|_{E(\Delta)\mathfrak{X}}. \end{aligned}$$

□

*Meanwhile, back at the Sado-masochist club . . .*
*"This is going to hurt me more than it will hurt you."*

## 3. The spectrum

**3.1. Spectrum relative to a subalgebra.** Suppose that $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras with $1 \in \mathcal{A} \subseteq \mathcal{B}$. For $a \in \mathcal{A}$, we have have two notions of spectrum, namely:

$$\sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} : (\lambda 1 - a)^{-1} \notin \mathcal{A}\}$$

and

$$\sigma_{\mathcal{B}}(a) = \{\lambda \in \mathbb{C} : (\lambda 1 - a)^{-1} \notin \mathcal{B}\}.$$

In general, it is clear that $\sigma_{\mathcal{B}}(a) \subseteq \sigma_{\mathcal{A}}(a)$. Our present intention is to exhibit a closure relation between the two spectra.

**3.2. Example.** Let $\mathcal{B} = \mathcal{C}(\mathbb{T})$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle in the complex plane. Let $\mathcal{A} = \mathcal{A}(\mathbb{D})$ be the disk algebra defined in Example 1.4. By the same Example, $\mathcal{A} \subseteq \mathcal{B}$.

Let $f$ be the identity function $f(z) = z$, so that clearly $f \in \mathcal{A}$. Then $\|f\| = 1$, so that $\sigma_{\mathcal{A}}(f),\ \sigma_{\mathcal{B}}(f) \subseteq \mathbb{D}$. Now if $|\lambda| < 1$, then the function $g_\lambda(z) = \frac{1}{\lambda - z} \notin \mathcal{A}(\mathbb{D})$, and so $\lambda \in \sigma_{\mathcal{A}}(f)$. Since the spectrum of an element is always compact and hence closed, $\sigma_{\mathcal{A}}(f) = \mathbb{D}$.

In contrast, $g_\lambda \in \mathcal{B} = \mathcal{C}(\mathbb{T})$, so that $\sigma_{\mathcal{B}}(f) \subseteq \mathbb{T}$. If $|\lambda| = 1$, then $g_\lambda$ is clearly not continuous on the circle, so that $\lambda \in \sigma_{\mathcal{B}}(f)$. We conclude that $\sigma_{\mathcal{B}}(f) = \mathbb{T}$.

This example proves to be prototypical of the phenomenon we wish to explore.

**3.3. Definition.** *Let $\mathcal{A}$ be a Banach algebra. An element $a \in \mathcal{A}$ is said to be a* ***right*** *(resp.* ***left; joint****)* ***topological divisor of zero*** *if there exists a sequence $\{x_n\}_{n=1}^\infty \subseteq \mathcal{A}$, $\|x_n\| = 1$ for all $n \geq 1$ such that $\lim_n x_n a = 0$ (resp. $\lim_n a x_n = 0$; $\lim_n \|x_n a\| + \|a x_n\| = 0$).*

**3.4. Theorem.** *Let $\mathcal{A}$ be a unital Banach algebra and let $a \in \partial(\mathcal{A}^{-1})$. Then $a$ is a joint topological divisor of zero.*

**Proof.** Since $a \in \partial(\mathcal{A}^{-1})$, there exists a sequence $\{b_n\} \subseteq \mathcal{A}^{-1}$ such that $\lim_n b_n = a$. Now we claim that $\{\|b_n^{-1}\|\}_{n=1}^\infty$ is unbounded, for if $\|b_n^{-1}\| \leq M$ for some $M > 0$ and for all $n \geq 1$, then

$$\begin{aligned} \|b_n^{-1} - b_m^{-1}\| &= \|b_n^{-1}(b_m - b_n)b_m^{-1}\| \\ &\leq M^2 \|b_m - b_n\|. \end{aligned}$$

Thus $\{b_n^{-1}\}_{n=1}^\infty$ is a Cauchy sequence. Let $c = \lim_n b_n^{-1}$. Then by the continuity of inversion, $c = a^{-1}$ and so $a \in \mathcal{A}^{-1}$. But $\mathcal{A}^{-1}$ is open, which contradicts the fact that $a \in \partial(\mathcal{A}^{-1})$.

Next, by choosing a suitable subsequence of $\{b_n\}_{n=1}^\infty$ and reindexing if necessary, we may assume that $\|b_n^{-1}\| \geq n,\ n \geq 1$. Let $x_n = b_n^{-1}/\|b_n^{-1}\|$ for each $n$, and

$$\begin{aligned}\|ax_n\| &= \|(a-b_n)x_n + b_n x_n\| \\ &\leq \|(a-b_n)x_n\| + \|b_n^{-1}\|^{-1}.\end{aligned}$$

Thus $\lim_n ax_n = 0$, and similarly, $\lim_n x_n a = 0$.

□

**3.5. Corollary.** *Let $\mathcal{A}$ be a Banach algebra and $a \in \mathcal{A}$. If $\lambda \in \partial(\sigma(a))$, then $(a-\lambda)$ is a joint topological divisor of $0$.*
**Proof.** Immediate.

□

**3.6. Proposition.** *Let $\mathcal{A}$ be a Banach algebra and suppose that $a \in \mathcal{A}$ is a joint topological divisor of $0$ in $\mathcal{A}$. Then $0 \in \sigma_{\mathcal{A}}(a)$.*
**Proof.** Suppose that there exists $b = a^{-1} \in \mathcal{A}$. Take $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$, $\|x_n\| = 1$ for all $n \geq 1$, such that $\lim_n ax_n = 0$. Then

$$\|x_n\| = \|b\, a\, x_n\| \leq \|b\|\, \|a\, x_n\|$$

so that $\lim_n \|x_n\| = 0$, a contradiction.

□

We note that if $a \in \mathcal{A}$ is a joint topological divisor of 0 in $\mathcal{A}$, and if $\mathcal{B}$ is a Banach algebra containing $\mathcal{A}$, then $a$ is a joint topological divisor of 0 in $\mathcal{B}$, and so $0 \in \sigma_{\mathcal{B}}(a)$ as well.

**3.7. Proposition.** *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and suppose $a \in \mathcal{A} \subseteq \mathcal{B}$. Then*

(i) *$\sigma_{\mathcal{B}}(a) \subseteq \sigma_{\mathcal{A}}(a)$; and*
(ii) *$\partial(\sigma_{\mathcal{A}}(a)) \subseteq \sigma_{\mathcal{B}}(a)$.*

**Proof.**

(i) Immediate.
(ii) If $\lambda \in \partial(\sigma_{\mathcal{A}}(a))$, then $a - \lambda$ is a topological divisor of 0 in $\mathcal{A}$ and so $a - \lambda$ is not invertible in $\mathcal{B}$, by Proposition 3.6.

□

**3.8. Remark.** The conclusion of Proposition 3.7 is that the most that can happen to the spectrum of an element $a$ when passing to a subalgebra that contains $a$ is that we "fill in" the "holes" of the spectrum, that is, the bounded components of the resolvent of $a$ in the larger algebra.

**3.9. Theorem.** *Let $\mathcal{B}$ be a Banach algebra and $a \in \mathcal{B}$. Let $\Omega$ be a subset of $\rho_{\mathcal{B}}(a)$ which has non-empty intersection with each bounded component of $\rho_{\mathcal{B}}(a)$. Finally, let $\mathcal{A}$ be the smallest closed subalgebra of $\mathcal{B}$ containing 1, $a$, and $(\lambda - a)^{-1}$ for each $\lambda \in \Omega$. Then $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$.*

**Proof.** Choose $\phi \in \mathcal{B}^*$ so that $\phi(x) = 0$ for all $x \in \mathcal{A}$. Define the function

$$\begin{array}{rccc} h_\phi : & \rho_{\mathcal{B}}(a) & \to & \mathbb{C} \\ & z & \mapsto & \phi((z-a)^{-1}) \end{array}$$

so that $h_\phi$ is holomorphic on its domain. We shall now show that $h_\phi \equiv 0$. Since this is true for all $\phi \in \mathcal{B}^*$ that annihilates $\mathcal{A}$, we can then invoke Corollary 1.2.4 to obtain the desired result.

Now if $|z| > \operatorname{spr}(a)$, then

$$(z-a)^{-1} = \sum_{n=0}^{\infty} z^{-n-1} a^n$$

converges uniformly and thus $(z-a)^{-1} \in \mathcal{A}$. Hence $h_\phi(z) \equiv 0$ for all $z$, $|z| > \operatorname{spr}(a)$. Thus $h_\phi \equiv 0$ on the unbounded component of $\rho_{\mathcal{B}}(a)$.

If $\lambda \in \Omega$ lies in a bounded component of $\rho_{\mathcal{B}}(a)$, then note that

$$(z-a) = (\lambda - a)\,(1 - (\lambda - z)\,(\lambda - a)^{-1}).$$

Thus if $|\lambda - z| < \|(\lambda - a)^{-1}\|^{-1}$, we have

$$(z-a)^{-1} = \sum_{n=0}^{\infty} (\lambda - z)^n\,(\lambda - a)^{-n-1}$$

which converges in norm and therefore lies in $\mathcal{A}$. As such, $h_\phi \equiv 0$ on an open neighbourhood of $\lambda$ and so $h_\phi \equiv 0$ on the entire component of $\rho_{\mathcal{B}}(a)$ containing $\lambda$.

Since $\Omega$ intersects every bounded component of $\rho_{\mathcal{B}}(a)$, $h_\phi \equiv 0$ on $\rho_{\mathcal{B}}(a)$. As this is true for all $\phi \in \mathcal{B}^*$ which annihilates the closed subspace $\mathcal{A}$, we conclude that $(z-a)^{-1} \in \mathcal{A}$ for all $z \in \rho_{\mathcal{B}}(a)$. That is, $\rho_{\mathcal{A}}(a) = \rho_{\mathcal{B}}(a)$, or equivalently, $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$.

□

It is worth pointing out that what we have shown is that $\mathcal{A}$ coincides with the closed algebra generated by 1, $a$, and $(z-a)^{-1}$ for all $z \in \rho_{\mathcal{B}}(a)$; in other words, the algebra generated by the rational functions with poles outside of $\sigma_{\mathcal{B}}(a)$. This algebra is often denoted by $\operatorname{Rat}(a)$ in the literature.

**3.10. Definition.** *If $\mathcal{B}$ is a Banach algebra, then a subalgebra $\mathcal{A}$ of $\mathcal{B}$ is said to be a* ***maximal abelian subalgebra*** *(or a* ***masa****) if it is commutative and it is not properly contained in any commutative subalgebra of $\mathcal{B}$.*

**3.11. Example.** We leave it to the reader to verify that if $\{e_k\}_{k=1}^n$ is an orthonormal basis for $\mathcal{H} = \mathbb{C}^n$, and if $\mathcal{D}_n$ denotes the set of diagonal matrices in $\mathbb{M}_n \simeq \mathcal{B}(\mathcal{H})$ (see Example 1.9) with respect to this basis, then $\mathcal{D}_n$ is a masa in $\mathcal{B}(\mathcal{H})$.

**3.12. Proposition.** *Let $\mathcal{B}$ be a unital Banach algebra, and suppose that $\mathcal{A}$ is a maximal abelian subalgebra of $\mathcal{A}$. Then $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$ for all $a \in \mathcal{A}$.*

**Proof.** First observe that $1 \in \mathcal{A}$, for otherwise the algebra generated by 1 and $\mathcal{A}$ is abelian and properly contains $\mathcal{A}$, a contradiction.

Clearly $\sigma_{\mathcal{B}}(a) \subseteq \sigma_{\mathcal{A}}(a)$. Suppose that $\lambda \in \rho_{\mathcal{B}}(a)$. Then for all $c \in \mathcal{A}$, $c\,(a - \lambda 1) = (a - \lambda 1)\,c$. If we let $b = (a - \lambda 1)^{-1} \in \mathcal{B}$, then multiplying this equation on the left and the right by $b$ yields $b\,c = c\,b$ for all $c \in \mathcal{A}$. Thus $b \in \mathcal{A}$, as $\mathcal{A}$ is maximal abelian. In other words, $\lambda \in \rho_{\mathcal{A}}(a)$, and we are done.

□

**3.13. The upper-semicontinuity of the spectrum.** We now turn to the question of determining in what sense the map that sends an element $a$ of a Banach algebra $\mathcal{A}$ to its spectrum $\sigma(a) \subseteq \mathbb{C}$ is continuous.

To do this, we shall first define a new metric, called the *Hausdorff metric* on $\mathbb{C}$. Our usual notion of distance between two compact sets $A$ and $B$ is

$$\mathrm{dist}(A, B) := \inf\{|a - b| \ : a \in A,\, b \in B\}.$$

Of course, if $A = \{a\}$ is a singleton , we simply write $\mathrm{dist}(a, B)$.

The problem (for our purposes) with this distance is the following. If we let $A = \{0\}$ and $B = \mathbb{D}$, the closed unit disk, then $\mathrm{dist}(A, B) = 0$. We are looking for a notion of distance that indicates how far two subsets of $\mathbb{C}$ are from being identical.

**3.14. Definition.** *Given two compact subsets $A$ and $B$ of $\mathbb{C}$, we define the* ***Hausdorff distance*** *between $A$ and $B$ to be*

$$d_{\mathrm{H}}(A, B) = \max\{\sup_{a \in A} \mathrm{dist}(a, B), \sup_{b \in B} \mathrm{dist}(b, A)\}.$$

We remark that the Hausdorff distance between $\{0\}$ and $\mathbb{D}$ is 1.

**3.15. Definition.** *Let $X$ and $Y$ be topological spaces and let $\Phi : X \to \mathcal{P}(Y)$ be a function, where $\mathcal{P}(Y)$ denotes the power set of $Y$. The mapping $\Phi$ is said to be* ***upper-semicontinuous*** *if for every $x_0 \in X$ and every neighbourhood $U$ of $\Phi(x_0)$ in $Y$, there exists a neighbourhood $V$ of $x_0$ such that $\Phi(x) \subseteq U$ for all $x \in V$.*

**3.16. Theorem. [The upper-semicontinuity of the spectrum]** *Let $\mathcal{A}$ be a Banach algebra. Then the mapping*

$$\begin{array}{rccc} \Phi : & \mathcal{A} & \to & \mathcal{P}(\mathbb{C}) \\ & a & \mapsto & \sigma(a) \end{array}$$

*is upper-semicontinuous.*

**Proof.** We must show that if $U$ is an open set in $\mathbb{C}$ containing $\sigma(a)$, then there exists $\delta > 0$ such that $\|x - a\| < \delta$ implies $\sigma(x) \subseteq U$.

Suppose otherwise. Then by choosing $\delta_n = 1/n$, $n \geq 1$, we can find $x_n \in \mathcal{A}$ with $\|x_n - a\| < \delta_n$ and $\lambda_n \in \sigma(x_n) \cap (\mathbb{C} \setminus U)$. Since $|\lambda_n| \leq \operatorname{spr}(x_n) \leq \|x_n\| \leq \|a\| + 1/n \leq \|a\| + 1$, we know that $\{\lambda_n\}_n$ is bounded, and so by the Bolzano-Weierstraß Theorem (by dropping to a subsequence if necessary), we may assume that $\lambda = \lim_n \lambda_n$ exists.

Clearly $\lambda \notin U$ as $\lambda_n \notin U$, $n \geq 1$, and $\mathbb{C} \setminus U$ is closed. Thus $\lambda - a \in \mathcal{A}^{-1}$. Since $\lambda - a = \lim_{n\to\infty} \lambda_n - x_n$ and $\mathcal{A}^{-1}$ is open, we must have $\lambda_n - x_n \in \mathcal{A}^{-1}$ for some $n \geq 1$, a contradiction.

This completes the proof.

□

It is worth noting that the map $\Phi$ above need not in general be continuous. For example, it is possible to find a sequence $(Q_n)_{n=1}^\infty$ of Hilbert space operators such that $\sigma(Q_n) = \{0\}$ for each $n \geq 1$, converging to an operator $T \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) = \{z \in \mathbb{C} : |z| \leq 1\}$.

The above theorem, while basic, is of extreme importance in the theory of approximation of Hilbert space operators. While this result in itself is sufficient for a large number of applications, sometimes we require a stronger result; one which implies the upper-semicontinuity of the "parts" or components of the spectrum.

The theorem we have in mind is due to Newburgh (see Theorem 3.18 below), and as a corollary we obtain a class of elements for which the spectrum is continuous, as opposed to just semi-continuous. We begin with the following proposition.

**3.17. Proposition.** *Let $a \in \mathcal{A}$, a unital Banach algebra, and let $\{a_n\}_{n=1}^\infty \subseteq \mathcal{A}$ be a sequence such that $a = \lim_n a_n$. Let $U \supseteq \sigma(a)$ be open and suppose*

(i) $\sigma(a_n) \subseteq U$ *for all* $n \geq 1$*;*
(ii) $f : U \to \mathbb{C}$ *is analytic.*

*Then* $\lim_n f(a_n) = f(a)$*.*

**Note:** Condition (i) can always be obtained simply by applying Theorem 3.16 and dropping to an appropriate subsequence.

**Proof.** Let $V \subseteq \mathbb{C}$ be an open subset satisfying $\sigma(a) \subseteq V \subseteq \overline{V} \subseteq U$. Without loss of generality, we may assume $\sigma(a_n) \subseteq V$ for all $n \geq 1$. Let $\Gamma$ be a finite system of closed contours satisfying

(a) $\mathrm{Ind}_\Gamma(\lambda) = 1$ for all $\lambda \in V$;
(b) $\{z \in \mathbb{C} : \mathrm{Ind}_\Gamma(z) \neq 0\} \subseteq U$.

Then $f(a)$, $f(a_n)$ are all well-defined. Moreover,

$$\begin{aligned}\|f(a) - f(a_n)\| &= \|\frac{1}{2\pi i}\int_\Gamma f(z)(z-a)^{-1} - f(z)(z-a_n)^{-1}\,dz\| \\ &= \frac{1}{2\pi}\|\int_\Gamma f(z)\left((z-a)^{-1} - (z-a_n)^{-1}\right)\,dz\| \\ &\leq \frac{1}{2\pi}\|\Gamma\|\,\|f\|_\Gamma \sup_{z\in\Gamma}\|(z-a)^{-1} - (z-a_n)^{-1}\|,\end{aligned}$$

where $\|\Gamma\|$ denotes the arclength of $\Gamma$, and $\|f\|_\Gamma = \sup\{|f(z)| : z \in \Gamma\}$.

Since inversion is continuous and $\Gamma$ is compact, the latter quantity tends to 0 as $n$ tends to infinity, and so we obtain

$$\lim_{n\to\infty}\|f(a) - f(a_n)\| = 0.$$

□

**3.18. Theorem.** **[Newburgh]** *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. Suppose that $U$ and $V$ are two disjoint open sets such that $\sigma(a) \subseteq U \cup V$ and that $\sigma(a) \cap U \neq \emptyset$. Then there exists $\delta > 0$ such that $\|x - a\| < \delta$ implies $\sigma(x) \cap U \neq \emptyset$.*

**Proof.** By the upper-semicontinuity of the spectrum, there exists $\varepsilon > 0$ such that $\|x - a\| < \varepsilon$ implies $\sigma(x) \subseteq U \cup V$. Suppose that our assertion is false. Then there exists a sequence $\{x_n\}_n \subseteq \mathcal{A}$ satisfying

(a) $\lim_{n\to\infty} x_n = a$; and
(b) $\sigma(x_n) \subseteq V$.

Consider the function $f : U \cup V \to \mathbb{C}$ defined to be 1 on $U$ and 0 on $V$. Then $f$ is clearly analytic on $U \cup V$, and so by Proposition 3.17, $\lim_n \|f(a) - f(x_n)\| = 0$. But $f(x_n) = 0$ for all $n \geq 1$, and $1 \in f(\sigma(a)) = \sigma(f(a))$. Thus $f(a) \neq 0$, a contradiction. We conclude that the assertion holds.

□

It follows that if $(a_n)_{n=1}$ is a sequence in a Banach algebra $\mathcal{A}$ converging to an element $a \in \mathcal{A}$, and if $\sigma(a_n)$ is connected for each $n \geq 1$, then $\sigma(a)$ is connected. While this is an easy consequence of Newburgh's Theorem, it is a useful one.

**3.19. Corollary.** **[Newburgh]** *Suppose that $\mathcal{A}$ is a unital Banach algebra and that $\sigma(a)$ is totally disconnected. Then the map $a \mapsto \sigma(a)$ is continuous at $a$.*

**Proof.** Let $\varepsilon > 0$. Since $\sigma(a)$ is totally disconnected, we can find a cover of $\sigma(a)$ consisting of disjoint open sets $U_1, U_2, \ldots, U_n$, each of which intersects $\sigma(a)$ non-trivially and has diameter less than $\varepsilon$. By the upper-semicontinuity of the spectrum, there exists $\delta_1 > 0$ such that $\|x - a\| < \delta_1$ implies $\sigma(x) \subseteq \cup_{j=1}^{n} U_j$.

By Newburgh's Theorem 3.18, there exists $\delta_2 > 0$ such that $\|x - a\| < \delta_2$ implies that $\sigma(x) \cap U_j \neq \emptyset$, $1 \leq j \leq n$. Thus the Hausdorff distance

$$d_{\mathrm{H}}(\sigma(a), \sigma(x)) < \varepsilon$$

for all $x \in \mathcal{A}$, $\|x - a\| < \min(\delta_1, \delta_2)$, implying that the map $a \mapsto \sigma(a)$ is indeed continuous at $a$.

$\square$

*Nature abhors a vacuum cleaner salesman.*

## Notes for Chapter Two

The examples of Banach algebras given in this Chapter are but a tiny fraction of those which are of interest in the theory of Operator Algebras. One particular class which we shall be examining in much greater detail is that of $C^*$-algebras. Even in this subclass there is a plethora of examples, including $\mathcal{B}(\mathcal{H})$ itself, *von Neumann algebras*, *UHF-algebras* and more generally *AF-algebras*, the *irrational rotation algebras*, *Toeplitz algebras*, *Bunce-Deddens algebras*, and many more.

The Riesz-Dunford functional calculus made its first appearance in a paper of Riesz [**Rie11**]. In his case, he studied only compact operators acting on a Hilbert space $\mathcal{H}$, and then the only functions he considered were the characteristic functions of an isolated point of the spectrum of the given operator. Indeed, alongside a number of related results by a number of authors, it was Dunford who presented the work in its most complete form. Recently, Conway and Morrel [**CM87**] and again Conway, Herrero and Morrel [**CHM89**] have considered what might be termed a "converse" to the Riesz-Dunford functional calculus.

As we have seen, in the Riesz-Dunford functional calculus, one begins with an element $a$ of a unital Banach algebra $\mathcal{A}$ and considers the class $\mathcal{F}(a)$ of functions $f$ which are analytic on some open neighbourhood of the spectrum of $a$. One then obtains an algebra homomorphism

$$\begin{array}{rccc} \tau : & \mathcal{F}(a) & \to & \mathcal{A} \\ & f & \mapsto & f(a). \end{array}$$

In the Conway, Herrero and Morrel approach, one begins with a subset $\Delta$ of the complex plane $\mathbb{C}$, and the class $\mathcal{S}(\Delta)$ of operators $T$ acting on a separable Hilbert space $\mathcal{H}$ and satisfying $\sigma(T) \subseteq \Delta$.

The aim of their program is to determine $f(\mathcal{S}(\Delta)) = \{f(T) : T \in \mathcal{S}(\Delta)\}$, where $f : \Delta \to \mathbb{C}$ is a *fixed* analytic function. As an example, suppose $\Delta = \mathbb{D}$ so that $\mathcal{S}(\Delta)$ contains an appropriate scalar multiple of every bounded linear operator on $\mathcal{H}$. If $f(z) = z^2$, then $\mathbb{C}f(\mathcal{S}(\Delta))$ coincides with the set of all operators possessing a square root. However, as noted in the Conway and Morrel paper [**CM87**], this proves beyond the scope of present day operator theory, even for such simple functions as $f(z) = z^p$ or $f(z) = e^z$. Because of this, they study the norm closure in $\mathcal{B}(\mathcal{H})$ of the set $\mathcal{S}(\Delta)$. This allows them to employ the elaborate machinery of the *Similarity Theorem* for Hilbert space operators, developed by Apostol, Herrero, and Voiculescu [**AFHV84**]. This theorem and its many consequences detail the structure of the closure of many similarity invariant subsets of $\mathcal{B}(\mathcal{H})$. In particular, much of the analysis may be applied to $\overline{f(\mathcal{S}(\Delta))}$, which is itself similarity invariant.

Examples of results found in [**CHM89**] are:

- If $\overline{f(\mathcal{S}(\Delta))} = \mathcal{B}(\mathcal{H})$, then $f(\mathcal{S}(\Delta)) = \mathcal{B}(\mathcal{H})$.
- If $\Delta = \mathbb{C}$ and $f(z) = z \sin z$ or $f(z) = \Pi_{n=1}^{\infty}(1 - a/n^2)$, then $f(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H})$.
- Let $\Delta = \{z \in \mathbb{C} : z \neq 2\}$ and $f(z) = z^2(2-z)$. If $U$ is the unilateral forward shift operator (cf. Definition 3.10), then $U \oplus U \in f(\mathcal{S}(\Delta))$, but $U \notin \overline{f(\mathcal{S}(\Delta))}$. On the other hand, $U \oplus 0 \in f(\mathcal{S}(\Delta))$.

Analysis of the spectrum and the functional calculus are key ingredients in Single Operator Theory, where one is often interested in studying a class of operators which may or may not possess an algebraic structure. For instance, on may begin with the set of *algebraic operators* on $\mathcal{H}$,

$$\mathrm{Alg}(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : p(T) = 0 \text{ for some polynomial } p\}.$$

The description of the closure of this set was obtained by Dan Voiculescu [**Voi74**] in terms of spectral conditions. More precisely, he showed that

$$\overline{\mathrm{Alg}(\mathcal{H})} = \{T \in \mathcal{B}(\mathcal{H}) : \dim \ker(T - \lambda) = \operatorname{codim} \operatorname{ran}(T - \lambda) \ \forall \lambda \in \rho_{\mathrm{sF}}(T)\}$$

Here, the $\rho_{\mathrm{sF}}(T)$ denotes the *semi-Fredholm domain* of $T$. It is defined as the set of complex numbers for which the range of $T$ is closed, and at least one of $\dim \ker T$ or $\operatorname{codim} \operatorname{ran} T$ is finite.

Another important notion of relative spectrum is that of the spectrum of the image of an element in a quotient algebra. As we have seen in Proposition 1.16, if $\mathcal{K}$ is a closed ideal of a Banach algebra $\mathcal{A}$, then $\mathcal{A}/\mathcal{K}$ is a Banach algebra. Letting $\pi$ denote the canonical projection map, it is clear that if $a \in \mathcal{A}$, then $\sigma_{\mathcal{A}/\mathcal{K}}(\pi(a)) \subseteq \sigma_{\mathcal{A}}(a)$.

One particular instance of quotient algebras deserves special mention. Recall from Example 1.17 that the quotient algebra $\mathcal{Q}(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$ by $\mathcal{K}(\mathcal{H})$ is referred to as the **Calkin algebra**. If $T \in \mathcal{B}(\mathcal{H})$, and $\pi$ is the canonical homomorphism from $\mathcal{B}(\mathcal{H})$ to $\mathcal{Q}(\mathcal{H})$, then the spectrum of $\pi(T)$ is called the **essential spectrum** of $T$, and is often denoted by $\sigma_e(T)$. In this connection, two of the most important results concerning the spectrum are:

**Theorem. [Putnam-Schechter]** *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Suppose $\lambda \in \partial(\sigma(T))$. Then either $\lambda$ is isolated, or $\lambda \in \sigma_e(T)$.*

**Corollary.** *Let $T \in \mathcal{B}(\mathcal{H})$. Then $\sigma(T) = \sigma_e(T) \cup \Omega$, where $\Omega$ consists of some bounded components of the resolvent of $\pi(T)$, and a sequence of isolated points in $\rho(\pi(T))$ converging to $\sigma_e(T)$.*

Proofs of the above results appear in Appendix A.

There are other results concerning the continuity of the spectrum and of the spectral radius of Banach algebra elements. In particular, Murphy [**Mur81**] has shown the following:

Suppose that $K \subseteq \mathbb{C}$ is compact, and $\mathcal{A}$ is a unital Banach algebra. Define $\alpha(K) = \sup\{\inf_{\lambda \in C} |\lambda| : C \text{ a component of } K\}$ and $r(K) = \sup_{\lambda \in K} |\lambda|$.

Then $\alpha(K) \leq r(K)$. Let $D$ be a *diagonal* operator on $\mathcal{B}(\ell^2(\mathbb{N}))$; that is, if $\{e_n\}_{n=1}^{\infty}$ denotes an orthonormal basis for $\ell^2(\mathbb{N})$, then $De_n = d_n e_n$ for some bounded sequence $\{d_n\}_{n=1}^{\infty}$ of complex numbers.

**Proposition.** [**Murphy**] *The following statements are equivalent:*

(i) *Every element of every unital Banach algebra $\mathcal{A}$ with spectrum $K$ is a point of continuity of the function $a \mapsto \sigma_{\mathcal{A}}(a)$;*
(ii) $\alpha(K) = r(K)$;
(iii) *$D$ is a point of continuity of the function $T \mapsto \sigma_{\mathcal{B}(\mathcal{H})}(T)$.*

As for the spectral radius, let

$$K_0 = \{\lambda \in K : \text{ the component of } \lambda \text{ in } K \text{ is } \{\lambda\}\}.$$

Thus $K = K_0$ if and only if $K_0$ is totally disconnected.

**Proposition.** [**Murphy**] *The following statements are equivalent:*

(i) *Every element of every unital Banach algebra $\mathcal{A}$ with spectrum $K$ is a point of continuity of the function $a \mapsto \operatorname{spr}_{\mathcal{A}}(a)$;*
(ii) $K = \overline{K_0}$;
(iii) *$D$ is a point of continuity of the function $T \mapsto \operatorname{spr}_{\mathcal{B}(\mathcal{H})}(T)$;*
(iv) *For each $\varepsilon > 0$ and for each $\lambda \in K$, $B(\lambda, \varepsilon) = \{\mu \in \mathbb{C} : |\mu - \lambda| < \varepsilon\}$ contains a component of $K$.*

CHAPTER 3

# Operator Algebras

*Mediocrity knows nothing higher than itself, but talent instantly recognizes genius*

**Arthur Conan Doyle**

## 1. The algebra of Banach space operators

**1.1.** As we have already seen there are myriads of examples of Banach algebras. We begin our study with a very important subclass, namely the class of *operator algebras*. We shall divide our analysis into the study of operators on general Banach spaces, and later we shall turn our attention to Hilbert space operators. The loss of generality in specifying the underlying space is made up for in the strength of the results we can obtain. We begin with a definition.

**1.2. Definition.** *Let $\mathfrak{X}$ be a Banach space. Then $\mathcal{B}(\mathfrak{X})$ consists of those linear maps $T : \mathfrak{X} \to \mathfrak{X}$ which are continuous in the norm topology. Given $T \in \mathcal{B}(\mathfrak{X})$, we define the* ***norm*** *of $T$ to be*

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1\}.$$

It follows immediately from the definition that $\|Tx\| \leq \|T\| \, \|x\|$ for all $x \in \mathfrak{X}$, and that $\|T\|$ is the smallest non-negative number with this property.

**1.3. Remark.** We assume that the reader is familiar with the fact that $\mathcal{B}(\mathfrak{X})$ is a Banach space. To verify that it is indeed a Banach algebra, we need only verify that the operator norm is submultiplicative, that is, that $\|AB\| \leq \|A\| \, \|B\|$ for all operators $A$ and $B$.

But

$$\begin{aligned} \|AB\| &= \sup\{\|ABx\| : \|x\| = 1\} \\ &\leq \sup\{\|A\| \, \|Bx\| : \|x\| = 1\} \\ &\leq \sup\{\|A\| \, \|B\| \, \|x\| : \|x\| = 1\} \\ &= \|A\| \, \|B\|. \end{aligned}$$

Since $\mathcal{B}(\mathfrak{X})$ is a Banach algebra, all of the results from Chapter Two apply. In particular, for $T \in \mathcal{B}(\mathfrak{X})$, the spectrum of $T$ is a non-empty, compact subset of $\mathbb{C}$. The function $R(\lambda, T) = (\lambda I - T)^{-1}$ is analytic on

$\rho(T)$, and we can (and do!) define the operator $f(T)$ when $f$ is analytic on a neighbourhood of $\sigma(T)$.

**1.4. Proposition.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces, and $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. The following are equivalent:*

(i) *$T$ is invertible.*
(ii) *$T$ is a bijection.*
(iii) *$T$ is bounded below and has dense range.*

**Proof.**

(i) $\Rightarrow$ (iii) Suppose $T$ is invertible. Let $x \in \mathfrak{X}$. Then $x = T^{-1}Tx$, and so $\|x\| \le \|T^{-1}\| \, \|Tx\|$, i.e. $\|Tx\| \ge \|T^{-1}\|^{-1} \, \|x\|$ and $T$ is bounded below. Since $T$ is onto, its range is trivially dense.

(iii) $\Rightarrow$ (ii) Suppose $T$ is bounded below by, say, $\delta > 0$. We shall first show that in this case, the range of $T$ is closed.

Indeed, suppose that there exists a sequence $y_n = Tx_n, \ n \ge 1$ and $y$ such that $\lim_{n\to\infty} y_n = y$. Then $\delta \|x_m - x_n\| \le \|y_m - y_n\|$, forcing $\{x_n\}_{n=1}^\infty$ to be a Cauchy sequence. Let $x = \lim_{n\to\infty} x_n$. By the continuity of $T$, we have $Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_n = y$. We have shown that $y \in \operatorname{ran} T$, and hence that $\operatorname{ran} T$ is closed.

It follows that if $T$ has dense range, as per our hypothesis, then $T$ is surjective.

As well, suppose that $x \in \ker T$. Then $\delta\|x\| \le \|Tx\| = 0$, forcing $x$ to be zero, and $T$ to be injective.

(ii) $\Rightarrow$ (i) Suppose that $T$ is a bijection. The Open Mapping Theorem 2.5 then implies that the inverse image map $T^{-1}$ is continuous, and thus that $T$ is invertible.

□

In general, for $T \in \mathcal{B}(\mathfrak{X})$, there are many subclassifications of the spectrum of $T$. Condition (ii) above leads to the following obvious ones.

**1.5. Definition.** *Let $\mathfrak{X}$ be a Banach space and $T \in \mathcal{B}(\mathfrak{X})$. Then the* ***point spectrum*** *of $T$ is*

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not injective}\}.$$

*These are the eigenvalues of $T$. The* ***approximate point spectrum*** *of $T$ is*

$$\sigma_a(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded below}\}.$$

*The* ***compression spectrum*** *is*

$$\sigma_c(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ does not have dense range}\}.$$

**1.6. Remarks.**

- If $\dim \mathfrak{X} < \infty$, then $\sigma_p(T) = \sigma_a(T) = \sigma_c(T) = \sigma(T)$.
- If $\lambda \in \sigma_a(T)$, then for all $n \geq 1$, there exists $0 \neq x_n \in \mathfrak{X}$ such that $\|(T - \lambda I)x_n\| \leq \frac{1}{n}\|x_n\|$. Let $y_n = x_n/\|x_n\|$. Then $\|y_n\| = 1$ for all $n \geq 1$, and $(T - \lambda I)y_n \to 0$.
- $\sigma(T) = \sigma_a(T) \cup \sigma_c(T)$ and in general, $\sigma_p(T) \subseteq \sigma_a(T)$.

**1.7. Example.** Let $\mathfrak{X} = \mathcal{C}([0,1])$. Let $f \in \mathfrak{X}$, and consider the bounded linear operator $M_f$ given by

$$\begin{array}{rlcl} M_f : & \mathcal{C}([0,1]) & \to & \mathcal{C}([0,1]) \\ & g & \mapsto & fg. \end{array}$$

$M_f$ is referred to as "multiplication by $f$". We leave it to the reader to verify that (i) $\lambda I - M_f = M_{\lambda 1 - f}$ for all $\lambda \in \mathbb{C}$, and (ii) $\|M_f\| = \|f\|$.
**Claim:** $\sigma(M_f) = \text{ran} f = f([0,1])$.

For if $\lambda \notin f([0,1])$, then $h = (\lambda 1 - f)^{-1}$ is continuous and $M_h(\lambda I - M_f) = M_h M_{\lambda 1 - f} = M_{h(\lambda 1 - f)} = M_1 = I = (\lambda I - M_f)M_h$. In particular, $\lambda \notin \sigma(M_f)$.

Now suppose $\lambda = f(t_0)$ for some $t_0 \in [0,1]$. Take

$$g_n(t) = \begin{cases} 0 & \text{if } |t - t_0| > \frac{1}{n}, \\ 1 - n|t - t_0| & \text{if } t \in [t_0 - \frac{1}{n}, t_0 + \frac{1}{n}]. \end{cases}$$

Let $\epsilon > 0$ and choose $\delta > 0$ such that $|f(t) - \lambda| < \epsilon$ for all $t \in (t_0 - \delta, t_0 + \delta)$. Then, when $\frac{1}{n} < \delta$,

$$\begin{aligned} \|M_{\lambda 1 - f} g_n\| &= \|(\lambda 1 - f)g_n\| \\ &\leq \sup_{|t - t_0| < \frac{1}{n}} |\lambda g_n(t) - f(t)g_n(t)| \\ &\leq \sup_{|t - t_0| < \delta} |\lambda 1 - f(t)|\, |g_n(t)| \\ &\leq \epsilon \|g_n(t)\|_\infty \\ &= \epsilon. \end{aligned}$$

Since $\|g_n\|_\infty = 1$ for $n \geq 1$, we see that $\lambda I - M_f$ is not bounded below. In other words, $\lambda \in \sigma_a(M_f)$.

Moreover, if $\lambda = f(t_0)$ for some $t_0 \in [0,1]$, then

$$\begin{aligned} \|1 - (\lambda I - M_f)g\| &= \|1 - (\lambda 1 - f)g\| \\ &\geq |1(t_0) - (\lambda - f(t_0))g(t_0)| \\ &= 1. \end{aligned}$$

Thus the range of $\lambda I - M_f$ is not dense; i.e. $\lambda \in \sigma_c(M_f)$.

Suppose now that $\lambda \in \sigma_p(M_f)$. Then $(\lambda I - M_f)g = M_{\lambda 1 - f} g = 0$ for some non-zero continuous function $g$. It follows that

$$(\lambda - f(t))g(t) = 0 \text{ for all } t \in [0,1].$$

Since $g \neq 0$, we can choose $t_0 \in [0,1]$ such that $g(t_0) \neq 0$. Since $g$ is continuous, there exists an open neighbourhood $U$ of $t_0$ such that $g(t) \neq 0$ for all $t \in U$. But then $\lambda - f(t) = 0$ for all $t \in U$. We conclude that if $\lambda \in \sigma_p(M_f)$, then $f$ must be constant on some interval. We leave it to the reader to check that the converse is also true.

In particular, if we choose $f(x) = x$ for all $x \in [0,1]$ and write $M_x$ for $M_f$ (as is usually done), then we see that

$$M_x : \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$$

has no eigenvalues!

**1.8. Example.** Let $\mathfrak{X} = \mathcal{C}([0,1])$, and consider $V \in \mathcal{B}(\mathfrak{X})$ given by

$$(Vf)(x) = \int_0^1 k(x,y)f(y)dy,$$

where

$$k(x,y) = \begin{cases} 0 & \text{if } x < y, \\ 1 & \text{if } x \geq y. \end{cases}$$

Then $(Vf)(x) = \int_0^x f(y)dy$. This is an example of a *Volterra operator*. The function $k(x,y)$ is referred to as the *kernel* of the integral operator. This should not be confused with the notion of a null space, also referred to as a kernel.

We wish to determine the spectrum of $V$. Now

$$\begin{aligned}
(V^2 f)(x) &= (V(Vf))(x) \\
&= \int_0^1 k(x,t)\,(Vf)(t)dt \\
&= \int_0^1 k(x,t) \int_0^1 k(t,y)\,f(y)dy\,dt \\
&= \int_0^1 f(y) \int_0^1 k(x,t)\,k(t,y)dt\,dy \\
&= \int_0^1 f(y)\,k_2(x,y)dy,
\end{aligned}$$

where $k_2(x,y) = \int_0^1 k(x,t)\,k(t,y)dt$ is a new kernel. Note that

$$\begin{aligned} |k_2(x,y)| &= |\int_0^1 k(x,t)\,k(t,y)dt| \\ &= |\int_y^x k(x,t)\,k(t,y)dt| \\ &= (x-y) \text{ for } x>y, \end{aligned}$$

while for $x<y$, $k_2(x,y)=0$.

In general, since $x-y<1-0=1$, we get

$$\begin{aligned} (V^n f)(x) &= \int_0^1 f(y)\,k_n(x,y)dy \\ k_n(x,y) &= \int_0^1 k(x,t)\,k_{n-1}(t,y)dt \\ |k_n(x,y)| &\le \frac{1}{(n-1)!}(x-y)^{n-1} \le \frac{1}{(n-1)!}. \end{aligned}$$

Thus if we take $\|f\|\le 1$, then

$$\begin{aligned} \|V^n\| &= \sup_{\|f\|=1} \|V^n f\| \\ &= \sup_{\|f\|=1} \|\int_0^1 f(y)\,k_n(x,y)dy\| \\ &\le \sup_{\|f\|=1} \|f\|\,\|k_n(x,y)\| \\ &\le 1/(n-1)!. \end{aligned}$$

Thus $\operatorname{spr}(V) = \lim_{n\to\infty}\|V^n\|^{\frac{1}{n}} \le \lim_{n\to\infty}(1/n!)^{\frac{1}{n}} = 0$. In particular, $\sigma(V)=\{0\}$.

Now let $f_n(x)=x^n,\ 0\le x\le 1$. Then $\|f_n\|_\infty = 1$. Also,

$$\begin{aligned} (Vf_n)(x) &= \int_0^x f_n(y)dy \\ &= \int_0^x y^n dy \\ &= \frac{y^{n+1}}{(n+1)}|_0^x \\ &= \frac{x^{n+1}}{n+1}. \end{aligned}$$

As such, $\|Vf_n\| = \frac{1}{n+1}$, and so $V$ is not bounded below. Hence $0\in\sigma_a(V)$.

Also, let $f\in\mathcal{C}([0,1])$ be arbitrary. Then $(Vf)(0)=0$, and so

$$\|1-Vf\|_\infty \ge |1(0)-Vf(0)| = 1.$$

Thus $0\in\sigma_c(V)$. It is a standard result that $0\notin\sigma_p(V)$.

**1.9. Definition.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. Let $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. We shall now define an operator $T^* \in \mathcal{B}(\mathfrak{Y}^*, \mathfrak{X}^*)$, called the **Banach space adjoint** of $T$.*

First, for $x^* \in \mathfrak{X}^*$, we adopt the notation $< x, x^* >= x^*(x)$. Then for $y^* \in \mathfrak{Y}^*$, define $T^*$ so that

$$< x, T^*(y^*) > = < Tx, y^* > .$$

That is, $(T^*y^*)(x) = y^*(Tx)$ for all $x \in \mathfrak{X},\ y^* \in \mathfrak{Y}^*$. It is not hard to verify that $T^*$ is linear.

**1.10. Proposition.** *Let $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ be Banach spaces, $S, T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$, and let $R \in \mathcal{B}(\mathfrak{Y}, \mathfrak{Z})$. Then*

(i) *for all $\alpha$, $\beta \in \mathbb{C}$, we have $(\alpha S + \beta T)^* = \alpha S^* + \beta T^*$;*
(ii) *$(R \circ T)^* = T^* \circ R^*$.*

**Proof.** Let $x \in \mathfrak{X},\ y^* \in \mathfrak{Y}^*$, and $z^* \in \mathfrak{Z}^*$. Then

(i)

$$\begin{aligned} < x, (\alpha S + \beta T)^* y^* > &= < (\alpha S + \beta T)x,\, y^* > \\ &= y^*((\alpha S + \beta T)x) \\ &= \alpha\, y^*(Sx) + \beta\, y^*(Tx) \\ &= \alpha < Sx,\, y^* > + \beta < Tx,\, y^* > \\ &= \alpha < x,\, S^*y^* > + \beta < x, T^*y^* > . \end{aligned}$$

Since this is true for all $x \in \mathfrak{X}$ and $y^* \in \mathfrak{Y}^*$, we conclude that $(\alpha S + \beta T)^* = \alpha S^* + \beta T^*$.

(ii)

$$\begin{aligned} < x, (R \circ T)^* z^* > &= < (R \circ T)x, z^* > \\ &= < R(Tx), z^* > \\ &= < Tx,\, R^* z^* > \\ &= < x, T^*(R^* z^*) > \\ &= < T^* \circ R^* z^* > . \end{aligned}$$

Again, this shows that $(R \circ T)^* = T^* \circ R^*$.

□

**1.11. Theorem.** *Let $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$, where $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces. Then $\|T^*\| = \|T\|$.*

**Proof.**

For any $y^* \in \mathfrak{Y}^*$, we have

$$\begin{aligned} \|T^* y^*\| &= \sup\{|T^* y^*(x)| : x \in \mathfrak{X},\, \|x\| = 1\} \\ &= \sup\{|y^*(Tx)| : x \in \mathfrak{X},\, \|x\| = 1\} \\ &\leq \sup\{\|y^*\|\,\|Tx\| : x \in \mathfrak{X},\, \|x\| = 1\} \\ &= \|y^*\|\,\|T\|. \end{aligned}$$

Thus we see that $\|T^*\| \le \|T\|$.

Next, let $x \in \mathfrak{X}$. By the Hahn-Banach Theorem, we can choose $y^* \in \mathfrak{Y}^*$ such that $y^*(Tx) = \|Tx\|$ and $\|y^*\| = 1$. Then

$$\begin{aligned} \|Tx\| &= y^*(Tx) \\ &= < Tx, y^* > \\ &= < x, T^*y^* > \\ &= (T^*y^*)(x) \\ &\le \|T^*y^*\| \, \|x\| \\ &\le \|T^*\| \|x\|. \end{aligned}$$

Thus $\|T\| \le \|T^*\|$.

Combining this with the previous estimate, we have that $\|T^*\| = \|T\|$.

□

**1.12. Proposition.** *Let $\mathfrak{X} = \mathbb{C}^n$ and $A \in \mathcal{B}(\mathfrak{X}) \simeq \mathbb{M}_n$. Then the matrix of the Banach space adjoint $A^*$ of $A$ with respect to the dual basis coincides with $A^t$, the transpose of $A$.*

**Proof.** Recall that $\mathfrak{X}^* \simeq \mathfrak{X}$. We then let $\{e_i\}_{i=1}^n$ be a basis for $\mathfrak{X}$ and let $\{f_j\}_{j=1}^n$ be the corresponding dual basis; that is, $f_j(e_i) = \delta_{ij}$, where $\delta_{ij}$ is the Dirac delta function. Let $x \in \mathfrak{X}$. Define $\lambda_j = f_j(x)$.

Writing the matrix of $A \in \mathcal{B}(\mathfrak{X})$ as $[a_{ij}]$, we have

$$Ae_j = [a_{ij}] \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ 1 \\ 0 \\ . \\ . \\ 0 \end{bmatrix} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ a_{j-1\,j} \\ a_{jj} \\ a_{j+1\,j} \\ . \\ . \\ a_{nj} \end{bmatrix} = \sum_{k=1}^n a_{kj} e_k.$$

Thus $a_{ij} = f_i(Ae_j)$.

Now $A^* \in \mathcal{B}(\mathfrak{X}^*) \simeq \mathbb{M}_n$, and so we can also write the matrix for $A^*$ with respect to $\{f_j\}_{j=1}^n$. As above, we have

$$A^* f_j = \sum_{k=1}^n \alpha_{kj} f_k.$$

Thus

$$\alpha_{ij} = (A^* f_j)(e_i) = f_j(Ae_i) = a_{ji}.$$

In particular, the matrix for $A^*$ with respect to $\{f_j\}_{j=1}^n$ is simply the transpose of the matrix for $A$ with respect to $\{e_j\}_{j=1}^n$.

□

**1.13. Proposition.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces and let $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. Then $T$ is invertible if and only if $T^*$ is invertible.*
**Proof.** First assume that $T$ is invertible, i.e., that $T^{-1} \in \mathcal{B}(\mathfrak{Y}, \mathfrak{X})$. Then $I_{\mathfrak{X}^*} = (I_{\mathfrak{X}})^* = (T^{-1} \circ T)^* = T^* \circ (T^{-1})^*$.

Also, $I_{\mathfrak{Y}^*} = (I_{\mathfrak{Y}})^* = (T \circ T^{-1})^* = (T^{-1})^* \circ T^*$. Thus $T^*$ is invertible and $(T^*)^{-1} = (T^{-1})^*$.

Now assume that $T^*$ is invertible. Then ran$T$ is dense, for otherwise by the Hahn-Banach Theorem we can take $y^* \in \mathfrak{Y}^*$ such that $\|y^*\| = 1$ and $y^*|_{\overline{(\mathrm{ran} T)}} = 0$. Then

$$(T^* y^*)(x) = y^*(Tx) = 0$$

for all $x \in \mathfrak{X}$. Thus $T^* y^* = 0$ but $y^* \neq 0$, implying that $T^*$ is not injective, a contradiction.

Moreover, $T$ is bounded below. For consider: $T^*$ invertible implies that $T^{**} = (T^*)^*$ is invertible from above. Thus $T^{**}$ is bounded below. Recall that $\mathfrak{X}$ embeds isometrically isomorphically into $\mathfrak{X}^{**}$ via the map

$$\begin{array}{ccc} \mathfrak{X} & \simeq & \widehat{\mathfrak{X}} \subseteq \mathfrak{X}^{**} \\ x & \mapsto & \hat{x} \end{array}$$

where $\hat{x}(x^*) = x^*(x)$ for all $x^* \in \mathfrak{X}^*$. (Recall that $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ and that $T^{**} \in \mathcal{B}(\mathfrak{X}^{**}, \mathfrak{Y}^{**})$.)

Now $T^{**}(\hat{x}) \in \mathfrak{Y}^{**}$, and

$$\begin{aligned} ((T^*)^*(\hat{x}))(y^*) &= \hat{x}(T^* y^*) \\ &= (T^* y^*)(x) \\ &= y^*(Tx) \text{ for all } y^* \in \mathfrak{Y}^*. \end{aligned}$$

Thus

$$\sup\{|(T^{**}\hat{x})(y^*)| : y^* \in \mathfrak{Y}^*, \ \|y^*\| = 1\} = \sup\{|y^*(Tx)| : y^* \in \mathfrak{Y}^*, \ \|y^*\| = 1\}.$$

In other words, $\|T^{**}\hat{x}\| = \|Tx\|$. Since $T^{**}$ is bounded below, say by $\delta > 0$,

$$\delta\|x\| = \delta\|\hat{x}\| \leq \|T^{**}\hat{x}\| = \|Tx\|.$$

In other words, $T$ is also bounded below.

Finally, $T$ bounded below and ran $T$ dense together imply that $T$ is invertible, by Proposition 1.4.

$\square$

**1.14. Corollary.** *Let $\mathfrak{X}$ be a Banach space and $T \in \mathcal{B}(\mathfrak{X})$. Then $\sigma(T) = \sigma(T^*)$.*

**1.15. Definition.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces, and $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. Let $B_1$ denote the unit ball of $\mathfrak{X}$, so that $B_1 = \{x \in \mathfrak{X} : \|x\| \le 1\}$. Then $T$ is said to be* ***compact*** *if $\overline{T(B_1)}$ is compact in $\mathfrak{Y}$. The set of compact operators from $\mathfrak{X}$ to $\mathfrak{Y}$ is denoted by $\mathcal{K}(\mathfrak{X}, \mathfrak{Y})$, and if $\mathfrak{Y} = \mathfrak{X}$, we simply write $\mathcal{K}(\mathfrak{X})$.*

Recall that a subset $K$ of a metric space $L$ is said to be **totally bounded** if for every $\epsilon > 0$ there exists a finite cover $\{B_\epsilon(y_i)\}_{i=1}^n$ of $K$ with $y_i \in K$, $1 \le i \le n$, where $B_\epsilon(y_i) = \{z \in L : \text{dist}\,(z, y_i) < \epsilon\}$..

**1.16. Proposition.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces, and $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. The following are equivalent:*

(i) *$T$ is compact;*
(ii) *$\overline{T(F)}$ is compact in $\mathfrak{Y}$ for all bounded subsets $F$ of $\mathfrak{X}$;*
(iii) *If $\{x_n\}_{n=1}^\infty$ is a bounded sequence in $\mathfrak{X}$, then $\{Tx_n\}_{n=1}^\infty$ has a convergent subsequence;*
(iv) *$T(B_1)$ is totally bounded.*

**Proof.** Exercise.

□

**1.17. Theorem.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. Then $\mathcal{K}(\mathfrak{X}, \mathfrak{Y})$ is a closed subspace of $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$.*

**Proof.** Let $\alpha, \beta \in \mathbb{C}$ and let $K_1, K_2 \in \mathcal{K}(\mathfrak{X}, \mathfrak{Y})$. Let $\{x_n\}_{n=1}^\infty$ be a bounded sequence in $\mathfrak{X}$. Then $K_1$ generates a convergent subsequence, say $\{K_1(x_{n(j)})\}_{j=1}^\infty$. Similarly, $K_2$ generates a convergent subsequence from $\{x_{n(j)}\}_{j=1}^\infty$, say $\{K_2(x_{n(j(i))})\}_{i=1}^\infty$.

Then $\{(\alpha K_1 + \beta K_2)(x_{n(j(i))})\}_{i=1}^\infty$ is a convergent subsequence in $\mathfrak{Y}$. From part (iii) of Proposition 1.16, $\alpha K_1 + \beta K_2 \in \mathcal{K}(\mathfrak{X}, \mathfrak{Y})$.

Now we show that $\mathcal{K}(\mathfrak{X}, \mathfrak{Y})$ is closed. Suppose $K_n \in \mathcal{K}(\mathfrak{X}, \mathfrak{Y})$ for $n \ge 1$ and $\lim_{n\to\infty} K_n = K \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. Letting $B_1$ denote the unit ball of $\mathfrak{X}$, we show that $K(B_1)$ is totally bounded. First let $\epsilon > 0$, and choose $N > 0$ such that $\|K_N - K\| < \epsilon/3$.

Since $K_N(B_1)$ is totally bounded, we can find $\{y_i = K_N(x_i)\}_{i=1}^M$ such that $\{B_{\epsilon/3}(y_i)\}_{i=1}^M$ is a finite cover of $K_N(B_1)$. Thus for all $x \in B_1$, $\|K_N(x) - K_N(x_j)\| < \epsilon/3$ for some $1 \le j = j(x) \le M$. Then

$$\begin{aligned}
\|K(x) - K(x_j)\| &= \|K(x) - K_N(x) + K_N(x) - K_N(x_j) + \\
&\qquad K_N(x_j) - K(x_j)\| \\
&\le \|K - K_N\|\,\|x\| + \|K_N(x) - K_N(x_j)\| + \\
&\qquad \|K_N - K\|\,\|x_j\| \\
&\le (\epsilon/3) + (\epsilon/3) + (\epsilon/3) \\
&= \epsilon.
\end{aligned}$$

Thus $K(B_1)$ is totally bounded and so $K \in \mathcal{K}(\mathfrak{X}, \mathfrak{Y})$.

□

**1.18. Theorem.** *Let $\mathfrak{W}$, $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ be Banach spaces. Suppose $R \in \mathcal{B}(\mathfrak{W}, \mathfrak{X})$, $K \in \mathcal{K}(\mathfrak{X}, \mathfrak{Y})$, and $T \in \mathcal{B}(\mathfrak{Y}, \mathfrak{Z})$. Then $TK \in \mathcal{K}(\mathfrak{X}, \mathfrak{Z})$ and $KR \in \mathcal{K}(\mathfrak{W}, \mathfrak{Y})$.*

**Proof.** Let $B_1$ denote the unit ball of $\mathfrak{X}$. Then

$$\begin{aligned} \overline{T \circ K(B_1)} &= \overline{T(K(B_1))} \\ &\subseteq \overline{T(\overline{K(B_1)})}. \end{aligned}$$

Since $\overline{K(B_1)}$ is compact and $T$ is continuous, $\overline{T \circ K(B_1)}$ is a closed subset of the compact set $\overline{T(\overline{K(B_1)})} = \overline{T(K(B_1))}$, and so it is compact as well. Thus $TK \in \mathcal{K}(\mathfrak{X}, \mathfrak{Z})$.

Now if $D_1$ is the unit ball of $\mathfrak{W}$, then

$$\overline{KR(D_1)} = \overline{K(R(D_1))};$$

but $R(D_1)$ is bounded since $R$ is, and so by Proposition 1.16, $\overline{KR(D_1)}$ is compact. Thus $KR \in \mathcal{K}(\mathfrak{W}, \mathfrak{Y})$.

□

**1.19. Corollary.** *If $\mathfrak{X}$ is a Banach space, then $\mathcal{K}(\mathfrak{X})$ is a closed, two-sided ideal of $\mathcal{B}(\mathfrak{X})$.*

*"For every auction, there is an equal and opposite re-auction."*
***Karl Marx***

## 2. The Fredholm Alternative

**2.1.** In this section we shall study compact operators acting on Banach spaces, and see to what extent their behaviour mirrors that of finite dimensional matrices.

**2.2. Proposition.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces and assume that $K \in \mathcal{K}(\mathfrak{X}, \mathfrak{Y})$. Then $K(\mathfrak{X})$ is closed in $\mathfrak{Y}$ if and only if* $\dim K(\mathfrak{X})$ *is finite.*
**Proof.** $K(\mathfrak{X})$ is easily seen to be a submanifold of $\mathfrak{Y}$. Since finite dimensional manifolds are always closed, we find that $\dim K(\mathfrak{X}) < \infty$ implies $K(\mathfrak{X})$ is closed.

Now assume that $K(\mathfrak{X})$ is closed. Then $K(\mathfrak{X})$ is a Banach space and the map

$$\begin{array}{rccc} K_0: & \mathfrak{X} & \to & K(\mathfrak{X}) \\ & x & \mapsto & Kx \end{array}$$

is a surjection. By the Open Mapping Theorem, 2.5, it is also an open map. In particular, if $B_1 = \{x \in \mathfrak{X} : \|x\| \leq 1\}$ is the unit ball of $\mathfrak{X}$, then $K_0(\operatorname{int} B_1)$ is open in $K(\mathfrak{X})$ and $0 \in K_0(\operatorname{int} B_1)$. Let $G$ be an open ball in $K(\mathfrak{X})$ centred at 0 and contained in $K_0(\operatorname{int} B_1)$. Then $\overline{K_0(B_1)} = \overline{K(B_1)}$ is compact, hence closed, and also contains $\overline{G}$. Thus $\overline{G}$ is compact in $K(\mathfrak{X})$ and so $\dim K(\mathfrak{X})$ is finite.

□

**2.3. Definition.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. Then $F \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ is said to be* ***finite rank*** *if* $\dim F(\mathfrak{X})$ *is finite. The set of finite rank operators from $\mathfrak{X}$ to $\mathfrak{Y}$ is denoted by $\mathcal{F}(\mathfrak{X}, \mathfrak{Y})$.*

**2.4. Proposition.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. Then $\mathcal{F}(\mathfrak{X}, \mathfrak{Y}) \subseteq \mathcal{K}(\mathfrak{X}, \mathfrak{Y})$.*
**Proof.** Suppose $F \in \mathcal{F}(\mathfrak{X}, \mathfrak{Y})$. Let $B_1(\mathfrak{X}) = \{x \in \mathfrak{X} : \|x\| \leq 1\}$. Then $\overline{FB_1(\mathfrak{X})}$ is closed and bounded in $\operatorname{ran} F$, but $\operatorname{ran} F$ is finite dimensional in $\mathfrak{Y}$, as $F$ is finite rank. Thus $\overline{FB_1(\mathfrak{X})}$ is compact in $\operatorname{ran} F$, and thus compact in $\mathfrak{Y}$ as well, showing that $F$ is compact.

□

**2.5. Proposition.** *Let $\mathfrak{X}$ be a Banach space. Then $\mathcal{K}(\mathfrak{X}) = \mathcal{B}(\mathfrak{X})$ if and only if $\mathfrak{X}$ is finite dimensional.*
**Proof.** If $\dim \mathfrak{X} < \infty$, then $\mathcal{B}(\mathfrak{X}) = \mathcal{F}(\mathfrak{X}) \subseteq \mathcal{K}(\mathfrak{X}) \subseteq \mathcal{B}(\mathfrak{X})$, and equality follows.

If $\mathcal{K}(\mathfrak{X}) = \mathcal{B}(\mathfrak{X})$, then $I \in \mathcal{K}(\mathfrak{X})$, so $\overline{I(B_1)} = I(B_1) = B_1$ is compact. In particular, $\mathfrak{X}$ is finite dimensional.

□

For the remainder of this section, unless explicitly stated otherwise, $\mathfrak{X}$ will denote an *infinite dimensional* Banach space.

**2.6. Theorem.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces and suppose $K \in \mathcal{K}(\mathfrak{X}, \mathfrak{Y})$. Then $K^* \in \mathcal{K}(\mathfrak{Y}^*, \mathfrak{X}^*)$.*

**Proof.** Let $\epsilon > 0$ and let $B_1$ denote the closed unit ball of $\mathfrak{X}$. Then $K(B_1)$ is totally bounded, so we can find $x_1, x_2, \ldots, x_n \in B_1$ such that if $x \in B_1$, then $\|Kx - Kx_i\| < \epsilon/3$ for some $1 \leq i \leq n$. Let

$$\begin{array}{rlll} R: & \mathfrak{Y}^* & \to \mathbb{C}^n & \\ & \phi & \mapsto & (\phi(K(x_1)), \phi(K(x_2)), \ldots, \phi(K(x_n))). \end{array}$$

Then $R \in \mathcal{F}(\mathfrak{Y}^*, \mathbb{C}^n) \subseteq \mathcal{K}(\mathfrak{Y}^*, \mathbb{C}^n)$, and so $R(D_1)$ is totally bounded, where $D_1$ is the unit ball of $\mathfrak{Y}^*$. Thus we can find $g_1, g_2, \ldots, g_m \in D_1$ such that if $g \in D_1$, then $\|Rg - Rg_j\| < \epsilon/3$ for some $1 \leq j \leq m$. Now

$$\begin{aligned} \|Rg - Rg_j\| &= \max_{1 \leq i \leq n} |g(K(x_i)) - g_j(K(x_i))| \\ &= \max_{1 \leq i \leq n} |K^*(g)(x_i) - K^*(g_j)(x_i)|. \end{aligned}$$

Suppose $x \in B_1$. Then $\|Kx - Kx_i\| < \epsilon/3$ for some $1 \leq i \leq n$, and $|K^*(g)(x_i) - K^*(g_j)(x_i)| < \epsilon/3$ for some $1 \leq j \leq m$, so

$$\begin{aligned} |K^*(g)(x) - K^*(g_j)(x)| &\leq |K^*(g)(x) - K^*(g)(x_i)| + \\ &\qquad |K^*(g)(x_i) - K^*(g_j)(x_i)| + \\ &\qquad\quad |K^*(g_j)(x_i) - K^*(g_j)(x)| \\ &\leq \|g\|\,\|Kx - Kx_i\| + \epsilon/3 + \|g_j\|\,\|Kx - Kx_i\| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

Thus $\|K^*g - K^*g_j\| \leq \epsilon$ and so $K^*(D_1)$ is totally bounded. We conclude that $K^* \in \mathcal{K}(\mathfrak{Y}^*, \mathfrak{X}^*)$.

□

**2.7. Lemma.** *Let $\mathfrak{X}$ be a Banach space and $\mathfrak{M}$ be a finite dimensional subspace of $\mathfrak{X}$. Then there exists a closed subspace $\mathfrak{N}$ of $\mathfrak{X}$ such that $\mathfrak{M} \oplus \mathfrak{N} = \mathfrak{X}$.*

**Proof.** Let $\{e_i\}_{i=1}^n$ be a basis for $\mathfrak{M}$ and let $\{f_i\}_{i=1}^n$ be the dual basis to $\{e_i\}_{i=1}^n$ (cf. Proposition 1.12). Then we can extend $\{f_i\}_{i=1}^n$ to $\{\phi_i\}_{i=1}^n \subseteq \mathfrak{X}^*$ by the Hahn-Banach Theorem. We then let $\mathfrak{N} = \cap_{i=1}^n \ker \phi_i$. It remains to check that $\mathfrak{N}$ is the desired space. Clearly it is closed.

If $x \in \mathfrak{X}$, then let $\lambda_i = \phi_i(x),\ 1 \leq i \leq n$, and set $y = \sum_{i=1}^n \lambda_i e_i \in \mathfrak{M}$. Let $z = x - y$ so that $x = y + z$. Then $\phi_i(z) = \phi_i(x) - \phi_i(y) = \lambda_i - \lambda_i = 0,\ 1 \leq i \leq n$. Hence $z \in \mathfrak{N}$, which shows that $\mathfrak{X} = \mathfrak{M} + \mathfrak{N}$.

If $x \in \mathfrak{M} \cap \mathfrak{N}$, write $x = \sum_{i=1}^n \lambda_i e_i$. Since $x \in \mathfrak{N}$, we have $0 = \phi_j(x) = \sum_{i=1}^n \lambda_i \phi_j(e_i) = \sum_{i=1}^n \lambda_i \delta_{ij} = \lambda_j,\ 1 \leq j \leq n$. Thus $x = 0$; that is, $\mathfrak{M} \cap \mathfrak{N} = \{0\}$, and so $\mathfrak{X} = \mathfrak{M} \oplus \mathfrak{N}$.

□

**2.8. Proposition.** *Let $\mathfrak{X}$ be a Banach space, and $K \in \mathcal{K}(\mathfrak{X})$. Suppose $0 \neq \lambda \in \mathbb{C}$. Then*

(i) $\mathfrak{M} = \ker(\lambda I - K)$ *is finite dimensional;*
(ii) $\mathfrak{R} = \operatorname{ran}(\lambda I - K)$ *is a closed subspace of* $\mathfrak{X}$*;*
(iii) $\dim(\mathfrak{X}/\mathfrak{R}) = \dim\ker(\lambda I - K^*) < \infty$.

**Proof.**

(i) Clearly $\mathfrak{M}$ is a closed subspace of $\mathfrak{X}$, and hence a Banach space itself. Consider

$$\begin{array}{rccc} K_0: & \mathfrak{M} & \to & \mathfrak{X} \\ & x & \mapsto & Kx \ (= \lambda x). \end{array}$$

Then $K_0$ is compact. Moreover, $K_0(\mathfrak{M}) = \mathfrak{M}$ is closed. By Proposition 2.2, $\mathfrak{M}$ is finite dimensional.

(ii) From above, $\mathfrak{M}$ is closed and finite dimensional, and so we can find $\mathfrak{N} \subseteq \mathfrak{X}$, a closed subspace such that $\mathfrak{X} = \mathfrak{M} \oplus \mathfrak{N}$. Consider

$$\begin{array}{rccc} T: & \mathfrak{N} & \to & \mathfrak{X} \\ & y & \mapsto & (\lambda I - K)y, \end{array}$$

(i.e. $T = (\lambda I - K)|_{\mathfrak{N}}$).

We claim that $T$ is bounded below, for otherwise, there exists a sequence $\{y_n\}_{n=1}^\infty$ of norm one vectors such that $\lim_{n\to\infty} Ty_n = \lim_{n\to\infty}(\lambda I - K)y_n = 0$.

Moreover, since $K$ is compact, there exists a subsequence $\{y_{n(j)}\}_{j=1}^\infty$ such that $\lim_{j\to\infty} Ky_{n(j)} = z \in \mathfrak{X}$ exists. But then

$$\begin{aligned} \lim_{j\to\infty}(\lambda I - K)y_{n(j)} &= \lim_{j\to\infty}\lambda y_{n(j)} - \lim_{j\to\infty} Ky_{n(j)} \\ &= \lim_{j\to\infty}\lambda y_{n(j)} - z \\ &= 0, \end{aligned}$$

and so $\lambda^{-1}z = \lim_{j\to\infty} y_{n(j)}$. Moreover, $z \in \mathfrak{N}$, since $\mathfrak{N}$ is closed. Then

$$\begin{aligned} (\lambda I - K)\lambda^{-1}z &= \lambda(\lambda^{-1}z) - K(\lambda^{-1})z \\ &= z - K(\lim_{j\to\infty} y_{n(j)}) \\ &= z - z \\ &= 0, \end{aligned}$$

so that $\lambda^{-1}z$ and hence $z \in \ker(\lambda I - K) = \mathfrak{M}$. But $z \in \mathfrak{M} \cap \mathfrak{N}$ implies $z = 0$, i.e.

$$0 = \|\lambda^{-1}z\| = \lim_{j\to\infty} y_{n(j)}.$$

This contradicts the fact that $\|y_{n(j)}\| = 1$ for all $j \geq 1$. The conclusion must be that $T$ is bounded below.

As in the proof of Proposition 1.4, we find that $\operatorname{ran} T$ is closed. But $\operatorname{ran} T = \operatorname{ran}(\lambda I - K)$, so that the latter is closed as well.

(iii) First note that

$$\begin{aligned}\tau \in \ker(\lambda I - K^*) &\iff (\lambda I - K^*)(\tau)x = 0 \text{ for all } x \in \mathfrak{X}\\ &\iff \tau((\lambda I - K)x) = 0 \text{ for all } x \in \mathfrak{X}\\ &\iff \tau|_{\mathfrak{R}} = 0.\end{aligned}$$

We define the map

$$\begin{array}{rccc}\Phi: & \ker(\lambda I - K^*) & \to & (\mathfrak{X}/\mathfrak{R})^*\\ & \tau & \mapsto & \Phi(\tau),\end{array}$$

where $\Phi(\tau)(x + \mathfrak{R}) := \tau(x)$.

If $x + \mathfrak{R} = y + \mathfrak{R}$, then

$$\begin{aligned}\Phi(\tau)(x + \mathfrak{R}) - \Phi(\tau)(y + \mathfrak{R}) &= \tau(x) - \tau(y)\\ &= \tau(x - y) \qquad (\text{but } x - y \in \mathfrak{R})\\ &= 0,\end{aligned}$$

so that $\Phi(\tau)$ is well-defined.

We wish to show that $\Phi$ is an isomorphism of $\ker(\lambda I - K^*)$ onto $(\mathfrak{X}/\mathfrak{R})^*$. Since $K$ is compact implies that $K^*$ is compact, using (i) above for $K^*$ will then imply that $\ker(\lambda I - K^*)$ is finite dimensional, so that $(\mathfrak{X}/\mathfrak{R})^*$ will be as well.

Suppose $0 \neq \tau \in \ker(\lambda I - K^*)$. Then, since $\tau|_{\mathfrak{R}} = 0$ and $\tau \neq 0$, we can find $x \in \mathfrak{X}\backslash\mathfrak{R}$ such that $\tau(x) \neq 0$. Then $\Phi(\tau)(x + \mathfrak{R}) = \tau(x) \neq 0$, so that $\Phi(\tau) \neq 0$. In particular, $\Phi$ is injective.

If $\overline{\phi} \in (\mathfrak{X}/\mathfrak{R})^*$ and $\pi : \mathfrak{X} \to (\mathfrak{X}/\mathfrak{R})$ is the canonical map, then define $\phi \in \mathfrak{X}^*$ via $\phi = \overline{\phi} \circ \pi$. Clearly $\phi|_{\mathfrak{R}} = 0$, so that $\phi \in \ker(\lambda I - K^*)$. Finally,

$$\begin{aligned}\Phi(\phi)(x + \mathfrak{R}) &= \phi(x)\\ &= \overline{\phi}(\pi(x))\\ &= \overline{\phi}(x + \mathfrak{R})\end{aligned}$$

so that $\Phi(\phi) = \overline{\phi}$, and hence $\Phi$ is surjective. Thus $\Phi$ is an isomorphism, and so from above, we conclude that

$$\dim(\mathfrak{X}/\mathfrak{R}) = \dim(\mathfrak{X}/\mathfrak{R})^* = \dim(\lambda I - K^*)$$

is finite.

□

**2.9. Remark.** The above proof actually shows that if $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces, $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$, and $\operatorname{ran} T$ is closed, then

$$\ker T^* \simeq (\mathfrak{Y}/\operatorname{ran} T)^*.$$

Compactness was only used to show that this was finite in the case we were considering.

**2.10.** Let $\mathfrak{X}$ be a Banach space and $T \in \mathcal{B}(\mathfrak{X})$. Then associated to $T$ are two linearly ordered sequences of linear manifolds:

$$\mathcal{C}_a = \{\ker T^n\}_{n=1}^\infty \text{ and } \mathcal{C}_d = \{\operatorname{ran} T^n\}_{n=1}^\infty.$$

**Definition.** *Let $\mathfrak{X}$ be a Banach space and $T \in \mathcal{B}(\mathfrak{X})$. If $\ker T^n \neq \ker T^{n+1}$ for all $n \geq 0$, then $T$ is said to have **infinite ascent**, and we write $\operatorname{asc} T = \infty$. Otherwise, we set $\operatorname{asc} T = p$, where $p$ is the least non-negative integer such that $\ker T^p = \ker T^n$, $n \geq p$.*

*If $\operatorname{ran} T^n \neq \operatorname{ran} T^{n+1}$ for all $n \geq 0$, then $T$ is said to have **infinite descent**, and we write $\operatorname{desc} T = \infty$. Otherwise, we set $\operatorname{desc} T = q$, where $q$ is the least non-negative integer such that $\operatorname{ran} T^q = \operatorname{ran} T^n$, $n \geq q$.*

**2.11. Lemma.** *Let $\mathfrak{X}$ be a Banach space and $K \in \mathcal{K}(\mathfrak{X})$. Suppose we can find $\{\lambda_n\}_{n=1}^\infty \subseteq \mathbb{C}$ and a sequence $\{V_n\}_{n=1}^\infty$ of closed subspaces of $\mathfrak{X}$ satisfying:*

(i) *$V_n \subset V_{n+1}$ for all $n \geq 1$, where $'' \subset ''$ denotes* proper *containment;*
(ii) *$KV_n \subseteq V_n$ for all $n \geq 1$;*
(iii) *$(K - \lambda_n)V_n \subseteq V_{n-1}$, for all $n \geq 1$.*

*Then $\lim_{n\to\infty} \lambda_n = 0$. Moreover, the same conclusion holds if $\{W_n\}_{n=1}^\infty$ is a sequence of closed subspaces of $\mathfrak{X}$ satisfying:*

(iv) *$W_n \supset W_{n+1}$ for all $n \geq 1$, where $'' \supset ''$ denotes* proper *containment;*
(v) *$KW_n \subseteq W_n$ for all $n \geq 1$;*
(vi) *$(K - \lambda_n)W_n \subseteq W_{n+1}$, for all $n \geq 1$.*

**Proof.** Let $\overline{z_n} \in V_n/V_{n-1}$, $\|\overline{z_n}\| = 1/2$ and choose $x_n \in V_n$ such that $\overline{x_n} = \overline{z_n}$. Since $\|\overline{z_n}\| = \inf\{\|x_n + y\| : y \in V_{n-1}\}$, we can find $y_n \in V_{n-1}$ such that if we let $w_n = x_n + y_n$, then $\overline{w_n} = \overline{z_n}$ and $1/2 \leq \|w_n\| < 1$.

Then $w_n \in V_n$, so $(K - \lambda_n)w_n \in V_{n-1}$. That is, $Kw_n = \lambda_n w_n + v_{n-1}$ for some $v_{n-1} \in V_{n-1}$.

If $m < n$,

$$\begin{aligned} \|Kw_n - Kw_m\| &= \|\lambda_n w_n + (v_{n-1} - \lambda_m w_m - v_{m-1})\| \\ &\geq \inf\{\|\lambda_n w_n + y\| : y \in V_{n-1}\} \\ &= |\lambda_n| \, \|\overline{w_n}\| \\ &= |\lambda_n|/2. \end{aligned}$$

Suppose $\lim_{n\to\infty} \lambda_n \neq 0$. Find $\{\lambda_{n(j)}\}_{j=1}^\infty$ such that $\inf\{|\lambda_{n(j)}| : j \geq 1\} = \delta > 0$. Then $\{Kw_{n(j)}\}_{j=1}^\infty$ has no convergent subsequence, although $\{w_{n(j)}\}_{j=1}^\infty$ is bounded. This contradicts the compactness of $K$. Thus $\lim_{n\to\infty} \lambda_n = 0$.

The second statement is proven in a similar fashion.

□

**2.12. Lemma.** *Let $\mathfrak{X}$ be a Banach space, $K \in \mathcal{K}(\mathfrak{X})$, and $0 \neq \lambda \in \mathbb{C}$. Then* $\operatorname{ran}(\lambda I - K)^n$ *is closed for all $n \geq 0$.*
**Proof.** Exercise.

□

**2.13. Theorem.** *Let $K$ be a compact operator on a Banach space $\mathfrak{X}$ and suppose $0 \neq \lambda \in \mathbb{C}$. Then $(\lambda I - K)$ has finite ascent and finite descent.*
**Proof.** Suppose that $(K - \lambda I)$ has infinite ascent. Then we can apply Lemma 2.11 with $\lambda_n = \lambda$ for all $n \geq 1$ and $V_n = \ker(K - \lambda I)^n$, $n \geq 1$ to conclude that $\lim_{n\to\infty} \lambda_n = \lambda = 0$, a contradiction. Thus $(K - \lambda I)$ has finite ascent.

Similarly, if $(K - \lambda I)$ has infinite descent, then putting $\lambda_n = \lambda$ and putting $W_n = \operatorname{ran}(K - \lambda I)^n$ for all $n \geq 1$ again implies that $\lim_{n\to\infty} \lambda_n = \lambda = 0$, a contradiction. Thus $(K - \lambda I)$ has finite descent.

□

**2.14. Theorem. [The Fredholm Alternative]** *Let $\mathfrak{X}$ be a Banach space and let $K \in \mathcal{K}(\mathfrak{X})$. Suppose $0 \neq \lambda \in \mathbb{C}$. Then $(\lambda I - K)$ is injective if and only if it is surjective.*
**Proof.** First assume that $(\lambda I - K)$ is surjective, and suppose that it is not injective. Let $V_n = \ker(\lambda I - K)^n$ for each $n \geq 1$. Each $(\lambda I - K)^n$ is onto. Let $0 \neq y \in \ker(\lambda I - K)$ and let $x \in \mathfrak{X}$ such that $y = (\lambda I - K)^n x$. Then $x \in V_{n+1}$ but $x \notin V_i$, $1 \leq i \leq n$. That is, $V_n$ is a proper subset of $V_{n+1}$ for all $n \geq 1$. But $(\lambda I - K)$ has finite ascent, by Theorem 2.13, a contradiction. Thus $(\lambda I - K)$ is injective.

Now assume that $(\lambda I - K)$ is injective. Let $\mathfrak{M} = \operatorname{ran}(\lambda I - K)$. By Proposition 2.8, $\mathfrak{M}$ is closed. Consider the operator

$$\begin{array}{rccc} R: & \mathfrak{X} & \to & \mathfrak{M} \\ & x & \mapsto & (\lambda I - K)x. \end{array}$$

Then $R$ is bijective and so by Proposition 1.4, $R$ is invertible. Moreover, $R^* : \mathfrak{M}^* \to \mathfrak{X}^*$ is invertible, and hence surjective.

Take $x^* \in \mathfrak{X}^*$ and choose $m^* \in \mathfrak{M}^*$ such that $R^* m^* = m^* \circ R = x^*$. We can extend $m^*$ to a functional $x^*_m \in \mathfrak{X}^*$ by the Hahn-Banach Theorem. Then for all $x \in \mathfrak{X}$,

$$\begin{aligned} ((\lambda I - K)^* x^*_m)(x) &= x^*_m((\lambda I - K)x) \\ &= (m^* \circ R)x \\ &= x^*(x). \end{aligned}$$

Thus $(\lambda I - K)^* x^*_m = x^*$, showing that $(\lambda I - K)^*$ is surjective. From the first half of the proof, it follows that $(\lambda I - K)^*$ is injective, and hence invertible. But then $(\lambda I - K)$ is invertible, and therefore surjective.

□

**2.15. Corollary.** *Let $\mathfrak{X}$ be an infinite dimensional Banach space and $K \in \mathcal{K}(\mathfrak{X})$. Then $\sigma(K) = \{0\} \cup \sigma_p(K)$.*

Note that $0 \in \sigma(K)$ since $K$ lies in a proper ideal, and hence can not be invertible.

Recall that eigenvectors corresponding to distinct eigenvalues of a linear operator $T \in \mathcal{B}(\mathfrak{X})$ are linearly independent.

**2.16. Theorem.** *Let $\mathfrak{X}$ be an infinite dimensional Banach space and $K \in \mathcal{K}(\mathfrak{X})$. Then for all $\epsilon > 0$, $\sigma(K) \cap \{z \in \mathbb{C} : |z| > \epsilon\}$ is finite. In other words, $\sigma(K)$ is a sequence of eigenvalues of finite multiplicity, and this sequence must converge to $0$.*

**Proof.** Let $\epsilon > 0$. Suppose $\sigma(K) \cap \{z \in \mathbb{C} : |z| > \epsilon\} \supseteq \{\lambda_n\}_{n=1}^\infty$ with $\lambda_i \neq \lambda_j$, $1 \leq i \neq j < \infty$. Let $\{v_n\}_{n=1}^\infty$ be eigenvectors corresponding to $\{\lambda_n\}_{n=1}^\infty$ and for each $n \geq 1$, let $V_n = \mathrm{span}_{1 \leq k \leq n}\{v_k\}$.

Then $\{V_n\}_{n=1}^\infty$ and $\{\lambda_n\}_{n=1}^\infty$ satisfy the conditions of Lemma 2.11. We conclude that $\lim_{n\to\infty} \lambda_n = 0$, a contradiction.

Thus $\sigma(K) = \{0\} \cup \{\lambda_n\}_{n=1}^r$, where $r$ is either finite or $\aleph_0$. Moreover, each $\lambda_n$ is an eigenvalue of $K$, and $\lim_{n\to\infty} \lambda_n = 0$ when $r$ is not finite.

□

**2.17. Definition.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. An operator $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ is said to be **Fredholm** if:*

(i) $\mathrm{ran}\, T$ *is closed;*
(ii) $\mathrm{nul}\, T = \dim \ker T$ *is finite; and*
(iii) $\mathrm{nul}\, T^* = \mathrm{codim}\, \mathrm{ran}\, T$ *is finite.*

*Given $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ a Fredholm operator, we define the **Fredholm index** of $T$ as follows:*

$$\mathrm{ind}\ T = \mathrm{nul}\, T - \mathrm{nul}\, T^*.$$

**2.18. Example.** Let $\mathfrak{X}$ be a Banach space, $K \in \mathcal{K}(\mathfrak{X})$ and $0 \neq \lambda \in \mathbb{C}$. Then $\lambda I - K$ is Fredholm.

In fact, we shall now show that $\mathrm{ind}\ (\lambda I - K) = 0$. We shall then return to Fredholm operators when we study $\mathcal{K}(\mathcal{H})$, the set of compact operators on a Hilbert space $\mathcal{H}$.

**2.19. Example.** Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^\infty$. Define an operator $U \in \mathcal{B}(\mathcal{H})$ via $Ue_n = e_{n+1}$ for all $n \geq 1$. (We extend this definition by linearity and continuity to all of $\mathcal{H}$. $U$ is referred to as *the unilateral forward shift.* Then $U$ is an isometry with range equal to the span of $\{e_n\}_{n=2}^\infty$. As such, the range of $U$ is closed, the nullity of $U$ is zero, and the codimension of the range of $U$ is 1. Hence $U$ is a Fredholm operator of index $-1$. We shall return to this example later.

**2.20. Lemma.** *Let $\mathfrak{X}$ be a Banach space and $\mathfrak{M}$ be a finite codimensional subspace of $\mathfrak{X}$. Then there exists a finite dimensional subspace $\mathfrak{N}$ of $\mathfrak{X}$ such that $\mathfrak{X} = \mathfrak{M} \oplus \mathfrak{N}$. Moreover,* $\dim \mathfrak{N} = \dim (\mathfrak{X}/\mathfrak{M})$.

**Proof.** Let $\{\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}\}$ be a basis for $\mathfrak{X}/\mathfrak{M}$, and choose $\{x_1, x_2, \ldots, x_n\} \subseteq \mathfrak{X}$ such that $\pi(x_j) = \overline{x_j}$, $1 \le j \le n$, where $\pi : \mathfrak{X} \to \mathfrak{X}/\mathfrak{M}$ is the canonical map.

Let $\mathfrak{N} = \overline{\text{span}}\{x_1, x_2, \ldots, x_n\} = \text{span}\{x_1, x_2, \ldots, x_n\}$. If $z \in \mathfrak{M} \cap \mathfrak{N}$, then $z \in \mathfrak{N}$ so that $z = \sum_{i=1}^n \lambda_i x_i$. But $z \in \mathfrak{M}$ and so $\overline{z} = 0 = \sum_{i=1}^n \lambda_i \overline{x_i}$. Thus $\lambda_i = 0$ for all $i$ and hence $z = 0$. In other words, $\mathfrak{M} \cap \mathfrak{N} = \{0\}$.

Now let $x \in \mathfrak{X}$. Then $\overline{x} = \sum_{i=1}^n \lambda_i \overline{x_i}$ and so $x = \sum_{i=1}^n \lambda_i x_i + y$ for some $y \in \mathfrak{M}$. Therefore $\mathfrak{X} = \mathfrak{M} + \mathfrak{N}$, so that $\mathfrak{X} = \mathfrak{M} \oplus \mathfrak{N}$.

□

**2.21. Remark.** If $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces and $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ is Fredholm, then there exists a closed subspace $\mathfrak{N}$ of $\mathfrak{Y}$ such that

$$\mathfrak{Y} = \text{ran}\, T \oplus \mathfrak{N}.$$

Moreover, $\dim \mathfrak{N} = \text{codim ran T} < \infty$.

**2.22. Theorem.** *Let $\mathfrak{X}$ be a Banach space, $K \in \mathcal{K}(\mathfrak{X})$, and $0 \ne \lambda \in \mathbb{C}$. Then*

$$\text{ind}\ (\lambda I - K) = 0.$$

**Proof.** Let $\mathfrak{M} = \ker(\lambda I - K)$. Then $\dim \mathfrak{M} < \infty$ and so $\mathfrak{M}$ has a closed complement $\mathfrak{N} \subseteq \mathfrak{X}$ such that $\mathfrak{M} \oplus \mathfrak{N} = \mathfrak{X}$. Let $\mathfrak{R} = \text{ran}\,(\lambda I - K)$. Then $\mathfrak{R}$ is closed and finite codimensional, so by Lemma 2.20, $\mathfrak{R}$ has a closed complement $\mathfrak{S} \subseteq \mathfrak{X}$ satisfying $\mathfrak{R} \oplus \mathfrak{S} = \mathfrak{X}$. Let $n = \min(\dim \mathfrak{M}, \dim \mathfrak{S})$.

Choose $\overline{\phi_1}, \overline{\phi_2}, \ldots, \overline{\phi_n}$ linearly independent in $(\mathfrak{X}/\mathfrak{N})^*$. Let $\pi : \mathfrak{X} \to \mathfrak{X}/\mathfrak{N}$ be the canonical map and define $\phi_i = \overline{\phi_i} \circ \pi$ so that $\phi_i \in \mathfrak{X}^*$, $1 \le i \le n$. Choose $\{f_i\}_{i=1}^n$ linearly independent in $\mathfrak{S}$. We shall define $Q \in \mathcal{K}(\mathfrak{X})$ via $Qx = \sum_{i=1}^n \phi_i(x) f_i$, $x \in \mathfrak{X}$. Then

$$K - Q \in \mathcal{K}(\mathfrak{X}) \text{ and } \ (\lambda I - (K - Q)) = (\lambda I - K) + Q$$

is either surjective (if $n = \dim \mathfrak{S}$), or injective (if $n = \dim \mathfrak{M}$). In either case, by the Fredholm Alternative , it is bijective.

We conclude that $\dim \mathfrak{M} = \dim \mathfrak{S}$, so that $\text{nul}\,(\lambda I - K) = \text{codim}\,(\lambda I - K)$, or equivalently,

$$\text{ind}\ (\lambda I - K) = 0.$$

□

**2.23.** We conclude this section by showing that although not all finite codimensional subspaces of a Banach space are closed, nevertheless, this is true for operator ranges. In particular, this means that in order to know if an operator $T$ is Fredholm, one need only verify that the nullity and the codimension of the range are finite.

**2.24. Proposition.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces and let $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. Suppose that $\mathfrak{N}$ is a closed subspace such that* $\operatorname{ran} T \oplus \mathfrak{N}$ *is closed in $\mathfrak{Y}$. Then* $\operatorname{ran} T$ *is closed.*

**Proof.** Define a norm on the space $(\mathfrak{X}/\ker T) \times \mathfrak{N}$ by $\|(\overline{x}, n)\| = \|\overline{x}\| + \|n\|$. Let $T_0$ denote the operator

$$\begin{aligned} T_0 : \quad (\mathfrak{X}/\ker T) \times \mathfrak{N} &\to \operatorname{ran} T \oplus \mathfrak{N} \\ (\overline{x}, n) &\mapsto Tx + n. \end{aligned}$$

It is easy to check that $T_0$ is well-defined, continuous, injective and that $\operatorname{ran} T_0$ is $\operatorname{ran} T \oplus \mathfrak{N}$. Since $\operatorname{ran} T \oplus \mathfrak{N}$ is closed, $T_0$ is invertible. This means that we can find $\delta > 0$ such that $\|Tx + n\| \geq \delta \,\|(\overline{x}, n)\|$. That is, $\|Tx + n\| \geq \delta\,(\|\overline{x}\| + \|n\|)$.

If $n = 0$, we get $\|Tx\| \geq \delta\,\|\overline{x}\|$, and so the restriction $T_1$ of $T_0$ to $(\mathfrak{X}/\ker T)$ is bounded below. But then $\operatorname{ran} T_1 = \operatorname{ran} T$ is closed, as pointed out in Proposition 1.4.

$\square$

**2.25. Corollary.** *If $\mathfrak{X}$ is a Banach space, $T \in \mathcal{B}(\mathfrak{X})$, and $\mathfrak{X}/\operatorname{ran} T$ is finite dimensional, then* $\operatorname{ran} T$ *is closed.*

**Proof.** By Lemma 2.20, there exists a finite dimensional (and therefore closed) subspace of $\mathfrak{X}$ such that $\operatorname{ran} T \oplus \mathfrak{N} = \mathfrak{X}$. Since $\mathfrak{X}$ is obviously closed, we may now apply the above Proposition 2.24 to conclude that $\operatorname{ran} T$ is closed, as desired.

$\square$

*"You are what you eat."*
***J. Dahmer (also attributed to A. Meiwes)***

## 3. The algebra of Hilbert space operators

**3.1.** We now consider the special case where the Banach space under consideration is in fact a Hilbert space, which we shall always denote by $\mathcal{H}$. The inner product on $\mathcal{H}$ will be denoted by $(\cdot, \cdot)$.

All of the results from the previous section of course apply to Hilbert space operators. On the other hand, the identification of a Hilbert space with its dual (an anti-isomorphism) allows us to consider a new version of adjoints, based on the Riesz Representation Theorem.

**3.2. Theorem. [The Riesz Representation Theorem]** *Let $\mathcal{H}$ be a Hilbert space and $\phi \in \mathcal{H}^*$. Then there exists a vector $y \in \mathcal{H}$ such that $\phi(x) = (x, y)$ for all $x \in \mathcal{H}$.*

**3.3. Theorem.** *Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$. Then there exists a unique operator $T^* \in \mathcal{B}(\mathcal{H})$, called the* ***Hilbert space adjoint*** *of $T$, satisfying*

$$(Tx, y) = (x, T^*y)$$

*for all $x$, $y \in \mathcal{H}$.*

**Proof.** Fix $y \in \mathcal{H}$. Then the map

$$\begin{array}{rccc} \phi_y : & \mathcal{H} & \to & \mathbb{C} \\ & x & \mapsto & (Tx, y) \end{array}$$

is a linear functional and so there exists a vector $z_y \in \mathcal{H}$ such that

$$\phi_y(x) = (Tx, y) = (x, z_y)$$

for all $x \in \mathcal{H}$. Define a map $T^* : \mathcal{H} \to \mathcal{H}$ by $T^*y = z_y$. We leave it to the reader to verify that $T^*$ is in fact linear, and we concentrate on showing that it is bounded.

To see that $T^*$ is bounded, consider the following. Let $y \in \mathcal{H}$, $\|y\| = 1$. Then $(Tx, y) = (x, T^*y)$ for all $x \in \mathcal{H}$, so

$$\begin{aligned} \|T^*y\|^2 &= (T^*y, T^*y) \\ &= (T\,T^*y, y) \\ &\leq \|T\|\,\|T^*y\|\,\|y\|. \end{aligned}$$

Thus $\|T^*y\| \leq \|T\|$, and so $\|T^*\| \leq \|T\|$.

$T^*$ is unique, for if there exists $A \in \mathcal{B}(\mathcal{H})$ such that $(Tx, y) = (x, T^*y) = (x, Ay)$ for all $x$, $y \in \mathcal{H}$, then $(x, (T^* - A)y) = 0$ for all $x$, $y \in \mathcal{H}$, and so $(T^* - A)y = 0$ for all $y \in \mathcal{H}$, i.e. $T^* = A$.

$\square$

**3.4. Corollary.** *Let $T \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. Then $(T^*)^* = T$. It follows that $\|T\| = \|T^*\|$.*
**Proof.** For all $x, y \in \mathcal{H}$, we get

$$\begin{aligned} (x, (T^*)^* y) &= (T^* x, y) \\ &= \overline{(y, T^* x)} \\ &= \overline{(Ty, x)} \\ &= (x, Ty), \end{aligned}$$

and so $(T^*)^* = T$. Applying Theorem 3.3, we get

$$\|T\| = \|(T^*)^*\| \leq \|T^*\| \leq \|T\|,$$

and so $\|T\| = \|T^*\|$.

□

**3.5. Proposition.** *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then $\sigma(T) = \sigma(T^*)^* := \{\overline{\lambda} : \lambda \in \sigma(T)\}$.*
**Proof.** If $\lambda \notin \sigma(T)$, let $R = (\lambda - T)^{-1}$. For all $x, y \in \mathcal{H}$,

$$\begin{aligned} (x, y) &= (R(\lambda - T)x, y) \\ &= ((\lambda - Tx, R^* y) \\ &= (x, (\lambda - T)^* R^* y). \end{aligned}$$

Thus $(\lambda - T)^* R^* = I$, and similarly, $R^* (\lambda - T)^* = I$. But $(\lambda - T)^* = \overline{\lambda} - T^*$, so that $R^* = (\overline{\lambda} - T^*)^{-1} = [(\lambda - T)^*]^{-1}$. Thus $\rho(T)^* \subseteq \rho(T^*)$.

Moreover, $\rho(T^*)^* \subseteq \rho(T^{**}) = \rho(T)$. In other words, $\rho(T^*) \subseteq \rho(T)^*$. We conclude that $\sigma(T) = \sigma(T^*)^*$.

□

**3.6. Remark.** The above proof also shows that for a Hilbert space $\mathcal{H}$ and $A, B \in \mathcal{B}(\mathcal{H})$, we have $(AB)^* = B^* A^*$. The adjoint operator

$$* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

is an example of an *involution* on a Banach algebra. Namely, for all $\alpha$, $\beta \in \mathbb{C}$ and $A, B \in \mathcal{B}(\mathcal{H})$, we obtain

(i) $(\alpha A)^* = \overline{\alpha} A^*$;
(ii) $(A + B)^* = A^* + B^*$; and
(iii) $(AB)^* = B^* A^*$.
(iv) $(A^*)^* = A$.

Involutions will appear again in our study of C*-algebras.

**3.7. Proposition.** *Let $\mathcal{H} = \mathbb{C}^n$ and $T \in \mathcal{B}(\mathcal{H}) \simeq \mathbb{M}_n$. Then the matrix of $T^*$ with respect to any orthonormal basis is the conjugate transpose of that of $T$.*

**Proof.** Let $\{e_i\}_{i=1}^n$ be an orthonormal basis for $\mathcal{H}$. With respect to this basis, $T$ has a matrix $[t_{ij}]_{1 \le i,j \le n}$ and $T^*$ has a matrix $[r_{ij}]_{1 \le i,j \le n}$.

But $t_{ij} = (Te_j, e_i) = (e_j, T^*e_i) = \overline{(T^*e_i, e_j)} = \overline{r_{ij}}$, completing the proof.

□

**3.8. Proposition.** *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then $(\operatorname{ran} T)^\perp = \ker T^*$. In particular, therefore:*

(i) $\overline{\operatorname{ran} T} = (\ker T^*)^\perp$;
(ii) *for* $\lambda \in \mathbb{C}$, $\lambda \in \sigma_c(T)$ *if and only if* $\overline{\lambda} \in \sigma_p(T^*)$;
(iii) $\operatorname{ran} T$ *is not dense in* $\mathcal{H}$ *if and only if* $\ker T^* \ne 0$.

**Proof.** Let $y \in \mathcal{H}$. Then

$$\begin{aligned} y \in \ker T^* &\iff \text{for all } x \in \mathcal{H},\ 0 = (x, T^*y) \\ &\iff \text{for all } x \in \mathcal{H},\ 0 = (Tx, y) \\ &\iff y \in (\operatorname{ran} T)^\perp. \end{aligned}$$

□

**3.9. Example.** Let $\mathcal{H}$ be a Hilbert space with orthomormal basis $\{e_n\}_{n=1}^\infty$. Define the operator $S \in \mathcal{B}(\mathcal{H})$ by first setting $Se_{n+1} = e_n$ for all $n \ge 1$ and $Se_1 = 0$, and then extending $S$ by linearity and continuity to all of $\mathcal{H}$.

$S$ is then called the *unilateral (backward) shift*, and with respect to the above basis for $\mathcal{H}$, the matrix for $S$ is:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

It is easily verified that $\|S\| = 1$. As for the spectral radius of $S$, note that $\|S^n\| \le \|S\|^n \le 1$, while $\|S^n e_{n+1}\| = \|e_1\| = 1$, so that $\|S^n\| \ge 1$. Hence $\operatorname{spr}(S) = \lim_{n\to\infty} \|S^n\|^{\frac{1}{n}} = 1$.

Let $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Consider $(\lambda I - S)$. Let $x_n = (1/\sqrt{n}) \sum_{i=1}^n \lambda^i e_i$. Then $\|x_n\| = 1$ for all $n \ge 1$, and

$$\|(\lambda I - S)x_n\| = \|(1/\sqrt{n})\lambda^{n+1} e_n\| = 1/\sqrt{n}.$$

Letting $n$ tend to $\infty$ yields $\lambda \in \sigma_a(S)$.

Now let $\lambda \in \mathbb{C}$, $0 < |\lambda| < 1$, and $y = \sum_{i=1}^{\infty} \lambda^i e_i$. Then

$$\begin{aligned}(\lambda I - S)y &= \sum_{i=1}^{\infty} \lambda^{i+1} e_i - \sum_{i=1}^{\infty} \lambda^{i+1} e_i \\ &= 0.\end{aligned}$$

As for $\lambda = 0$, $e_1$ lies in the kernel of $S$, and hence of $0 - S$. Hence $\sigma_p(S) \supseteq \{z \in \mathbb{C} : |z| < 1\}$. In particular, $\overline{\lambda} \in \sigma_c(S^*)$, i.e. $\sigma_c(S^*) \supseteq \{z \in \mathbb{C} : |z| < 1\}$.

**Note:** An easy calculation which is left as an exercise shows that $S^* e_n = e_{n+1}$, $n \geq 1$, and hence

$$S^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Clearly, $\operatorname{ran} S = \mathcal{H}$, so $\ker S^* = \{0\}$. In fact, $S^*$ is an isometry! Finally, $\sigma(S) = \{z \in \mathbb{C} : |z| \leq 1\} = \sigma(S^*)$. Indeed, $S^* = U$, where $U$ is the operator we defined in Example 2.19.

**3.10. Definition.** *Given an infinite dimensional, separable Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_n\}_{n=1}^{\infty}$, a **unilateral (forward) weighted shift** $W$ on $\mathcal{H}$ is an operator satisfying $We_n = w_n e_{n+1}$, $n \geq 1$, where $\{w_n\}_{n=1}^{\infty} \in \ell^{\infty}(\mathbb{N})$ is called the sequence of **weights** of $W$. The adjoint of a unilateral forward weighted shift is referred to as a **unilateral backward weighted shift**.*

*A **bilateral weighted shift** is an operator $V \in \mathcal{B}(\mathcal{H})$ such that $Vf_n = Vf_{n+1}$ for all $n \in \mathbb{Z}$, where $\{f_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $\mathcal{H}$ and $\{v_n\}_{n\in\mathbb{Z}} \in \ell^{\infty}(\mathbb{Z})$.*

Weighted shifts are of interest because they provide one of the few tractable classes of operators which exhibit a reasonably wide variety of phenomena typical of more general operators. As such, they are an excellent test case for conjectures about general operators.

In the case where all of the weights are constant and equal to 1, the shift in question is referred to as *the* forward (backward, bilateral) shift. The terms **unilateral shift** and **unweighted shift** are also used, it usually being clear from the context whether the shift is forward or backward.

**3.11. Definition.** *Let $\mathcal{H}$ be a Hilbert space, and $N \in \mathcal{B}(\mathcal{H})$. Then*

(i) *If $N = N^*$, $N$ is said to be **self-adjoint**, or **hermitian**;*
(ii) *if $N = N^*$ and $(Nx, x) \geq 0$ for all $x \in \mathcal{H}$, then $N$ is said to be **positive**;*
(iii) *if $N N^* = N^* N$, then $N$ is said to be **normal**;*
(iv) *if $N^* = N^{-1}$, then $N$ is said to be **unitary**; observe that all unitary operators are automatically normal.*

(v) *if $N = N^* = N^2$, then $N$ is called an* ***(orthogonal) projection***.

**3.12. Remark.** Suppose that $U \in \mathcal{B}(\mathcal{H})$ is unitary. Since $U$ is invertible, it must be bijective. Moreover, given $x$ and $y$ in $\mathcal{H}$, we find that

$$(Ux, Uy) = (U^* Ux, y) = (U^{-1}Ux, y) = (Ix, y) = (x, y).$$

In particular, unitaries preserve inner products, and therefore preserve both angles and lengths. Indeed, they serve as the isomorphisms in the category of Hilbert spaces.

**3.13. Example.** Let $\mathcal{H}$ be a Hilbert space and let $B$ denote the unweighted bilateral shift. It is straightforward to verify that $B$ is unitary. On the other hand, if $S$ denotes the backward shift, then $S\,S^* - S^*\,S = P$, where $P$ is a rank one projection. Thus $S$ is not normal.

**3.14. Example.** Let $\mathcal{H} = L^2([0,1], dx)$ and define, for $f \in L^\infty([0,1], dx)$, the operator

$$\begin{array}{rccc} M_f : & \mathcal{H} & \to & \mathcal{H} \\ & g & \mapsto & fg. \end{array}$$

Then

$$\begin{aligned} (M_f g, h) &= \int_0^1 (M_f g)\overline{h} dx \\ &= \int_0^1 (gf)\overline{h} dx \\ &= \int_0^1 g\overline{(\overline{f}h)} \\ &= (g, M_{\overline{f}} h) \\ &= (g, (M_f)^* h). \end{aligned}$$

Thus $M_f^* = M_{\overline{f}}$. We leave it as an exercise for the reader to verify that $M_f$ is always normal.

$M_f$ will be self-adjoint precisely if $M_f = M_{\overline{f}}$, and it is readily seen that this happens if and only if $\overline{f} = f$; namely if $f$ is real-valued.

$M_f$ will be positive if and only if $(M_f g, g) \geq 0$ for all $g \in \mathcal{H}$. But this happens precisely when

$$\int_0^1 f(x)\,|g(x)|^2 dx \geq 0$$

for all $g \in L^2([0,1], dx)$, which in turn is equivalent to the condition that $f(x) \geq 0$ almost everywhere in $[0,1]$.

Finally, $M_f$ will be unitary if $M_{\overline{f}} = {M_f}^{-1}$, which is equivalent to $\overline{f} = f^{-1}$. In other words, $|f(x)| = 1$ almost everywhere in $[0,1]$.

**3.15. Example.** Let $\mathcal{H} = \ell^2(\mathbb{N})$ and let $D = \text{diag}\,\{d_n\}_{n=1}^\infty$, where $\{d_n\}_{n=1}^\infty \in \ell^\infty(\mathbb{N})$. Suppose that $\{e_n\}_{n=1}^\infty$ is the standard orthonormal basis for $\mathcal{H}$, and that $De_n = d_n\, e_n$ for all $n \geq 1$. Then $\|D\| = \sup_{n\geq 1} |d_n|$. Then $D^* = \text{diag}\,\{\overline{d_n}\}_{n=1}^\infty$, and it is not hard to check that $D$ is normal. In fact, $D$ can be thought of as a multiplication operator on an $L^2$-space with respect to counting measure.

Furthermore, $\sigma_p(D) = \{d_n\}_{n=1}^\infty$, while $\sigma_a(D) = \sigma(D) = \overline{\{d_n\}_{n=1}^\infty}$. Finally, $\sigma_c(D) = \overline{\sigma_p(D^*)} = \{d_n\}_{n=1}^\infty$.

Again, $D$ is self-adjoint precisely when $d_n \in \mathbb{R}$ for all $n \geq 1$, $D$ is positive if and only if $d_n \geq 0$ for all $n \geq 1$, and $D$ is unitary if and only if $|d_n| = 1$ for all $n \geq 1$.

**3.16. Lemma.** *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$. If $N$ is normal, then $\|Nx\| = \|N^*x\|$ for all $x \in \mathcal{H}$. In particular, therefore,* $\ker\, N = \ker\, N^*$.

**Proof.** Let $x \in \mathcal{H}$. Then

$$\begin{aligned}
\|Nx\|^2 &= (Nx, Nx) \\
&= (N^*Nx, x) \\
&= (N\,N^*x, x) \\
&= (N^*x, N^*x) \\
&= \|N^*x\|^2.
\end{aligned}$$

That is, $\|Nx\| = \|N^*x\|$.

□

**3.17. Proposition.** *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$. If $N$ is normal, then $\sigma(N) = \sigma_a(N)$.*

**Proof.** Clearly $\sigma_a(N) \subseteq \sigma(N) = \sigma_a(N) \cup \sigma_c(N)$. Assume $\lambda \in \sigma_c(N)$. Then $\overline{\lambda} \in \sigma_p(N^*)$, by Proposition 3.8. Let $0 \neq x \in \ker\,(N^* - \overline{\lambda})$. Then $x \in \ker\,(N^* - \overline{\lambda})^* = \ker\,(N - \lambda I)$ by the above Lemma. This means that $\lambda \in \sigma_p(N) \subseteq \sigma_a(N)$. We conclude that $\sigma(N) \subseteq \sigma_a(N)$.

□

*Meanwhile, back at the Skywalker's dinner table...*
*"Use the fork, Luke."*

## 4. The spectral theorem for compact normal operators

**4.1.** The set of compact operators acting on a Hilbert space is more tractable in general than the set of compact operators acting on an arbitrary Banach space. One of the reasons for this is the characterization given below. Recall that the set of finite rank operators acting on a Banach space $\mathfrak{X}$ is denoted by $\mathcal{F}(\mathfrak{X})$.

**4.2. Theorem.** *Let $\mathcal{H}$ be a Hilbert space and let $K \in \mathcal{B}(\mathcal{H})$. The following are equivalent:*

(i) *$K$ is compact;*
(ii) *$K^*$ is compact;*
(iii) *There exists a sequence $\{F_n\}_{n=1}^\infty \subseteq \mathcal{F}(\mathcal{H})$ such that $K = \lim_{n\to\infty} F_n$.*

**Proof.**

(i) $\Rightarrow$ (iii) Let $B_1$ denote the unit ball of $\mathcal{H}$, and let $\epsilon > 0$. Since $\overline{K(B_1)}$ is compact, it must be separable (i.e. it is totally bounded). Thus $\mathcal{M} = \overline{\operatorname{ran} K}$ is a separable subspace of $\mathcal{H}$, and thus possesses an orthonormal basis $\{e_n\}_{n=1}^\infty$.

Let $P_n$ denote the orthogonal projection of $\mathcal{H}$ onto span $\{e_k\}_{k=1}^n$. Set $F_n = P_n K$, noting that each $F_n$ is finite rank. We now show that $K = \lim_{n\to\infty} F_n$.

Let $x \in \mathcal{H}$ and consider $y = Kx \in \mathcal{M}$, so that $\lim_{n\to\infty} \|P_n y - y\| = 0$. Thus $\lim_{n\to\infty} \|F_n x - Kx\| = \lim_{n\to\infty} \|P_n y - y\| = 0$. Since $K$ is compact, $K(B_1)$ is totally bounded, so we can choose $\{x_k\}_{k=1}^m \subseteq B_1$ such that $K(B_1) \subseteq \cup_{k=1}^m B(Kx_k, \epsilon/3)$, where given $z \in \mathcal{H}$ and $\delta > 0$, $B(z, \delta) = \{w \in \mathcal{H} : \|w - z\| < \delta\}$.

If $\|x\| \le 1$, choose $i$ such that $\|Kx_i - Kx\| < \epsilon/3$. Then for any $n > 0$,

$$\begin{aligned} \|Kx &- F_n x\| \\ &\le \|Kx - Kx_i\| + \|Kx_i - F_n x_i\| + \|F_n x_i - F_n x\| \\ &< \epsilon/3 + \|Kx_i - F_n x_i\| + \|P_n\|\,\|Kx_i - Kx\| \\ &< 2\epsilon/3 + \|Kx_i - F_n x_i\|. \end{aligned}$$

Choose $N > 0$ such that $\|Kx_i - F_n x_i\| < \epsilon/3$, $1 \le i \le m$ for all $n > N$. Then $\|Kx - F_n x\| \le 2\epsilon/3 + \epsilon/3 = \epsilon$. Thus $\|K - F_n\| < 3$ for all $n > N$. Since $\epsilon > 0$ was arbitrary, $K = \lim_{n\to\infty} F_n$.

(iii) $\Rightarrow$ (ii) Suppose $K = \lim_{n\to\infty} F_n$, where $F_n$ is finite rank for all $n \ge 1$. Note that $F_n^*$ is also finite rank (why?), and that $\|K^* - F_n^*\| = \|K - F_n\|$ for all $n \ge 1$, which clearly implies that $K^* = \lim_{n\to\infty} F_n^*$, and hence that $K^*$ is compact.

(ii) $\Rightarrow$(i) Since $K$ compact implies $K^*$ is compact from above, we deduce that $K^*$ compact implies $(K^*)^* = K$ is compact, completing the proof.

$\square$

We can restate the above Theorem more succinctly by saying that $\mathcal{K}(\mathcal{H})$ is the norm closure of the set of finite rank operators on $\mathcal{H}$. This is an extraordinarily useful result.

**4.3. Remark.** Contained in the above proof is the following interesting observation. If $K$ is a compact operator acting on a separable Hilbert space $\mathcal{H}$, then for *any* sequence $\{P_n\}_{n=1}^\infty$ of finite rank projections tending strongly (i.e. pointwise) to the identity, $\|K - P_n K\|$ tends to zero. By considering adjoints, we find that $\|K - KP_n\|$ also tends to zero.

Let $\epsilon > 0$, and choose $N > 0$ such that $n \geq N$ implies $\|K - KP_n\| < \epsilon/2$ and $\|K - P_n K\| < \epsilon/2$. Then for all $n \geq N$ we get

$$\begin{aligned} \|K - P_n K P_n\| &\leq \|K - KP_n\| + \|KP_n - P_n K P_n\| \\ &\leq \|K - KP_n\| + \|K - P_n K\| \, \|P_n\| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

It follows that if $\mathcal{H}$ has an orthonormal basis indexed by the natural numbers, say $\{e_n\}_{n=1}^\infty$, then the matrix for $K$ with respect to this basis comes within $\epsilon$ of the matrix for $P_N K P_N$. In other words, $K$ "virtually lives" on the "top left-hand corner".

Alternatively, if $\mathcal{H}$ has an orthonormal basis indexed by the integers, say $\{f_n\}_{n\in\mathbb{Z}}$, and we let $P_n$ denote the orthogonal projection onto span $\{e_k\}_{k=-n}^n$, then the matrix for $K$ with respect to this basis can be arbitrarily well estimated by a sufficiently large but finite "central block".

**4.4. Example.** Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^\infty$. Let $\{d_n\}_{n=1}^\infty$ be a bounded sequence and consider the diagonal operator $D \in \mathcal{B}(\mathcal{H})$ defined locally by $De_n = d_n e_n$ and extended to all of $\mathcal{H}$ by linearity and continuity.

Then $D \in \mathcal{K}(\mathcal{H})$ if and only if $\lim_{n\to\infty} d_n = 0$.

**4.5. Example.** Let $\mathcal{H} = L^2([0,1], dx)$, and consider the function $k(x,t) \in L^2([0,1] \times [0,1], dm)$, where $dm$ represents Lebesgue planar measure. Then we define a Volterra operator

$$\begin{array}{llll} V: & L^2([0,1], dx) & \to & L^2([0,1], dx) \\ & (Vf)(x) & = & \int_0^1 f(t)\, k(x,t)\, dt. \end{array}$$

(The classical Volterra operator has $k(x,t) = 1$ if $x \geq t$, and $k(x,t) = 0$ if $x < t$.)

Now for $f \in L^2([0,1], dx)$ we have

$$\begin{aligned}
\|Vf\|^2 &= \int_0^1 |Vf(x)|^2 dx \\
&= \int_0^1 \left| \int_0^1 f(t)\, k(x,t) dt \right|^2 dx \\
&\leq \int_0^1 \left( \int_0^1 |f(t)\, k(x,t)| dt \right)^2 dx \\
&\leq \int_0^1 \|f\|_2^2 \int_0^1 |k(x,t)|^2 dt dx \text{ by the Cauchy-Schwartz Inequality} \\
&= \|f\|_2^2\, \|k\|_2^2,
\end{aligned}$$

so that $\|V\| \leq \|k\|_2$.

Let $\mathcal{A}$ denote the algebra of continuous functions on $[0,1] \times [0,1]$ which can be resolved as $g(x,t) = \sum_{i=1}^n u_i(x)\, w_i(t)$. Then $\mathcal{A}$ is an algebra which separates points, contains the constant functions, and is closed under complex conjugation. By the Stone-Weierstraß Theorem, given $\epsilon > 0$ and $h \in \mathcal{C}([0,1] \times [0,1])$, there exists $g \in \mathcal{A}$ such that $\|h - g\|_2 \leq \|h - g\|_\infty < \epsilon$. But since $\mathcal{C}([0,1] \times [0,1])$ is dense (in the $L^2$-topology) in $L^2([0,1] \times [0,1], dm)$, $\mathcal{A}$ must also be dense (in the $L^2$-topology) in $L^2([0,1] \times [0,1], dm)$.

Let $\epsilon > 0$. For $k$ as above, choose $g \in \mathcal{A}$ such that $\|k - g\|_2 < \epsilon$. Define

$$\begin{array}{rlcl}
V_0 : & L^2([0,1], dx) & \rightarrow & L^2([0,1], dx) \\
& V_0 f(x) & = & \int_0^1 f(t)\, g(x,t) dt.
\end{array}$$

From above, we find that $\|V - V_0\| \leq \|k - g\|_2 < \epsilon$.

To see that $V_0$ is finite rank, consider the following; first, $g(x,t) = \sum_{i=1}^n u_i(x)\, w_i(t)$. If we set $\mathcal{M} = \text{span}_{\,1 \leq i \leq n}\{u_i\}$, then $\mathcal{M}$ is a finite dimensional subspace of $L^2([0,1], dx)$. Moreover,

$$\begin{aligned}
V_0 f(x) &= \int_0^1 f(t)\, g(x,t) dt \\
&= \sum_{i=1}^n \left( \int_0^1 f(t)\, w_i(t) dt \right) u_i(x),
\end{aligned}$$

so that $V_0 f \in \mathcal{M}$.

Thus $V$ can be approximated arbitrarily well by elements of the form $V_0 \in \mathcal{F}(L^2([0,1], dx)$, and so $V$ is compact.

**4.6. Definition.** *Let $\mathfrak{X}$ be a Banach space and $T \in \mathcal{B}(\mathfrak{X})$. Then $T$ is said to be* ***quasinilpotent*** *if $\sigma(T) = 0$. By the spectral mapping theorem 2.13, it is easily seen that every nilpotent operator is automatically quasinilpotent.*

**4.7. Example.** Let $V$ denote the classical Volterra operator defined in Example 4.5 above. We shall show that $V$ is quasinilpotent. (Note that we have seen that the Volterra operator acting in $\mathcal{B}(\mathcal{C}[0,1])$ is quasinilpotent in Example 1.8.)

Since $V \in \mathcal{K}(\mathcal{H})$, we know that $\sigma(V) = \{0\} \cup \sigma_p(V)$. Suppose $0 \neq \lambda \in \sigma_p(V)$, and that $f \in \ker(\lambda - V)$. Then

$$\begin{aligned} |\lambda|\,|f(x)| &= \left| \int_0^x f(t)dt \right| \\ &\leq \int_0^x |f(t)|dt \\ &\leq \int_0^1 |f(t)|dt \\ &\leq \|f\|_2\,\|1\|_2 \\ &= \|f\|_2. \end{aligned}$$

Then for $0 \leq x \leq 1$,

$$\begin{aligned} |f(x)| &\leq (1/|\lambda|) \int_0^x |f(t_1)|dt_1 \\ &\leq (1/|\lambda|) \int_0^x (1/|\lambda|) \int_0^{t_1} |f(t_2)|dt_2 dt_1 \\ &\leq \cdots \\ &\leq (1/|\lambda|^{n+1}) \int_0^x \int_0^{t_1} \cdots \int_0^{t_n} |f(t_{n+1})|dt_{n+1}dt_n \ldots dt_1 \\ &\leq (1/|\lambda|^{n+1}) \int_0^x \int_0^{t_1} \cdots \int_0^{t_n} (\|f\|_2/|\lambda|)dt_{n+1}dt_n \ldots dt_1 \\ &= (1/|\lambda|^{n+2})\,\|f\|_2 \int_0^x \int_0^{t_1} \cdots \int_0^{t_n} 1\,dt_{n+1}dt_n \ldots dt_1 \\ &\leq (1/|\lambda|^{n+2})\,\|f\|_2\, x^{n+1}/(n+1)! \text{ for all } n \geq 1. \end{aligned}$$

Thus $f(x) = 0$ for all $x \in [0,1]$, and hence $f = 0$. But then $\lambda \notin \sigma_p(V)$. Therefore $\sigma(V) = \{0\}$, and so $V$ is quasinilpotent as claimed.

**4.8. Definition.** *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{M}$ be a subspace of $\mathcal{H}$, and suppose that $T \in \mathcal{B}(\mathcal{H})$. Recall that $\mathcal{M}$ is called* ***invariant*** *for $T$ provided that $T\mathcal{M} \subseteq \mathcal{M}$. We say that $\mathcal{M}$ is* ***reducing*** *for $T$ if $\mathcal{M}$ is invariant both for $T$ and for $T^*$.*

**4.9. Notation.** Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ be a subspace of $\mathcal{H}$. By $P(\mathcal{M})$ we shall denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$.

**4.10. Proposition.** *Let $\mathcal{H}$ be a Hilbert space, $T \in \mathcal{B}(\mathcal{H})$, and $\mathcal{M}$ be a subspace of $\mathcal{H}$. Then $\mathcal{M}$ is reducing for $T$ if and only if both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant for $T$. When this is the case, we can write*

$$T = T_1 \oplus T_2 = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$$

*with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. Furthermore, $T$ is compact if and only if both $T_1$ and $T_2$ are compact, and $T$ is normal if and only if $T_1$ and $T_2$ are.*

**Proof.** First suppose that $\mathcal{M}$ is reducing for $T$. Then $(I - P(\mathcal{M}))\, T\, P(\mathcal{M}) = 0$. Since $T^*\mathcal{M} \subseteq \mathcal{M}$, we also get $(I - P(\mathcal{M}))\, T^*\, P(\mathcal{M}) = 0$, and so after taking adjoints, $P(\mathcal{M})\, T\, (I - P(\mathcal{M})) = 0$. (Note that $P(\mathcal{M})$ is self-adjoint.) It follows that both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant for $T$.

Now suppose that $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant for $T$, so that

$$(I - P(\mathcal{M}))\, T\, P(\mathcal{M}) = 0 = P(\mathcal{M})\, T\, (I - P(\mathcal{M})).$$

By taking adjoints once more, $(I - P(\mathcal{M}))\, T^*\, P(\mathcal{M}) = 0$, and so $\mathcal{M}$ is reducing for $T$. The matrix form for $T$ follows directly from these observations.

If $T_1$ and $T_2$ are compact, then they are limits of finite rank operators $F_n$ and $G_n$ respectively, from which we conclude that $T$ is a limit of the finite rank operators $F_n \oplus G_n$. Thus $T$ is compact.

If $T$ is compact, then the compression of $T$ to any subspace is compact, and so both $T_1$ and $T_2$ are compact.

We leave it as an exercise to the reader to show that $T^* = T_1^* \oplus T_2^*$. Given this, it is easy to see that $T$ is normal if and only if $0 = [T, T^*] = [T_1, T_1^*] \oplus [T_2, T_2^*]$, which is equivalent to the simultaneous normality of $T_1$ and $T_2$.

□

**4.11. Proposition.** *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ be normal. Then* $\ker N = \ker N^*$ *is reducing for $N$.*

**Proof.** That $\ker N = \ker N^*$ is the second half of Lemma 3.16. Now let $x \in \ker N$. Then $N^2 x = N(Nx) = 0$, and $NN^*x = N^*Nx = 0$. Thus $\ker N$ is invariant for both $N$ and $N^*$, and hence is reducing for $N$.

□

**4.12. Proposition.** *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$. If $N$ is normal and $\lambda \neq \mu \in \sigma_p(N)$, then* $\ker(N - \lambda I)$ *is orthogonal to* $\ker(N - \mu I)$.

**Proof.** Let $x \in \ker(N - \lambda I)$ and $y \in \ker(N - \mu I)$. Then

$$\lambda(x, y) = (Nx, y) = (x, N^* y) = (x, \overline{\mu} y) = \mu(x, y).$$

Thus $(\lambda - \mu)(x, y) = 0$. Since $\lambda - \mu \neq 0$, we must have $x \perp y$.

□

**4.13. Proposition.** *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$. If $N$ is normal, then* spr $(N) = \|N\|$.

**Proof.** Consider first:

$$\begin{aligned} \|N^2\| &= \sup_{\|x\|=1} \|N^2 x\| \\ &= \sup_{\|x\|=1} \|N^* N x\| \\ &\geq \sup_{\|x\|=1} |(N^* N x, x)| \\ &= \sup_{\|x\|=1} (Nx, Nx) \\ &= \sup_{\|x\|=1} \|Nx\|^2 \\ &= \|N\|^2. \end{aligned}$$

By induction, $\|N^{2^n}\| \geq \|N\|^{2^n}$ for all $n \geq 1$. The reverse inequality follows immediately from the submultiplicativity of the norm in a Banach algebra. Thus $\|N^{2^n}\| = \|N\|^{2^n}$ for all $n \geq 1$. By Beurling's Spectral Radius Formula, Theorem 2.1.36,

$$\text{spr } (N) = \lim_{n \to \infty} \|N^{2^n}\|^{1/2^n} = \|N\|.$$

□

**4.14. Corollary.** *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$. If $N$ is normal and $\sigma(N) = \{\lambda\}$, then $N = \lambda I$.*

**Proof.** Now $\sigma(N - \lambda I) = \{0\}$ by the Spectral Mapping Theorem. Since $N - \lambda I$ is also normal, $\|N - \lambda I\| = \text{spr } (N - \lambda I) = 0$.

□

**4.15. Lemma.** *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$. Suppose $N$ is compact and normal and that $\{\lambda_i\}_{i=1}^n \subseteq \sigma_p(N)$. Let $\mathcal{M} = \oplus_{i=1}^n \ker (N - \lambda_i I)$. Then $\mathcal{M}$ is a reducing subspace for $N$ and if $N_1 = (I - P(\mathcal{M}))N|_{\mathcal{M}^\perp} \in \mathcal{B}(\mathcal{M}^\perp)$, then $\sigma_p(N_1) = \sigma_p(N) \backslash \{\lambda_i\}_{i=1}^n$.*

**Proof.**

That $\mathcal{M}$ is reducing for $N$ follows from the fact that each $\ker (N - \lambda_i I)$ is reducing for $N$, $1 \leq i \leq n$. Now $N_1$ is both compact and normal by Proposition 4.10.

Suppose $\lambda \in \rho(N)$. Then $(N_1 - \lambda I)^{-1} = (I - P(\mathcal{M}))(N - \lambda I)^{-1}|_{\mathcal{M}^\perp}$, so that $\lambda \in \rho(N_1)$.

Let $\lambda \in \{\lambda_i\}_{i=1}^n$. Then $\ker (N - \lambda I) \subseteq \mathcal{M}$ by definition. Thus $N_1 - \lambda I$ is injective, so that $\lambda \notin \sigma_p(N_1)$. If $\lambda \in \sigma_p(N) \backslash \{\lambda_i\}_{i=1}^n$, then $\ker (N - \lambda I)$ is orthogonal to $\mathcal{M}$, so there exists $0 \neq x \in \ker (N - \lambda I)$ and then $(N_1 - \lambda I)x = (N - \lambda I)x = 0$. Hence $\lambda \in \sigma_p(N_1)$.

We now have $\sigma_p(N) \backslash \{\lambda_i\}_{i=1}^n \subseteq \sigma_p(N_1) \subseteq \sigma(N) \backslash \{\lambda_i\}_{i=1}^n$.

Finally, if $\lambda \in \sigma_p(N_1)$, then there exists $0 \neq x \in \mathcal{M}^\perp$ such that $(N_1 - \lambda I)x = 0$. But then $(N - \lambda I)x = 0$, so that $\lambda \in \sigma_p(N)$, completing the proof.

□

**4.16. Proposition.** *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$. Suppose $N$ is compact and normal and that $\sigma_p(N) = \{\lambda_i\}_{i=1}^n$. Then $\mathcal{H} = \oplus_{n=1}^\infty \ker(N - \lambda_n I)$.*

**Proof.** Let $\mathcal{M} = \oplus_{n=1}^\infty \ker(N - \lambda_n I)$. As above, $\mathcal{M}$ is reducing for $N$. Let $N_1 = P(\mathcal{M}^\perp)N|_{\mathcal{M}^\perp}$, viewed as an element of $\mathcal{B}(\mathcal{M}^\perp)$. Then $\sigma_p(N_1)$ is empty, for if $\lambda \in \sigma_p(N_1)$, then as in the previous lemma, we see that $\lambda \in \sigma_p(N)$, and hence $\ker(N_1 - \lambda I) \subseteq \ker(N - \lambda I) \subseteq \mathcal{M}$, a contradiction.

Since $N_1$ is compact, $\sigma(N_1) = \{0\}$, but $0 \notin \sigma_p(N_1)$ implies that $N_1$ is injective. On the other hand, $N_1$ is also normal, so $\|N_1\| = \text{spr }(N_1) = 0$, and hence $N_1 = 0$. Since it is injective, we are forced to conclude that $\mathcal{M}^\perp = \{0\}$, completing the proof.

□

.

**4.17. Theorem. The spectral theorem for compact normal operators.** *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ be a compact, normal operator. Suppose $\{\lambda_k\}_{k=1}^\infty$ are the distinct eigenvalues of $N$ and that $P(\mathcal{M}_k)$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}_k = \ker(N - \lambda_k I)$. Then $P(\mathcal{M}_k)\,P(\mathcal{M}_j) = 0 = P(\mathcal{M}_j)\,P(\mathcal{M}_k)$ if $j \neq k$, and*

$$N = \sum_{k=1}^\infty \lambda_k P(\mathcal{M}_k),$$

*where the series converges in the norm topology in $\mathcal{B}(\mathcal{H})$.*

**Proof.** That $P(\mathcal{M}_k)\,P(\mathcal{M}_j) = 0 = P(\mathcal{M}_j)\,P(\mathcal{M}_k)$ if $j \neq k$ is simply the statement that $\mathcal{M}_k$ is orthogonal to $\mathcal{M}_j$ for $j \neq k$, and this we saw in Proposition 4.12.

Recall also that $\lim_{k\to\infty} \lambda_k = 0$, by Theorem 2.16.

Consider $n > 0$, and $N - \sum_{k=1}^n \lambda_k P(\mathcal{M}_k)$. If $x \in \mathcal{M}_j$ for some $1 \leq j \leq n$, then

$$(N - \sum_{k=1}^\infty \lambda_k P(\mathcal{M}_k))x = Nx - \lambda_j x = 0.$$

Thus $\oplus_{k=1}^n \mathcal{M}_k \subseteq \ker(N - \sum_{k=1}^\infty \lambda_k P(\mathcal{M}_k))$. If $x$ is orthogonal to $\oplus \mathcal{M}_k$, then $P(\mathcal{M}_k)x = 0$, $1 \leq k \leq n$, so that $(N - \sum_{k=1}^\infty \lambda_k P(\mathcal{M}_k))x = Nx$. Moreover, $\oplus_{k=1}^n \mathcal{M}_k$ reduces $N$, so we let $N_n = P((\oplus_{k=1}^n \mathcal{M}_k)^\perp)N|_{\oplus_{k=1}^n \mathcal{M}_k{}^\perp}$.

Then $\|N - \sum_{k=1}^\infty \lambda_k P(\mathcal{M}_k)\| = \|N_n\|$. Also, $N_n$ is compact and normal by Proposition 4.10, and from Lemma 4.15,

$$\sigma_p(N_n) = \{\lambda_{n+1}, \lambda_{n+2}, \lambda_{n+3}, \ldots\}.$$

Thus $\|N_n\| = \text{spr}\,(N_n) = \sup_{k>n} |\lambda_k|$. In particular, $\lim_{n\to\infty} \|N_n\| = 0$, so that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_k P(\mathcal{M}_k) = \sum_{k=1}^{\infty} \lambda_k P(\mathcal{M}_k) = N.$$

□

**4.18. Corollary.** *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ be a compact, normal operator. Then there exists an orthonormal basis $\{e_\alpha\}_{\alpha\in\Lambda}$ for $\mathcal{H}$ such that each $e_\alpha$ is an eigenvector for $N$.*

**Proof.** Let $\{\lambda_n\}_{n=1}^\infty$ be the set of eigenvalues of $N$. For each $n \ge 1$, choose an orthonormal basis $\{e_{(n,\beta)}\}_{\beta\in\Lambda_n}$ for $\ker(N - \lambda_n I)$. (Note that if $\lambda_n \ne 0$, then the cardinality of $\Lambda_n$ is finite.) Then each $e_{(n,\beta)}$, $\beta \in \Lambda_n$, $n \ge 1$ is an eigenvector for $N$ corresponding to $\lambda_n$, the $e_{(n,\beta)}$'s are all orthogonal since all of the $\ker(N-\lambda_n I)$'s are. Finally, $\overline{\text{span}}\{e_{(n,\beta)}\}_{\beta\in\Lambda_n,\, n\ge1} = \oplus_{n=1}^\infty \ker(N-\lambda_n I) = \mathcal{H}$ by Proposition 4.16. Let $\{e_\alpha\}_{\alpha\in\Lambda} = \{e_{(n,\beta)}\}_{\beta\in\Lambda_n,\, n\ge1}$.

□

**4.19. Corollary.** *Let $\mathcal{H}$ be a Hilbert space and let $N \in \mathcal{B}(\mathcal{H})$. Then $N$ is compact and normal if and only if there exist an orthonormal set $\{f_n\}_{n=1}^\infty$ and a sequence of scalars $\{\beta_n\}_{n=1}^\infty$ such that*

(i) $\lim_{n\to\infty} \beta_n = 0$;
(ii) $N f_n = \beta_n f_n$, $n \ge 1$;
(iii) $Nx = 0$ *if* $x \in \mathcal{H}$, $x$ *orthogonal to* $\overline{\text{span}}\{f_n\}_{n=1}^\infty$.

**Proof.** Suppose the sets $\{f_n\}_{n=1}^\infty$ and $\{\beta\}_{n=1}^\infty$ as above exist. Then $N$ is seen to be compact, using the arguments of Theorem 4.2.

Now if $N$ is normal and compact, let $\{e_\alpha\}_{\alpha\in\Lambda}$ be an orthonormal basis for $\mathcal{H}$ consisting of eigenvectors of $N$, the existence of which is guaranteed by the preceding Corollary. Let $\{f_n\}_{n=1}^\infty$ be the subset of $\{e_\alpha\}_{\alpha\in\Lambda}$ comprised of those vectors whose corresponding eigenvalues $\{\beta_n\}_{n\ge1}$ are different from zero. That $\{f_n\}_{n\ge1}$ is at most countable follows from the fact that $\sigma_p(N)$ is countable, and $\text{nul}\,(N - \lambda_n I) < \infty$ for all $0 \ne \lambda_n \in \sigma_p(N)$.

Clearly $N f_n = \beta_n f_n$ for all $n \ge 1$, and $\lim_{n\to\infty} \beta_n = 0$ from the argument above combined with the fact that $\sigma_p(N)$ is a sequence tending to zero when $N$ is compact. Finally, $(\overline{\text{span}}\{f_n\}_{n=1}^\infty)^\perp = \ker(N - 0I) = \ker N$, from which condition (iii) also follows.

□

*A rolling Stone gathers no satisfaction.*

## 5. Fredholm theory in Hilbert space

**5.1.** The notion of a Fredholm operator was introduced in Section Two of this Chapter, where it was shown that if $K$ is a compact operator acting on a Banach space $\mathfrak{X}$ and if $\lambda$ is a non-zero scalar, then $\lambda I - K$ is Fredholm of index zero. We now wish to consider Fredholm operators acting on a Hilbert space. We shall establish the fact that the Fredholm operators are precisely the operators which are invertible modulo the compact operators, and that the index function serves to classify components of the set of invertible elements in the Calkin algebra .

**5.2.** Recall from Definition 2.17 that an operator $T$ acting on a Hilbert space $\mathcal{H}$ is said to be *Fredholm* if

(i) $\operatorname{ran} T$ is closed;
(ii) $\operatorname{nul} T$ is finite; and
(iii) $\operatorname{codim} \operatorname{ran} T$ is finite.

As before, when $T$ is Fredholm we may define the *Fredholm index* of $T$ to be

$$\operatorname{ind} T = \operatorname{nul} T - \operatorname{codim} \operatorname{ran} T.$$

From Remark 2.9, we see that when $T$ is Fredholm, we may replace $\operatorname{codim} \operatorname{ran} T$ by $\operatorname{nul} T^t$, where $T^t$ now denotes the *Banach space adjoint* of $T$, as opposed to the *Hilbert space adjoint* of $T$, which we denote by $T^*$. The distinction is important, since it is not *a priori* obvious that we may replace $\operatorname{codim} \operatorname{ran} T$ by $\operatorname{nul} T^*$. On the other hand, since $\operatorname{ran} T$ is closed, we obtain the decomposition $\mathcal{H} = \operatorname{ran} T \oplus (\operatorname{ran} T)^\perp$, and so

$$\operatorname{codim} \operatorname{ran} T = \dim (\mathcal{H}/\operatorname{ran} T) = \dim (\operatorname{ran} T)^\perp.$$

Since $(\operatorname{ran} T)^\perp = \ker T^*$, it follows that $\operatorname{nul} T^* = \operatorname{codim} \operatorname{ran} T = \operatorname{nul} T^t$, and so, as in the Banach space setting, we retrieve the equation

$$\operatorname{ind} T = \operatorname{nul} T - \operatorname{nul} T^*.$$

**5.3. Example.** Let $\mathcal{H}$ be a separable Hilbert space and let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis for $\mathcal{H}$. Let $S \in \mathcal{B}(\mathcal{H})$ denote the unilateral shift operator acting on this basis as defined in Example 3.9. That is, $Se_{n+1} = e_n$ if $n \geq 1$, and $Se_1 = 0$.

Then $\operatorname{ran} S = \mathcal{H}$, so that $\operatorname{ran} S$ is closed. Also, $\ker S = \operatorname{span}\{e_1\}$, so that $\operatorname{nul} S = 1$. Finally, $\ker S^* = \{0\}$, so that $\operatorname{nul} S^* = 0$. Thus $S$ is Fredholm and $\operatorname{ind} S = \operatorname{nul} S - \operatorname{nul} S^* = 1 - 0 = 1$.

Note also that $S^*$ is Fredholm as well, and that $\operatorname{ind} S^* = \operatorname{nul} S^* - \operatorname{nul} S^* * = \operatorname{nul} S^* - \operatorname{nul} S = -\operatorname{ind} S = 1$.

Finally, $S^n$ and $(S^*)^n$ are both Fredholm as well, and $\operatorname{ind} S^n = -n = -\operatorname{ind} (S^*)^n$ for each $n \geq 1$.

**5.4. Example.** Let $\mathcal{H}$ be a Hilbert space and $K \in \mathcal{K}(\mathcal{H})$. As we have seen, if $0 \neq \lambda \in \mathbb{C}$, then $\lambda I - K$ is Fredholm of index zero. It follows that so is any operator of the form $T + L$ where $T$ is invertible and $L$ is compact. Indeed, $T + L = T(I - (-T^{-1}L)$. The verification of the index is left to the reader.

It follows that if $S$ is the unilateral forward shift, then $S$ is not of the form $T + L$ for any $T$ invertible and $L$ compact.

**5.5. Proposition.** *Suppose $\mathcal{H}$ is a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ is Fredholm. Then $T|_{(\ker T)^\perp}$ is bounded below.*

**Proof.** Suppose $x, y \in (\ker T)^\perp$. Then $0 = Tx - Ty = T(x - y)$ implies $x - y \in \ker T$ and hence $x = y$. In particular, the map

$$\begin{array}{rlcl} T_0 : & (\ker T)^\perp & \to & \operatorname{ran} T \\ & x & \mapsto & Tx \end{array}$$

is a 1-1, onto map, and thus it is invertible. Let $R : \operatorname{ran} T \to (\ker T)^\perp$ denote the inverse of $T_0$. Then for $x \in (\ker T)^\perp$,

$$\begin{aligned} \|x\| = \|R T_0 x\| &= \|R T x\| \\ &\leq \|R\| \, \|Tx\| \end{aligned}$$

and so $\|Tx\| \geq \|R\|^{-1} \|x\|$.

Thus $T$ is bounded below on $(\ker T)^\perp$, as claimed.

□

**5.6. Remark.** Let $\mathcal{H}$ be a Hilbert space. Recall from Example 1.1.17 that the Calkin algebra $\mathcal{A}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the quotient of $\mathcal{B}(\mathcal{H})$ by the closed, two-sided ideal of compact operators. It follows from Proposition 2.1.16 that $\mathcal{A}(\mathcal{H})$ is a Banach algebra.

**5.7. Remark.** Our present goal is to establish a relationship between the set of Fredholm operators acting on a Hilbert space $\mathcal{H}$, and the set of invertible elements in the Calkin algebra. In fact, the relation we wish to establish is *equality*!

We record here a couple of facts which will prove useful:

- $(\mathcal{A}(\mathcal{H}))^{-1}$ is open in $\mathcal{A}(\mathcal{H})$.
- The involution on $\mathcal{B}(\mathcal{H})$ naturally gives rise to an involution in the Calkin algebra. Given $t \in \mathcal{A}(\mathcal{H})$, $t = \pi(T)$ for some $T \in \mathcal{B}(\mathcal{H})$. We then set $t^* = \pi(T^*)$. If $R \in \mathcal{B}(\mathcal{H})$ and $\pi(R) = t$, then $K = R - T \in \mathcal{K}(\mathcal{H})$. Thus $K^* = R^* - T^* \in \mathcal{K}(\mathcal{H})$, and so $\pi(R^*) = \pi(T^*)$, from which it follows that our involution is indeed well-defined. We then have that $\mathcal{A}(\mathcal{H})$ and $(\mathcal{A}(\mathcal{H}))^{-1}$ are self-adjoint. Indeed, for $t \in (\mathcal{A}(\mathcal{H}))^{-1}$, $(t^*)^{-1} = (t^{-1})^*$.

**5.8. Theorem.** *Let $\mathcal{H}$ be a Hilbert space, and $T \in \mathcal{B}(\mathcal{H})$. Then $T$ is Fredholm if and only if $t = \pi(T)$ is invertible in the Calkin algebra.*

**Proof.** Suppose $T$ is Fredholm. Then $\operatorname{ran} T$ is closed, $\operatorname{nul} T$ is finite and $\operatorname{nul} T^* = \operatorname{codim} \operatorname{ran} T$ is finite. Let us once again decompose

$$\begin{array}{rclcl} \mathcal{H} & = & \ker T & \oplus & (\ker T)^\perp \\ & = & \operatorname{ran} T & \oplus & (\operatorname{ran} T)^\perp. \end{array}$$

As in the proof of Proposition 5.5, we see that

$$\begin{array}{rlcl} T_0 : & (\ker T)^\perp & \to & \operatorname{ran} T \\ & x & \mapsto & Tx \end{array}$$

is invertible. Let $R_0 : \operatorname{ran} T \to (\ker T)^\perp$ denote the inverse of $T_0$, and define $R \in \mathcal{B}(\mathcal{H})$ via $Rx = \begin{cases} R_0 x & \text{if } x \in \operatorname{ran} T \\ 0 & \text{if } x \in (\operatorname{ran} T)^\perp. \end{cases}$.

Then $RTx = (I - P(\mathcal{M}))x$, where $\mathcal{M} = \ker T$, and $TRx = (I - P(\mathcal{N}))x$, where $\mathcal{N} = (\operatorname{ran} T)^\perp$. Since both $P(\mathcal{M})$ and $P(\mathcal{N})$ are finite rank, we obtain:

$$\pi(R)\,\pi(T) = \pi(RT) = \pi(I) = \pi(TR) = \pi(T)\,\pi(R)$$

, and so $t$ is invertible with inverse $r = \pi(R)$.

Next, suppose that $t = \pi(T) \in \mathcal{A}(\mathcal{H})$ is invertible. Then there exists $r \in \mathcal{A}(\mathcal{H})$ with $rt = 1 = \pi(I) = tr$, and choosing $R \in \mathcal{B}(\mathcal{H})$ with $\pi(R) = r$, we get

$$RT = I + K_1, \qquad TR = I + K_2$$

for some $K_1,\ K_2 \in \mathcal{K}(\mathcal{H})$.

Since $\operatorname{nul}(I + K_1) < \infty$ by Proposition 2.8, $\operatorname{nul} T < \infty$. Since $\operatorname{ran} T \supseteq \operatorname{ran}(I + K_2)$ and $\operatorname{codim} \operatorname{ran}(I + K_2) < \infty$ by Proposition 2.8, $\operatorname{codim} \operatorname{ran} T < \infty$.

By Corollary 2.25, $\operatorname{ran} T$ is closed, and so we are done.

□

**5.9.** We now wish to consider some of the stability properties of Fredholm operators and the index function. We mention that most, if not all, of the following results are true for Fredholm operators acting on a Banach space. On the other hand, certain arguments simplify when looking at Hilbert spaces, and we have made use of these simplifications. For the most general results, we refer the reader to the book of Caradus, Pfaffenberger and Yood [**CPY74**].

**5.10. Lemma.** *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ be Fredholm. If $R$ is invertible, then*

$$\operatorname{ind} TR = \operatorname{ind} RT = \operatorname{ind} T.$$

**Proof.** Exercise.

□

**5.11. Lemma.** *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. If $T$ is Fredholm and* $\operatorname{ind} T = 0$*, then there exists a finite rank operator $F$ such that $T + F$ is invertible.*

**Proof.** As we saw in Theorem 5.8, we can decompose $\mathcal{H}$ in two ways, namely:

$$\begin{aligned} \mathcal{H} &= \ker T \oplus (\ker T)^{\perp} \\ &= \operatorname{ran} T \oplus (\operatorname{ran} T)^{\perp}. \end{aligned}$$

Since $\operatorname{ind} T = 0$ by hypothesis, $\operatorname{nul} T = \operatorname{codim} \operatorname{ran} T$. Let $\{e_k\}_{k=1}^n$ and $\{f_k\}_{k=1}^n$ be orthonormal bases for $\ker T$ and $(\operatorname{ran} T)^{\perp}$ respective and let $F \in \mathcal{B}(\mathcal{H})$ be defined via $Fe_k = f_k$, $1 \le k \le n$, $Fz = 0$ if $z$ is orthogonal to $\ker T$. Then $F$ is clearly finite rank. We claim that $T + F$ is bijective, and hence invertible.

If $0 \neq x \in \mathcal{H}$, then $x = x_1 + x_2$, where $x_1 \in \ker T$, $x_2 \in (\ker T)^{\perp}$, and $\|x_1\| + \|x_2\| \neq 0$. If $x_1 \neq 0$, then

$$\begin{aligned} (T + F)x &= Tx + Fx \\ &= Tx_2 + Fx_1 \end{aligned}$$

and $0 \neq Fx_1 \in (\operatorname{ran} T)^{\perp}$ forces $(T + F)x \neq 0$. If $x_2 \neq 0$, then $(T + F)x = Tx_2 + Fx_1$ and $0 \neq Tx_2 \in (\operatorname{ran} F)^{\perp}$ forces $(T + F)x \neq 0$.

In either case, we see that $T + F$ is injective.

Now choose $y \in \mathcal{H}$ and decompose $y$ as $y = y_1 + y_2$ where $y_1 \in \operatorname{ran} T$ and $y_2 \in (\operatorname{ran} T)^{\perp}$. Choose $x_1 \in (\ker T)^{\perp}$ such that $Tx_1 = y_1$ and $x_2 \in \ker T$ such that $Fx_2 = y_2$. Then

$$\begin{aligned} (T + F)(x_1 + x_2) &= T(x_1 + x_2) + F(x_1 + x_2) \\ &= Tx_1 + Fx_2 \\ &= y_1 + y_2 \\ &= y. \end{aligned}$$

Thus $T$ is surjective, and therefore bijective, completing the proof.

□

**5.12. Theorem.** *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ be Fredholm. If $K \in \mathcal{K}(\mathcal{H})$, then*

$$\operatorname{ind}(T + K) = \operatorname{ind} T.$$

**Proof.** Suppose $\operatorname{ind} T = 0$. Then there exists $F$ finite rank such that $T + F$ is invertible. Moreover, $T + K = (T + F) + (K - F)$ and $K - F \in \mathcal{K}(\mathcal{H})$. By Lemma 5.10, $\operatorname{ind}(T + K) = 0 = \operatorname{ind} T$.

Suppose next that $\operatorname{ind} T = n > 0$. Letting $S$ denote the forward unilateral shift, $\operatorname{ind}(T \oplus S^n) = \operatorname{ind} T + \operatorname{ind} S^n = 0$. If $K \in \mathcal{K}(\mathcal{H})$, then $K \oplus 0 \in \mathcal{K}(\mathcal{H} \oplus \mathcal{H})$, and

$$\begin{bmatrix} T & 0 \\ 0 & S^n \end{bmatrix} + \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T + K & 0 \\ 0 & S^n \end{bmatrix}.$$

From above, $\mathrm{ind}\,((T+K)\oplus S^n) = 0 = \mathrm{ind}\,(T+K)+\mathrm{ind}\,S^n = \mathrm{ind}\,(T+K)-n$. Thus $\mathrm{ind}\,(T+K) = n = \mathrm{ind}\,T$.

If $\mathrm{ind}\,T = n < 0$, then $\mathrm{ind}\,T^* = -n > 0$. From above, for all $K \in \mathcal{K}(\mathcal{H})$,

$$\mathrm{ind}\,(T^* + K^*) = -n = -\mathrm{ind}\,(T+K),$$

and so $\mathrm{ind}\,(T+K) = n = \mathrm{ind}\,T$.

□

**5.13. Theorem.** *Let $\mathcal{H}$ be a Hilbert space and suppose that $T$, $R \in \mathcal{B}(\mathcal{H})$ are Fredholm. Then*

$$\mathrm{ind}\,TR = \mathrm{ind}\,T + \mathrm{ind}\,R.$$

**Proof.** First suppose that $\mathrm{ind}\,T = n > 0$ and $\mathrm{ind}\,R = m > 0$. Let $S$ denote the unilateral forward shift. Then

$$\mathrm{ind}\begin{bmatrix} T & 0 \\ 0 & S^n \end{bmatrix} = 0 = \begin{bmatrix} R & 0 \\ 0 & S^m \end{bmatrix}.$$

Thus there exists $K \in \mathcal{K}(\mathcal{H}\oplus\mathcal{H})$ such that $\begin{bmatrix} T & 0 \\ 0 & S^n \end{bmatrix} + K$ is invertible.

By Lemma 5.10,

$$0 = \mathrm{ind}\begin{bmatrix} R & 0 \\ 0 & S^m \end{bmatrix} = \mathrm{ind}\left(\begin{bmatrix} T & 0 \\ 0 & S^n \end{bmatrix} + K\right)\left(\begin{bmatrix} R & 0 \\ 0 & S^m \end{bmatrix}\right),$$

and by Theorem 5.12, this is equal to

$$\mathrm{ind}\begin{bmatrix} T & 0 \\ 0 & S^n \end{bmatrix}\begin{bmatrix} R & 0 \\ 0 & S^m \end{bmatrix} = \mathrm{ind}\begin{bmatrix} TR & 0 \\ 0 & S^{n+m} \end{bmatrix}.$$

Thus $0 = \mathrm{ind}\,(TR) + \mathrm{ind}\,S^{n+m} = \mathrm{ind}\,TR + (-n-m)$, and so $\mathrm{ind}\,(TR) = n+m = \mathrm{ind}\,T + \mathrm{ind}\,R$.

The cases where $n < 0$ (resp. $m < 0$) are handled similarly using $(S^*)^n$ (resp. $(S^*)^m$) instead of $S^n$ (resp. $S^m$), and Theorem 5.12 if necessary.

□

**5.14. Notation.** Let $\mathrm{Fred}(\mathcal{H}) = \pi^{-1}(\mathcal{A}(\mathcal{H})^{-1})$ denote the set of Fredholm operators, and for each $n \in \mathbb{Z}$, set

$$\mathrm{Fred}_n(\mathcal{H}) = \{T \in \mathrm{Fred}(\mathcal{H}) : \mathrm{ind}\,T = n\}.$$

**5.15. Theorem.** *Let $\mathcal{H}$ be a Hilbert space. Then for each $n \in \mathbb{Z}$, $\mathrm{Fred}_n(\mathcal{H})$ is open. In particular, therefore, $\mathrm{ind}(\cdot)$ is a continuous function on $\mathrm{Fred}(\mathcal{H})$.*

**Proof.** Of course, since $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{A}(\mathcal{H})$ is continuous, we see that $\mathrm{Fred}(\mathcal{H}) = \pi^{-1}((\mathcal{A}(\mathcal{H})^{-1})$ is open. Suppose $n \in \mathbb{Z}$.

Let $T \in \mathrm{Fred}_n(\mathcal{H})$. Since $\mathrm{Fred}(\mathcal{H})$ is open, there exists $\epsilon_1 > 0$ such that $\|U\| < \epsilon_1$ implies $T + U \in \mathrm{Fred}(\mathcal{H})$. Moreover, by Theorem 5.8, there exists $R \in \mathcal{B}(\mathcal{H})$ (in fact, $R \in \mathrm{Fred}_{-n}(\mathcal{H})$) such that $TR = I + K$ for some $K \in \mathcal{K}(\mathcal{H})$. Note that

$$\begin{aligned}(T+U)R &= TR + UR \\ &= (I+K) + UR \\ &= (I+UR) + K.\end{aligned}$$

Now take $\epsilon_2 = 1/\|R\|$. If $\|U\| < \epsilon_2$, then $I + UR$ is invertible in $\mathcal{B}(\mathcal{H})$. By Theorem 5.12, we conclude that if $\|U\| < \min(\epsilon_1, \epsilon_2)$, then $(T + U)R = (I + UR) + K$ satisfies

$$\begin{aligned}\mathrm{ind}\,(T+U)R &= \mathrm{ind}\,(I+UR) + K \\ &= (I+UR) \\ &= 0 \\ &= \mathrm{ind}\,(T+U) + \mathrm{ind}\,R \\ &= \mathrm{ind}\,T + \mathrm{ind}\,R.\end{aligned}$$

Thus $\mathrm{ind}\,(T + U) = \mathrm{ind}\,T$ and so $T + U \in \mathrm{Fred}_n(\mathcal{H})$. In other words, $\mathrm{Fred}_n(\mathcal{H})$ is open.

□

*Monkeys can't buy happiness.*

## Notes for Chapter Three

Theorem 4.2 shows us that in a Hilbert space $\mathcal{H}$, every compact operator $K$ is a norm limit of finite operators $F_n$. Since $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H})$, it follows that $\mathcal{K}(\mathcal{H}) = \overline{\mathcal{F}(\mathcal{H})}$.

In the Banach space setting, the inclusion $\mathcal{F}(\mathfrak{X}) \subseteq \mathcal{K}(\mathfrak{X})$ remains valid. The question of whether the reverse inclusion holds remained open for some time, and was referred to as the Finite Approximation Problem. In 1973, Per Enflo [**Enf73**] resolved this question by constructing an example of a Banach space $\mathfrak{X}$ and a compact operator on $\mathfrak{X}$ which cannot be approximated by finite rank operators.

One of the most famous open problems in Operator Theory today is the Invariant Subspace Problem.

- Given $\mathcal{H}$, a Hilbert space, and $T \in \mathcal{B}(\mathcal{H})$, does there exist a closed subspace $\mathcal{M}$ of $\mathcal{H}$ such that $\mathcal{M} \neq \{0\}$, $\mathcal{H}$ and $T\mathcal{M} \subseteq \mathcal{M}$?

Such a space is called a *non-trivial invariant subspace for* $T$. It is a standard exercise that if $T \in \mathcal{B}(\mathcal{H})$ and $\mathcal{H}$ is *not* separable, then we can decompose $\mathcal{H} = \oplus_{\alpha\in\Lambda}\mathcal{H}_\alpha$, where each $\mathcal{H}_\alpha$ is a separable, reducing subspace for $T$. Also, if $\mathcal{H}$ is finite dimensional, every operator can be upper triangularized, and thus has invariant subspaces. As such, the proper context in which to examine the Invariant Subspace Problem is in separable, infinite dimensional Hilbert spaces.

While the answer is not known in general, many results have been obtained. One of the strongest results is a generalization of a result of Lomonosov [**Lom73**] from 1973.

**Theorem.** *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ be a non-scalar operator. Suppose there exists $0 \neq K \in \mathcal{K}(\mathcal{H})$ such that $TK = KT$. Then there exists a closed subspace $\mathcal{M}$ of $\mathcal{H}$ which is* ***hyperinvariant*** *for $T$, that is: $\mathcal{M}$ is a non-trivial invariant subspace for every operator that commutes with $T$.*

**Corollary.** *Every compact operator on $\mathcal{H}$ has a non-trivial hyperinvariant subspace.*

A natural question that arises from this theorem is whether or not every operator in $\mathcal{B}(\mathcal{H})$ commutes with a non-scalar operator which in turn commutes with a non-zero compact operator. In other words, does Lomonosov's Theorem solve the Invariant Subspace Problem? That the answer is *no* was first shown by D.H. Hadwin, E.A. Nordgren, H. Radjavi, and P. Rosenthal [**HNRR80**].

Results are known for other classes of operators as well.

**Definition.** *An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be* ***subnormal*** *if there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a normal operator $N \in \mathcal{B}(\mathcal{K})$ of the form*

$$\begin{bmatrix} T & N_2 \\ 0 & N_4 \end{bmatrix}.$$

An example of a subnormal operator is the forward unilateral shift $U$. (We can take $N$ to be the bilateral shift by letting $N_4$ be the backward unilateral shift and $N_2$ the appropriate rank one operator. We then have the following theorem of Scott Brown from [**Bro78**].

**Theorem.** [**Brown**] *Every subnormal operator possesses a non-trivial invariant subspace.*

More recent results include:

**Theorem. [Brown, Chevreau, Pearcy] 1987** *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Suppose that*

- $\|T\| \le 1$*; and*
- $\sigma(T) \supseteq \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

*Then $T$ has a non-trivial invariant subspace.*

The corresponding question has been answered (negatively) for Banach spaces. In particular, in 1984, C.J. Read [**Rea84**] gave an example of a Banach space $\mathfrak{X}$ and a bounded linear operator $T$ on $\mathfrak{X}$ such that $fX$ and $\{0\}$ are the only closed subspaces of $\mathfrak{X}$ which are invariant for $T$. In 1985 [**Rea85**], he modified the construction to produce a bounded linear operator $T \in \mathcal{B}(\ell^1)$ such that $T$ does *not* have any non-trivial invariant subspace. The question remains open for reflexive Banach spaces.

If one considers *reducing* rather than invariant subspaces, then more is known. A major result of D. Voiculescu's [**Voi76**] known as his non-commutative Weyl-von Neumann Theorem implies that given $T \in \mathcal{B}(\mathcal{H})$ and $\epsilon > 0$, there exist an isometric, involution preserving map $\rho$ from $C^*(\pi(T))$, the closed Banach algebra generated by $\pi(T)$ and $\pi(T^*)$ in the Calkin algebra, into some $\mathcal{B}(\mathcal{H}_\rho)$, a unitary operator $U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}_\rho^{(\infty)})$ and $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \epsilon$ such that

$$T = U^*(T \oplus \rho(\pi(T))^{(\infty)})U + K.$$

It follows that every operator is a *limit* of operators with non-trivial reducing subspaces.

On the other hand, P. Halmos [**Hal68**] has shown that the set of *irreducible operators* (i.e. those with no non-trivial reducing subspaces) is dense in $\mathcal{B}(\mathcal{H})$.

The spectral theorem for compact normal operators shows that every such operator can be diagonalized. As such, it mimicks the finite dimensional result. For general normal operators on an infinite dimensional Hilbert space, this fails miserably. For instance, if $M_x$ is the multiplication operator acting on $L^2([0,1], dx)$, where $dx$ represents Lebesgue measure, then we have seen that $M_x$ is normal, but has no eigenvalues. It follows immediately from this observation that $M_x$ can not be diagonalizable. A wonderful result due known as the Weyl-von Neumann-Berg/Sikonia Theorem [**Ber71**] shows that once again, the result is true *up to a small compact perturbation.* More precisely,

**Theorem. [The Weyl-von Neumann-Berg/Sikonia Theorem]** *Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ be normal. Then, given $\epsilon > 0$, there exists $U \in \mathcal{B}(\mathcal{H})$ unitary, $K \in \mathcal{K}(\mathcal{H})$ satisfying $\|K\| < \epsilon$ and $D \in \mathcal{B}(\mathcal{H})$ diagonal such that*

$$T = U^* D U + K.$$

*Moreover, $D$ can be chosen to have the same spectrum and* essential spectrum *(see Appendix A) as $T$.*

Using this, we are now in a position to give a very simple proof of Halmos' result on the density of the irreducibles. This proof is due to H. Radjavi and P. Rosenthal [**RR69**]. Let us agree to say that an operator $D \in \mathcal{B}(\mathcal{H})$ is *diagonalizable* if there exists a unitary operator $U$ such that $U^* D U$ is diagonal.

**Theorem.** *Let $T \in \mathcal{B}(\mathcal{H})$ and $\epsilon > 0$. Then there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \epsilon$ such that $T + K$ is irreducible.*

**Proof.** By the Weyl-von Neumann-Berg/Sikonia Theorem, there exists a self-adjoint operator $D$ whose matrix is diagonal with respect to an orthonormal basis $\{e_n\}_{n=1}^\infty$ such that

$$\|D - (T^* - T)/2\| < \frac{\epsilon}{4}.$$

Then there is a self-adjoint operator $D_1$ diagonal with respect to $\{e_n\}_{n=1}^\infty$ such that all of the eigenvalues of $D_1$ are distinct and $\|D - D_1\| < \frac{\epsilon}{4}$. Now let $D_2$ be any self-adjoint compact operator within $\epsilon/2$ of $(T - T^*)/2i$ whose matrix with respect to $\{e_n\}_{n=1}^\infty$ has all entries different from 0 (such operators exist in profusion - why?). Then the operator $D_1 + iD_2$ is within $\epsilon$ of $T$. Also, $D_1 + iD_2$ is irreducible, since the invariant subspaces of $D_1$ are the subspaces spanned by subcollections of $\{e_n\}_{n=1}^\infty$, and none of these are invariant under $D_2$ except $\{0\}$ and $\mathcal{H}$. $\square$

CHAPTER 4

# Abelian Banach Algebras

*Conceit in weakest bodies strongest works.*

**William Shakespeare: Hamlet**

## 1. The Gelfand Transform

**1.1.** In this chapter we return to the study of abstract Banach algebras, this time focussing our attention on those which are abelian. The reader may refer back to Chapter Two, Section One for examples.

In any algebra, normed or otherwise, it is of interest to study the ideal structure of the algebra. Banach algebras are no exception.

**1.2. Definition.** *Let $\mathcal{A}$ be an abelian Banach algebra. An ideal $\mathcal{I}$ of $\mathcal{A}$ is said to be* ***modular*** *(also called* ***regular****) if we can find an element $e \in \mathcal{A}$ such that $ex - x \in \mathcal{I}$ for all $x \in \mathcal{A}$.*

This definition is readily seen to be equivalent to saying that the quotient algebra $\mathcal{A}/\mathcal{I}$ admits an identity element, namely $\overline{e} = e + \mathcal{I}$. Clearly every proper ideal in a unital Banach algebra is modular.

Given a Banach algebra $\mathcal{A}$ and an ideal $\mathcal{I}$ of $\mathcal{A}$, we shall use $\pi_{\mathcal{I}}$ to denote the canonical algebra map from $\mathcal{A}$ onto $\mathcal{A}/\mathcal{I}$. If $\mathcal{I}$ is understood, then we shall write only $\pi$.

**1.3. Example.** Let $\mathcal{A} = \mathcal{C}_0(\mathbb{R})$, the set of complex-valued continuous functions on $\mathbb{R}$ vanishing at infinity. Define $\mathcal{M} = \{f \in \mathcal{A} : f(x) = 0 \text{ if } x \in [-1, 1]\}$. It is readily seen that $\mathcal{A}$ is a non-unital Banach algebra and $\mathcal{M}$ is an ideal of $\mathcal{A}$.

Let $e \in \mathcal{A}$ be the function $e(x) = \begin{cases} 0 & \text{if } 2 < |x| \\ 2 - |x| & \text{if } 1 \le |x| \le 2 \\ 1 & \text{if } |x| \le 1. \end{cases}$

We leave it to the reader to verify that $e$ is an identity for $\mathcal{A}/\mathcal{M}$, and that $\mathcal{M}$ is therefore a regular ideal of $\mathcal{A}$.

**1.4. Proposition.** *Let $\mathcal{I}$ be a proper regular ideal of an abelian Banach algebra $\mathcal{A}$. If $e$ is an identity modulo $\mathcal{I}$, then $\|\pi(e)\| \ge 1$.*

**Proof.** First note that if $\mathcal{I}$ is closed, then $\mathcal{A}/\mathcal{I}$ is a Banach algebra by Proposition 1.16. But then $\|\pi(e)\| \ge 1$ by the submultiplicativity of the quotient norm.

If $\mathcal{I}$ is not closed, then $\|\pi(e)\| := \inf_{m\in\mathcal{I}} \|e - m\|$, and this defines a seminorm on the quotient algebra.

Suppose $\|e-m\| < 1$ for some $m \in \mathcal{I}$. Then $x = \sum_{n=1}^{\infty}(e-m)^n$ converges in $\mathcal{A}$. Now $(e-m)x = \sum_{n=2}^{\infty}(e-m)^n$, so

$$\begin{aligned} x &= (e-m)x + (e-m) \\ &= ex - mx + e - m; \end{aligned}$$

thus $e = x - ex + mx - m \in \mathcal{I}$. Since $ea - a \in \mathcal{I}$ for all $a \in \mathcal{A}$, we conclude that $a \in \mathcal{I}$ for all $a \in \mathcal{A}$, i.e. $\mathcal{A} \subseteq \mathcal{I}$, a contradiction.

Thus $\|\pi(e)\| \geq 1$.

□

**1.5. Definition.** *A proper ideal $\mathcal{I}$ of an algebra $\mathcal{A}$ is said to be* ***maximal*** *if it is not contained in any ideal of $\mathcal{A}$ except itself, and the entire algebra $\mathcal{A}$.*

**1.6. Example.** Let $\mathcal{A} = \mathcal{C}_0(\mathbb{R})$, and set $\mathcal{I} = \{f \in \mathcal{A} : f(0) = 0\}$. Then $\mathcal{I}$ is a maximal ideal of $\mathcal{A}$.

**1.7. Corollary.** *Let $\mathcal{A}$ be an abelian Banach algebra. If $\mathcal{I}$ is a proper modular ideal of $\mathcal{A}$, then $\mathcal{I}$ is contained in some maximal (modular) ideal $\mathcal{M}$ of $\mathcal{A}$. Furthermore, all maximal modular ideals of $\mathcal{A}$ are closed.*

**Proof.** First we observe that if $\mathcal{I}$ is a proper modular ideal of $\mathcal{A}$, and if $\mathcal{J}$ is any proper ideal of $\mathcal{A}$ containing $\mathcal{I}$, then $\mathcal{J}$ is also modular. Indeed, if $e$ is the identity modulo $\mathcal{I}$, then $e$ also serves as an identity modulo $\mathcal{J}$.

Consider the set

$$\mathfrak{J} = \{\mathcal{J} \subseteq \mathcal{A} : \mathcal{I} \subseteq \mathcal{J} \text{ and } \mathcal{J} \text{ is a proper ideal of } \mathcal{A}\},$$

partially ordered with respect to inclusion. Choose an increasing chain $\mathfrak{C}$ in $\mathfrak{J}$, say

$$\mathfrak{C} = \{\mathcal{J}_\alpha\}_{\alpha\in\Lambda}.$$

Let $\mathcal{J} = \cup_{\alpha\in\Lambda}\mathcal{J}_\alpha$, and $e \in \mathcal{A}$ be an identity modulo $\mathcal{I}$.

Then $\mathcal{J}$ is an ideal in $\mathcal{A}$. Also, $e \notin \mathcal{J}_\alpha$ for all $\alpha \in \Lambda$, and so $e \notin \mathcal{J}$. Thus $\mathcal{J}$ is proper. Clearly $\mathcal{J}$ is an upper bound for $\mathfrak{C}$. By Zorn's Lemma, there exists a maximal element $\mathcal{M}$ in $\mathfrak{J}$, and $\mathcal{I} \subseteq \mathcal{M}$. Clearly $e \notin \mathcal{M}$ since $e \notin \mathcal{K}$ for any $\mathcal{K} \in \mathfrak{J}$. Thus $\mathcal{M}$ is a proper maximal ideal of $\mathcal{A}$ containing $\mathcal{I}$.

Suppose that $\mathcal{L}$ is a maximal ideal of $\mathcal{A}$. Then the norm closure of $\mathcal{L}$ is also seen to be an ideal of $\mathcal{A}$. By maximality, $\mathcal{L} = \overline{\mathcal{L}}$ or $\mathcal{L} = \mathcal{A}$. But $\|\pi_{\mathcal{L}} e\| \geq 1$, and so $\operatorname{dist}(e, \mathcal{L}) \geq 1$. Thus $e \notin \overline{\mathcal{L}}$, and $\mathcal{L} = \overline{\mathcal{L}}$ is closed.

□

**1.8. Proposition.** *Let $\mathcal{A}$ be a commutative, unital Banach algebra and let $a \in \mathcal{A}$. If $a$ is not invertible, then $a$ is an element of some maximal ideal $\mathcal{M}$ of $\mathcal{A}$.*

**Proof.** Now $a\mathcal{A} = \mathcal{A}a$ is an ideal of $\mathcal{A}$ and $1 \notin a\mathcal{A}$. Thus $a \in a\mathcal{A} \subseteq \mathcal{M}$ for some maximal ideal $\mathcal{M}$ by Corollary 1.7.

□

**1.9. Definition.** *Let $\mathcal{A}$ be a Banach algebra. A non-zero complex linear functional $\phi : \mathcal{A} \to \mathbb{C}$ is said to be **multiplicative** if $\phi(ab) = \phi(a)\,\phi(b)$ for all $a, b \in \mathcal{A}$. The set of all non-zero multiplicative linear functionals on $\mathcal{A}$ is denoted by $\sum_{\mathcal{A}}$, and is called the **spectrum** of $\mathcal{A}$.*

Note that if $1 \in \mathcal{A}$, then $\phi(1) = \phi(1^2) = \phi(1)^2$, and so $\phi(1) \in \{0, 1\}$. If $\phi(1) = 0$, then $\phi(a) = \phi(1a) = \phi(1)\,\phi(a) = 0$ for all $a \in \mathcal{A}$, contradicting the fact that $\phi \neq 0$. Thus $\phi(1) = 1$.

**1.10. Proposition.** *Let $\mathcal{A}$ be a Banach algebra and $\phi$ be a multiplicative linear functional on $\mathcal{A}$. Then $\phi$ is bounded; in fact, $\|\phi\| \leq 1$.*

**Proof.** If $1 \notin \mathcal{A}$, then we may consider

$$\begin{array}{rccl} \phi^+ : & \mathcal{A}^+ & \to & \mathbb{C} \\ & (\lambda, a) & \mapsto & \lambda + \phi(a), \end{array}$$

which is a linear functional on $\mathcal{A}^+$, the unitization of $\mathcal{A}$ as defined in Remark 1.1.18. It is not hard to verify that $\phi$ is bounded if and only if $\phi^+$ is. As such, we may assume that $1 \in \mathcal{A}$.

Let $\mathcal{M} = \ker \phi$ and $a \in \mathcal{A}$. Then $\phi(a - \phi(a)1) = 0$, so that $a = \phi(a)1 + (a - \phi(a)1)$. Write $\lambda = \phi(a)$ and $b = (a - \phi(a)1)$ so that $\lambda \in \mathbb{C}$, $b \in \mathcal{M}$. Then

$$\begin{aligned} \|\phi\| &= \sup\{\frac{|\phi(x)|}{\|x\|} : \|x\| \neq 0\} \\ &= \sup\{\frac{|\phi(\lambda + b)|}{\|\lambda + b\|} : \lambda \neq 0, b \in \ker\,\phi\} \\ &= \sup\{\frac{|\lambda|}{\|\lambda + b\|} : \lambda \neq 0, b \in \ker\,\phi\} \\ &= \sup\{\frac{1}{\|1 + b'\|} : b' \in \ker\,\phi\} \\ &= 1, \end{aligned}$$

since otherwise $\|1 + b'\| < 1$ would imply that $b'$ is invertible, contradicting the fact that $b' \in \mathcal{M}$, a proper ideal of $\mathcal{A}$.

□

**1.11. Proposition.** *Let $\mathcal{A}$ be an abelian Banach algebra. Then there is a one-to-one correspondence between the spectrum $\sum_{\mathcal{A}}$, and the set of maximal modular ideals of $\mathcal{A}$.*

**Proof.** Let $\mathcal{M}$ be a maximal modular ideal of $\mathcal{A}$. Then $\mathcal{A}/\mathcal{M}$ is a unital Banach algebra with no proper ideals. Thus every non-zero element of $\mathcal{A}/\mathcal{M}$ is invertible, by Proposition 1.8. By the Gelfand-Mazur Theorem 2.1.33, there exists a unique isometric isomorphism $\tau : \mathcal{A}/\mathcal{M} \to \mathbb{C}$. The map

$$\begin{array}{rccl} \phi_{\mathcal{M}} : & \mathcal{A} & \to & \mathbb{C} \\ & a & \mapsto & \tau(\pi_{\mathcal{M}}(a)) \end{array}$$

is easily seen to be a multiplicative linear functional, and $\ker \phi_{\mathcal{M}} = \mathcal{M}$. Moreover, if $\mathcal{M}_1 \neq \mathcal{M}_2$ are two maximal modular ideals of $\mathcal{A}$, then $\phi_{\mathcal{M}_1} \neq \phi_{\mathcal{M}_2}$, since their kernels are distinct.

Conversely, if $\phi \in \sum_{\mathcal{A}}$, let $\mathcal{M} = \ker \phi$. Then $\mathbb{C} \simeq \phi(\mathcal{A}) \simeq \mathcal{A}/\ker \phi = \mathcal{A}/\mathcal{M}$, so $\mathcal{M}$ is a maximal regular ideal, as $\mathbb{C}$ is unital and has no non-trivial ideals. Consider $\phi_{\mathcal{M}}$ defined as above. Since the isomorphism between $\mathcal{A}/\mathcal{M}$ and $\mathbb{C}$ is unique, $\phi_{\mathcal{M}} = \phi$.

□

Because of this result, $\sum_{\mathcal{A}}$ is also referred to as the **maximal ideal space** of $\mathcal{A}$.

**1.12. Proposition.** *Let $\mathcal{A}$ be an abelian Banach algebra. Then $\sum_{\mathcal{A}}$ is locally compact in the weak*-topology on the unit ball of $\mathcal{A}^*$. If $\mathcal{A}$ is unital, then $\sum_{\mathcal{A}}$ is in fact compact.*

**Proof.** Let $\sum_{\mathcal{A}}^0 = \sum_{\mathcal{A}} \cup \{0\}$. Then $\sum_{\mathcal{A}}^0$ is clearly contained in the unit ball of $\mathcal{A}^*$. Let $\{\phi_\alpha\}_{\alpha\in\Lambda}$ be a net in $\sum_{\mathcal{A}}^0$ such that weak$*$-$\lim_{\alpha\in\Lambda} \phi_\alpha = \phi \in \mathcal{A}^*$.

Then for all $x$, $y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$,

$$\begin{aligned} \phi(\lambda x + y) &= \lim_\alpha \phi_\alpha(\lambda x + y) \\ &= \lim_\alpha \lambda\phi_\alpha(x) + \phi_\alpha(y) \\ &= \lambda\phi(x) + \phi(y) \end{aligned}$$

and

$$\begin{aligned} \phi(xy) &= \lim_\alpha \phi_\alpha(xy) \\ &= \lim_\alpha \phi_\alpha(x)\,\phi_\alpha(y) \\ &= \phi(x)\,\phi(y). \end{aligned}$$

Thus $\phi \in \sum_{\mathcal{A}}^0$. In particular, therefore, $\sum_{\mathcal{A}}^0$ is compact, being a closed subset of the weak*-compact unit ball of $\mathcal{A}^*$. Clearly $\{0\}$ is closed in $\sum_{\mathcal{A}}^0$. Since $\sum_{\mathcal{A}}$ is an open subset of a compact set $\sum_{\mathcal{A}}^0$, $\sum_{\mathcal{A}}$ is locally compact.

If $\mathcal{A}$ is unital, then $\{0\}$ is isolated in $\sum_{\mathcal{A}}^0$ since $\phi(1) = 1$ for all $\phi \in \sum_{\mathcal{A}}$. Thus $\sum_{\mathcal{A}}$ is closed in $\sum_{\mathcal{A}}^0$, and thus is weak*-compact itself.

□

**1.13. Definition.** *Let $\mathcal{A}$ be an abelian Banach algebra. Given $a \in \mathcal{A}$, we define the* ***Gelfand Transform*** *$\hat{a}$ of $a$ as follows:*

$$\begin{array}{rccc} \hat{a}: & \sum_{\mathcal{A}} & \to & \mathbb{C} \\ & \phi & \mapsto & \phi(a). \end{array}$$

It is readily verified that $\hat{a} \in \mathcal{C}(\sum_{\mathcal{A}})$. If $\epsilon > 0$, then $\{\phi \in \sum_{\mathcal{A}} : |\hat{a}(\phi)| \geq \epsilon \}$ is closed in $\sum_{\mathcal{A}}^0 = \sum_{\mathcal{A}} \cup \{0\}$, and hence it is compact. Thus $\hat{a} \in \mathcal{C}_0(\sum_{\mathcal{A}})$.

**1.14. Theorem.** **[The Gelfand Transform]** *Let $\mathcal{A}$ be an abelian Banach algebra.*

(i) *The map*

$$\begin{array}{rccc} \Gamma: & \mathcal{A} & \to & \mathcal{C}_0(\sum_{\mathcal{A}}) \\ & a & \mapsto & \hat{a} \end{array}$$

*is a contractive algebra homomorphism, and*

(ii) $\hat{\mathcal{A}} = \operatorname{ran} \Gamma$ *separates the points of* $\sum_{\mathcal{A}}$.

**Proof.**

(i) We have seen that $\hat{a}$ is continuous and vanishes at infinity. Now

$$\begin{aligned} \|\Gamma(a)\| &= \|\hat{a}\| \\ &= \sup_{\phi \in \sum_{\mathcal{A}}} |\hat{a}(\phi)| \\ &= \sup_{\phi \in \sum_{\mathcal{A}}} |\phi(a)| \\ &\leq \|a\|. \end{aligned}$$

Thus $\|\Gamma\| \leq 1$. That $\Gamma$ is indeed a homomorphism is left to the reader.

(ii) If $\phi_1 \neq \phi_2 \in \sum_{\mathcal{A}}$, then there exists $a \in \mathcal{A}$ such that $\phi_1(a) \neq \phi_2(a)$. But then $\hat{a}(\phi_1) \neq \hat{a}(\phi_2)$, and so $\hat{\mathcal{A}}$ indeed separates the points of $\sum_{\mathcal{A}}$, as claimed.

□

**1.15. Theorem.** *Let $\mathcal{A}$ be an abelian Banach algebra, and let $\sum_{\mathcal{A}}$ be its spectrum. Let $\Gamma : \mathcal{A} \to \mathcal{C}_0(\sum_{\mathcal{A}})$ be the Gelfand transform of $\mathcal{A}$. Then*

(i) $\sigma_{\mathcal{A}}(a) = \operatorname{ran} \hat{a}$ *if* $1 \in \mathcal{A}$;

(ii) $\sigma_{\mathcal{A}}(a) = \operatorname{ran} \hat{a} \cup \{0\}$ *if* $1 \notin \mathcal{A}$;

(iii) $\operatorname{spr}(a) = \|\hat{a}\|$.

**Proof.**

(i) If $\mathcal{A}$ is unital, then $\mathcal{C}_0(\sum_{\mathcal{A}}) = \mathcal{C}(\sum_{\mathcal{A}})$. Thus

$$\begin{aligned} \lambda \in \sigma_{\mathcal{A}}(a) &\iff (\lambda - a) \notin \mathcal{A}^{-1} \\ &\iff (\lambda - a) \text{ lies in a maximal ideal } \mathcal{M} \text{ of } \mathcal{A} \\ &\iff \phi_{\mathcal{M}}(\lambda - a) = 0 \text{ where } \mathcal{M} \text{ is a maximal ideal of } \mathcal{A} \\ &\iff \lambda - \phi_{\mathcal{M}}(a) = 0 \text{ where } \mathcal{M} \text{ is a maximal ideal of } \mathcal{A} \\ &\iff \lambda - \hat{a}(\phi_{\mathcal{M}}) = 0 \text{ where } \mathcal{M} \text{ is a maximal ideal of } \mathcal{A} \\ &\iff \lambda \in \operatorname{ran} \hat{a}. \end{aligned}$$

(ii) By Proposition 1.10, there is a bijective correspondence between $\sum_{\mathcal{A}}^0$ and $\sum_{\mathcal{A}^+}$. Moreover, $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{A}^+}(a) \cup \{0\}$. But by (i) above, $\sigma_{\mathcal{A}^+}(a) = \operatorname{ran} \widehat{j(a)}$, where

$$\begin{aligned} \widehat{j(a)} : \sum_{\mathcal{A}^+} &\mapsto \mathbb{C} \\ \widehat{j(a)} &= \varphi(j(a)) \\ &= \varphi|_{\mathcal{A}}(a) \\ &= \operatorname{ran} \hat{a} \cup \{0\}. \end{aligned}$$

(iii)

$$\begin{aligned} \|\hat{a}\| &= \operatorname{spr}(\hat{a}) \\ &= \sup\{|\lambda| : \lambda \in \sigma(\hat{a}) = \operatorname{ran} \hat{a}\} \\ &= \sup\{|\lambda| : \lambda \in \sigma(\hat{a}) \cup \{0\}\} \\ &= \sup\{|\lambda| : \lambda \in \sigma_{\mathcal{A}}(a)\} \\ &= \operatorname{spr}(a). \end{aligned}$$

□

*Meanwhile, back on the Titanic . . .*
*"It ain't over till the fat lady sinks."*

## 2. The radical

**2.1.** The kernel of the Gelfand transform plays a particular role in the study of homomorphims between Banach algebras.

**2.2. Definition.** *Let $\mathcal{A}$ be an abelian Banach algebra. Then the* ***Jacobson radical*** *of $\mathcal{A}$ is the kernel of the Gelfand transform . As such,*

$$\begin{aligned} \operatorname{rad} \mathcal{A} &= \cap\{\ker \phi : \phi \in \sum_{\mathcal{A}}\} \\ &= \cap\{\mathcal{M} : \mathcal{M} \textit{ a maximal ideal of } \mathcal{A}\}. \end{aligned}$$

*We say that $\mathcal{A}$ is* ***semisimple*** *if $\operatorname{rad} \mathcal{A} = \{0\}$.*

**2.3. Proposition.** *Let $\mathcal{A}$ be an abelian Banach algebra. Then $\operatorname{rad} \mathcal{A} = \{a \in \mathcal{A} : \operatorname{spr}(a) = 0\}$ and the following are equivalent:*

(i) *$\mathcal{A}$ is semisimple, i.e. the Gelfand transform $\Gamma : \mathcal{A} \to \mathcal{C}(\sum_{\mathcal{A}})$ is injective;*
(ii) *$\sum_{\mathcal{A}}$ separates the points of $\mathcal{A}$;*
(iii) *the spectral radius is a norm on $\mathcal{A}$.*

**Proof.** First note that $a \in \operatorname{rad}\mathcal{A}$ if and only if $\Gamma(a) = \hat{a} = 0$. But $0 = \hat{a} \in \mathcal{C}_0(\sum_{\mathcal{A}})$ if and only if $\operatorname{spr}(\hat{a}) = 0$, i.e. if and only if $\operatorname{spr}(a) = 0$.

(i) $\Longrightarrow$ (ii) Suppose $\mathcal{A}$ is semisimple. Let $a_1 \neq a_2 \in \mathcal{A}$. Then $0 \neq a_1 - a_2$, and so $\operatorname{spr}(a_1 - a_2) \neq 0$ from above. Thus there exists $0 \neq \lambda \in \operatorname{ran}(\widehat{a_1 - a_2})$. Let $\phi \in \sum_{\mathcal{A}}$ such that $\widehat{a_1 - a_2}(\phi) = \lambda$. Then $\widehat{a_1}(\phi) - \widehat{a_2}(\phi) = \phi(a_1 - a_2) = \lambda \neq 0$, so that $\sum_{\mathcal{A}}$ separates points.

(ii) $\Longrightarrow$ (i) Suppose that $\sum_{\mathcal{A}}$ separates the points of $\mathcal{A}$. Let $a_1 \neq a_2 \in \mathcal{A}$ and choose $\phi \in \sum_{\mathcal{A}}$ such that $\phi(a_1) \neq \phi(a_2)$. Then $\widehat{a_1}(\phi) \neq \widehat{a_2}(\phi)$, so that $\widehat{a_1} \neq \widehat{a_2}$, and the Gelfand transform is injective.

(i) $\Longrightarrow$ (iii) Suppose that the Gelfand transform $\Gamma$ is injective. In general, we have $\|\hat{a}\| = \operatorname{spr}(\hat{a}) = \operatorname{spr}(a)$. Then for all $a, b \in \mathcal{A}$,

- $\operatorname{spr}(\lambda a + b) = \|\widehat{\lambda a + b}\| \leq |\lambda| \|\hat{a}\| + \|\hat{b}\| = |\lambda| \operatorname{spr}(a) + \operatorname{spr}(b)$.
- $\operatorname{spr}(ab) = \|\widehat{ab}\| \leq \|\hat{a}\| \, \|\hat{b}\| = \operatorname{spr}(a) \operatorname{spr}(b)$.
- $\operatorname{spr}(a) = \|\hat{a}\| \geq 0$.
- Finally, $\operatorname{spr}(a) = 0$ if and only if $\|\hat{a}\| = 0$. But since $\Gamma$ is injective, this happens if and only if $a = 0$.

It follows that $\operatorname{spr}(\cdot)$ is a norm on $\mathcal{A}$.

(iii) $\Longrightarrow$ (i) Finally, suppose $\operatorname{spr}(\cdot)$ is a norm on $\mathcal{A}$. Then $\operatorname{spr}(a) = 0$ implies $a = 0$, so that $\operatorname{rad} \mathcal{A} = \{0\}$, and $\mathcal{A}$ is semisimple.

□

**2.4. Theorem.** *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian Banach algebras and suppose $\mathcal{B}$ is semisimple. Let $\tau : \mathcal{A} \to \mathcal{B}$ be an algebra homomorphism. Then $\tau$ is continuous.*

**Proof.** Let $\phi \in \sum_{\mathcal{B}}$, the maximal ideal space of $\mathcal{B}$. Then $\phi \circ \tau$ is a multiplicative linear functional on $\mathcal{A}$, and so $\|\phi \circ \tau\| = 1$, implying that $\phi \circ \tau$ is continuous.

The Closed Graph Theorem tells us that if $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces and $T : \mathfrak{X} \to \mathfrak{Y}$ is a linear map such that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} Tx_n = y$ together imply $y = 0$, then $T$ is continuous.

Suppose therefore that $\{a_n\}_{n=1}^\infty \subseteq \mathcal{A}$, that $\lim_{n\to\infty} a_n = 0$ and that $\lim_{n\to\infty} \tau(a_n) = b$. Then for $\phi \in \sum_{\mathcal{B}}$,

$$\begin{aligned} \phi(b) &= \phi(\lim_{n\to\infty} \tau(a_n)) \\ &= \lim_{n\to\infty} \phi \circ \tau(a_n) \\ &= (\phi \circ \tau)(\lim_{n\to\infty} (a_n)) \\ &= (\phi \circ \tau)(0) \\ &= 0. \end{aligned}$$

Thus $b \in \operatorname{rad} \mathcal{B} = \{0\}$. By the Closed Graph Theorem, $\tau$ is continuous.

$\square$

**2.5. Definition.** *A Banach algebra $\mathcal{A}$ has* ***uniqueness of norm*** *if all norms on $\mathcal{A}$ making it into a Banach algebra are equivalent.*

**2.6. Theorem.** *Let $\mathcal{A}$ be an abelian Banach algebra. If $\mathcal{A}$ is semisimple, then $\mathcal{A}$ has uniqueness of norm.*

**Proof.** With $\mathcal{A}$ abelian and semisimple, let $\|\cdot\|_1$ and $\|\cdot\|_2$ denote two Banach algebra norms on $\mathcal{A}$. Consider the natural injection

$$\begin{array}{rccc} i: & (\mathcal{A}, \|\cdot\|_1) & \to & (\mathcal{A}, \|\cdot\|_2) \\ & a & \mapsto & a. \end{array}$$

Then clearly $i$ is an algebra isomorphism, and hence from Theorem 2.4, $i$ is continuous. By the Banach Isomorphism Theorem, $i$ is a topological isomorphism, and so the two norms are equivalent.

$\square$

**2.7. Corollary.** *Let $\mathcal{A}$ be a semisimple abelian Banach algebra and $\alpha : \mathcal{A} \to \mathcal{A}$ be an algebra automorphism. Then $\alpha$ is also a homeomorphism.*

**Proof.** Theorem 2.4 implies that both $\alpha$ and $\alpha^{-1}$ are continuous.

$\square$

*Warts is hell.*

## 3. Examples

**3.1.** Depending upon the algebra $\mathcal{A}$ in question, the Gelfand Transform might not yield as much information as we might otherwise hope for. Here is an example for which $\sum_{\mathcal{A}}$ does not separate the points of $\mathcal{A}$.

**3.2. Example.** Let $n > 0$ and $\mathcal{B} = \mathbb{M}_n(\mathbb{C})$. Consider the algebra $\mathcal{A} \subseteq \mathbb{M}_{2n}(\mathbb{C})$, where

$$A = \left\{ \begin{bmatrix} \lambda I_n & B \\ 0 & \lambda I_n \end{bmatrix} : B \in \mathcal{B}, \lambda \in \mathbb{C} \right\}.$$

Then $\mathcal{A}$ is commutative. Let $\phi \in \sum_{\mathcal{A}}$. Then $\phi(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}) = 1$, and so $\phi(\lambda I_{2n}) = \lambda,\ \lambda \in \mathbb{C}$.

Also,

$$\begin{aligned} 0 &= \phi(0) \\ &= \phi(\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}^2) \\ &= \phi(\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix})^2 \end{aligned}$$

and so $\phi(\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}) = 0$.

Thus $\phi(\begin{bmatrix} \lambda I_n & B \\ 0 & \lambda I_n \end{bmatrix}) = \lambda$ for all $\lambda \in \mathbb{C}$ and $B \in \mathbb{M}_n$. In other words, $\sum_{\mathcal{A}} = \{\phi\}$, a singleton.

**3.3.** Let $X$ be a compact, Hausdorff space. We wish to consider the spectrum of the algebra $\mathcal{C}(X)$ of continuous functions on $X$. To do this, we first recall a preliminary result from topology.

**3.4. Proposition.** *Let $X$ be a compact space and $Y$ be a Hausdorff space. Suppose that $\tau : X \to Y$ is a bijective, continuous map. Then $\tau$ is a homeomorphism, i.e., $\tau^{-1}$ is also continuous.*

**3.5. Theorem.** *Let $X$ be a compact, Hausdorff space. Then $\sum_{\mathcal{C}(X)}$ equipped with its weak*-topology as a subset of $\mathcal{C}(X)^*$ is homeomorphic to $X$ with its given topology.*

**Proof.** Let $x \in X$, and consider the map

$$\begin{array}{llll} \delta_x : & \mathcal{C}(X) & \to & \mathbb{C} \\ & f & \mapsto & f(x). \end{array}$$

It is easy to see that $\delta_x \in \sum_{\mathcal{C}(X)}$. Such maps are called *evaluation functionals*. Note that the corresponding maximal ideal is $\mathcal{M}_x = \ker\, \delta_x = \{f \in \mathcal{C}(X) : f(x) = 0\}$. It is clear that given $x \neq y \in X$, $\delta_y \neq \delta_x$ since $\mathcal{C}(X)$

separates the points of $X$. Thus the map $x \mapsto \delta_x$ is injective. Our next goes is to show that it is surjective.

Let $\mathcal{M}$ be a maximal ideal of $\mathcal{C}(X)$. We shall show that there exists $x \in X$ such that $\mathcal{M} = \mathcal{M}_x$, where $\mathcal{M}_x$ is defined as above.

Suppose that for any $x \in X$, there exists $f_x \in \mathcal{M}$ such that $f_x(x) \neq 0$. Since $f$ is continuous, we can find an open neighbourhood $\mathcal{O}_x$ of $x$ such that $y \in \mathcal{O}_x$ implies $f_x(y) \neq 0$. Then the family $\{\mathcal{O}_x : x \in X\}$ is an open cover of the compact space $X$, and as such, we can find a finite subcover $\{\mathcal{O}_{x_i} : 1 \leq i \leq n\}$. Consider the function $g := \sum_{i=1}^n f_{x_i} \overline{f_{x_i}} \in \mathcal{M}$. Then clearly $g \geq 0$ and for any $x \in X$, there exists $x_i$ such that $f_{x_i}(x) \neq 0$. Thus $g(x) \geq |f_{x_i}(x)|^2 > 0$, and so $g$ is in fact invertible! This contradicts the fact that $\mathcal{M}$ is a maximal ideal, and thus is proper. It follows that there exists $x \in X$ such that $f(x) = 0$ for all $f \in \mathcal{M}$. But then $\mathcal{M} \subseteq \mathcal{M}_x$, and so by maximality, we conclude that $\mathcal{M} = \mathcal{M}_x$, and hence the map $x \mapsto \delta_x$ is surjective.

By Proposition 3.4, it remains only to show that the map $x \to \delta_x$ is continuous. Let $\{x_\alpha\}_{\alpha \in \Lambda}$ be a net in $X$ converging to the element $x$. Then $f(x_\alpha)$ converges to $f(x)$ for each $f \in \mathcal{C}(X)$. But then $\delta_{x_\alpha}(f)$ converges to $\delta_x(f)$ for all $f \in \mathcal{C}(X)$, and so $\delta_{x_\alpha}$ converges to $\delta_x$ in the weak*-topology on $\mathcal{C}(X)^*$. Thus $x \mapsto \delta_x$ is continuous, and our result is proved.

□

**3.6. Corollary.** *Let $X$ be a compact, Hausdorff space. Then $\mathcal{C}(X)$ has uniqueness of norm.*

**Proof.** The Gelfand map $\Gamma$ is the identity map, so it is injective, and thus $\mathcal{C}(X)$ is semisimple. We now apply Theorem 2.6.

□

**3.7.** Let $G$ be a locally compact abelian group equipped with a Haar measure $\mu$. It is well-known that if $\lambda$ is any other Haar measure on $G$, then $\lambda$ is a positive multiple of $\mu$. [See, for example, the book of Folland [**Fol95**][Thm 2.10, 2.20].] Moreover, since $G$ is abelian, it is *unimodular*, from which it follows that $d\mu(x^{-1}) = d\mu(x)$, as measures on $G$. Consider $f, g \in L^1(G, \mu)$. Then for $x \in G$,

$$
\begin{aligned}
(f * g)(x) &= \int f(y) g(y^{-1}x) d\mu(y) \\
&= \int f(xv) g(v^{-1}) d\mu(v) \quad (v = x^{-1}y) \\
&= \int f(xz^{-1}) g(z) d\mu(z) \quad (z = v^{-1}) \\
&= \int g(z) f(z^{-1}x) d\mu(z) \quad (xz^{-1} = z^{-1}x) \\
&= (g * f)(x).
\end{aligned}
$$

Thus $L^1(G,\mu)$ is abelian.

To verify that the norm on $L^1(G,\mu)$ is indeed a Banach algebra norm, consider

$$\begin{aligned}
\|f\|_1\,\|g\|_1 &= \int |f(y)|\,\|g\|_1 dy \\
&\geq \int |f(y)| \int |g(y^{-1}x)|dx\,dy \\
&\geq \int\int |f(y)\,g(y^{-1}x)|dx\,dy \\
&= \int\int |f(y)\,g(y^{-1}x)|dy\,dx \\
&\geq \int | \int f(y)g(y^{-1}x)dy|dx \\
&\geq \int |(f*g)(x)|dx \\
&= \|f*g\|_1.
\end{aligned}$$

**3.8. Definition.** *Given a locally compact abelian group $G$, we consider the set $\hat{G}$ of continuous homomorphisms of $G$ into $\mathbb{T}=\{z\in\mathbb{C} : |z|=1\}$. Such homomorphisms are called* ***characters*** *of $G$, and $\hat{G}$ is referred to as the* ***dual group*** *of $G$.*

**3.9.** We leave it to the reader to verify that $\hat{G}$ is indeed a group. In fact, $\hat{G}$ corresponds to the set of irreducible representations of $G$, which are always one dimensional when $G$ is abelian.

For the sake of convenience, let us write $\sum_G$ for $\sum_{L^1(G,\mu)}$, and $dx$ for $d\mu(x)$. Given $\phi\in\hat{G}$, we can define an element $\Phi\in\sum_G$ via

$$\Phi(f)=\int_G \phi(x)\,f(x)\,d\mu(x).$$

Indeed, for each $f$, $g\in L^1(G,\mu)$,

$$\begin{aligned}
\Phi(f*g) &= \int \phi(x)\,(f*g)(x)dx \\
&= \int \phi(x) \int f(y)\,g(y^{-1}x)dy\,dx \\
&= \int\int f(y)g(z)\phi(yz)dz\,dy \qquad (z=y^{-1}x) \\
&= \int\int f(y)\phi(y)\phi(z)g(z)dz\,dy \\
&= (\int f(y)\phi(y)dy)\,(\phi(z)g(z)dz) \\
&= \Phi(f)\,\Phi(g).
\end{aligned}$$

If $\phi_1 \neq \phi_2 \in \hat{G}$, then $0 \neq \phi_1 - \phi_2 \in \mathcal{C}_0(G) \subseteq L^\infty(G,\mu)$. Thus there exists $g \in L^1(G,\mu)$ such that $\int g(x)(\phi_1 - \phi_2)(x)dx \neq 0$. In particular, therefore, if $\Phi_1$ (resp. $\Phi_2$) is the element of $\sum_G$ corresponding to $\phi_1$ (resp. $\phi_2$) as above, then $\Phi_1(g) \neq \Phi_2(g)$, so that the map $\phi \mapsto \Phi$ is injective.

**3.10. Theorem.** *Let $G$ be a locally compact abelian group with Haar measure $\mu$. Then $\sum_G \simeq \hat{G}$.*

**Proof.** From above, we see that $\hat{G}$ embeds injectively into $\sum_G$. Next suppose that $\Phi \in \sum_G$. Since $\Phi \in L^1(G,\mu)^* \simeq L^\infty(G,\mu)$, there exists $\phi \in L^\infty(G,\mu)$ such that

$$\Phi(f) = \int f(x)\phi(x)dx \text{ for all } f \in L^1(G,\mu).$$

Choose $f \in L^1(G,\mu)$ such that $0 \neq \Phi(f)$. Then for any $g \in L^1(G,\mu)$,

$$\begin{aligned}
\Phi(f) \int \phi(y)g(y)dy &= \Phi(f)\,\Phi(g) \\
&= \Phi(f * g) \\
&= \int\int \phi(x)f(xy^{-1})g(y)dy\,dx \\
&= \int \Phi(L_y f)g(y)dy.
\end{aligned}$$

Thus $\phi(y) = \Phi(L_y f)/\Phi(f)$ a.e. . Redefine $\phi(y) = \Phi(L_y f)/\Phi(f)$ for every $y$, so that $\phi$ is continuous. Then

$$\begin{aligned}
\phi(xy)\Phi(f) &= \Phi(L_{xy} f) \\
&= \Phi(L_x L_y f) \\
&= \phi(x)\Phi(L_y f) \\
&= \phi(x)\phi(y)\Phi(f),
\end{aligned}$$

and hence $\phi(xy) = \phi(x)\,\phi(y)$.

Finally, $\phi(x^n) = \phi(x)^n$ for every $n \geq 1$, and $\phi$ bounded implies that $|\phi(x)| \leq 1$, while $\phi(x^{-n})$ bounded implies that $|\phi(x)| = 1$ for all $x \in G$. Thus $\phi \in \hat{G}$, and so the map $\phi \mapsto \Phi$ is onto, as claimed.

The topology we consider on $\hat{G}$ is that of uniform convergence on compact sets. Since $\hat{G}$ consists of continuous functions, this is the same as pointwise convergence, under which the operations of multiplication and inversion are clearly continuous. *Although we shall not show it here, it can be demonstrated that this topology coincides with the weak*-topology on $\hat{G}$ inherited from $L^\infty(G,\mu)$.*

But $\hat{G} \cup \{0\}$ is the set of all homomorphisms from $L^1(G,\mu)$ into $\mathbb{C}$, which is closed in the unit ball of $L^\infty(G,\mu)$, and hence is weak*-compact, by Alaoglu's Theorem. Thus $\hat{G}$ must be locally compact, as $\{0\}$ is closed.

□

**3.11. Theorem.**

(i) $\hat{\mathbb{Z}} \simeq \mathbb{T}$, *and thus* $\sum_{\ell^1(\mathbb{Z})} \simeq \mathbb{T}$;

(ii) $\hat{\mathbb{R}} \simeq \mathbb{R}$, *and thus* $\sum_{L^1(\mathbb{R},dx)} \simeq \mathbb{R}$;

(iii) $\hat{\mathbb{T}} \simeq \mathbb{Z}$, *and thus* $\sum_{L^1(\mathbb{T},dm)} \simeq \mathbb{Z}$, *where* $dm$ *represents normalised Lebesgue measure on the unit circle.*

**Remark:** We shall content ourselves here with the algebraic calculation, and omit the explicit determination of the underlying topologies, which are the natural topologies on the spaces involved.

**Proof.**

(i) For each $\alpha \in \mathbb{T}$, define $\phi_\alpha \in \hat{\mathbb{Z}}$ via $\phi_\alpha(1) = \alpha$. Suppose $\phi \in \hat{\mathbb{Z}}$. If $\alpha = \phi(1)$, then $\alpha \in \mathbb{T}$, and $\phi(n) = \phi(1)^n = \alpha^n$ for all $n \in \mathbb{Z}$. Thus $\phi = \phi_\alpha$. It follows that the map $\alpha \mapsto \phi_\alpha$ is surjective. That it is injective is trivial.

(ii) If $\phi \in \hat{\mathbb{R}}$, then we have $\phi(0) = 1$, so there exists $a > 0$ so that $\int_0^a \phi(t)dt \neq 0$. Setting $B = \int_0^a \phi(t)dt$, we have

$$B\phi(x) = \int_0^a \phi(t)dt\phi(x) = \int_0^a \phi(x)\phi(t)dt = \int_0^a \phi(x+t)dt = \int_x^{x+a} \phi(t)dt.$$

It follows that $\phi$ is differentiable and

$$\begin{aligned}
\phi'(x) &= \lim_{h\to 0} \frac{\phi(x+h)-\phi(x)}{h} \\
&= B^{-1} \lim_{h\to 0} \left( \frac{1}{h} \int_{x+h}^{x+a+h} \phi(t)dt - \frac{1}{h} \int_x^{x+a} \phi(t)dt \right) \\
&= B^{-1} \lim_{h\to 0} \left( \frac{1}{h} \int_{x+a}^{x+a+h} \phi(t)dt - \frac{1}{h} \int_x^{x+h} \phi(t)dt \right) \\
&= B^{-1}(\phi(x+a) - \phi(x)) \qquad \text{as } \phi \text{ is continuous} \\
&= B^{-1}\phi(x)\,(\phi(a) - 1) \\
&= c\phi(x),
\end{aligned}$$

where $c = B^{-1}(\phi(a) - 1)$. Thus $\phi(x) = e^{cx}$, and since $|\phi(x)| = 1$ for all $x$, $c = 2\pi i b$ for some $b \in \mathbb{R}$.

Conversely, for any $b \in \mathbb{R}$, $\phi_b(x) = e^{(2\pi i b)x}$ determines an element of $\hat{\mathbb{R}}$. Clearly the map $b \mapsto \phi_b$ is injective.

(iii) Since $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$ via the identification of $x \in \mathbb{R}/\mathbb{Z}$ with $\alpha = e^{(2\pi i)x}$, the characters of $\mathbb{T}$ are just the characters of $\mathbb{R}$ that vanish on $\mathbb{Z}$. But $\phi_b(1) = 1$ implies that $e^{2\pi i b} = 1$, and so $b \in \mathbb{Z}$. Thus $\hat{\mathbb{T}} \simeq \mathbb{Z}$.

□

**3.12. Definition.** *Let $\mathcal{A}$ be a Banach algebra and $a \in \mathcal{A}$. Then $a$ is said to* ***generate*** *$\mathcal{A}$ if the smallest closed subalgebra of $\mathcal{A}$ containing $a$ is $\mathcal{A}$ itself.*

The next theorem provides some justification for the term *spectrum* when referring to the set of non-zero multiplicative linear functionals on a Banach algebra.

**3.13. Theorem.** *Let $\mathcal{A}$ be a commutative unital Banach algebra and let $a$ be a generator for $\mathcal{A}$. Then the mapping $\hat{a} : \sum_{\mathcal{A}} \mapsto \sigma(a)$ is a homeomorphism.*

**Proof.** We already know that $\hat{a} \in \mathcal{C}(\sum_{\mathcal{A}})$ and that $\operatorname{ran} \hat{a} = \sigma(a)$. Since both $\sum_{\mathcal{A}}$ and $\sigma(a)$ are compact and Hausdorff, it suffices to show that $\hat{a}$ is injective. We can then apply Proposition 3.4 to obtain the desired result.

Suppose that $\phi_1, \phi_2 \in \sum_{\mathcal{A}}$ and that $\hat{a}(\phi_1) = \hat{a}(\phi_2)$. Then $\phi_1(a) = \phi_2(a)$. Let $\mathcal{B} = \{x \in \mathcal{A} : \phi_1(x) = \phi_2(x)\}$. Since $\phi_1$, $\phi_2$ are continuous, multiplicative and linear, $\mathcal{B}$ is an algebra that contains 1 and $a$, and $\mathcal{B}$ is closed. Thus $\mathcal{B} = \mathcal{A}$ and so $\phi_1 = \phi_2$, proving that $\hat{a}$ is injective, as required.

□

**3.14. Example.** Let $\mathcal{A} = \mathcal{A}(\mathbb{D})$, the disk algebra. Now it is a classical result that $\mathcal{A}$ is generated by 1 and $f$, where $f(z) = z$ for all $z \in \mathbb{D}$. (Indeed, this is the solution to the Dirichlet Problem for the circle.) By Theorem 3.13, $\sum_{\mathcal{A}}$ is homeomorphic to $\sigma(f)$. But as we have seen in Example 2.3.2, $\sigma(f) = \{z \in \mathbb{C} : |z| \leq 1\}$. We conclude that $\sum_{\mathcal{A}(\mathbb{D})} = \mathbb{D}$.

**3.15. Example.** Let us revisit $\ell^1(\mathbb{Z})$. For a function $f \in \mathcal{C}(\mathbb{T})$, consider the sequence $\{\hat{f}(n)\}_{n \in \mathbb{Z}}$ of *Fourier coefficients* of $f$ given by

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, e^{-in\theta} d\theta.$$

Define the *Wiener algebra*

$$\mathcal{AC}(\mathbb{T}) = \{f \in \mathcal{C}(\mathbb{T}) : \{\hat{f}(n)\}_{n \in \mathbb{Z}} \in \ell^1(\mathbb{Z})\},$$

equipped with the norm $\|f\| = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|$.

Clearly $\mathcal{AC}(\mathbb{T})$ is abelian. Let $f$ and $g$ lie in $\mathcal{AC}(\mathbb{T})$, so that

$$f(\theta) = \sum_{n \in \mathbb{Z}} a_n e^{in\theta} \text{ and } g(\theta) = \sum_{n \in \mathbb{Z}} b_n e^{in\theta}.$$

Then $(\hat{fg})(n) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, g(\theta)\, e^{-in\theta}\, d\theta$. Next,

$$\begin{aligned} f(\theta)\, g(\theta) &= \left( \sum_{k \in \mathbb{Z}} a_k e^{ik\theta} \right) \left( \sum_{n \in \mathbb{Z}} b_n e^{in\theta} \right) \\ &= \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} a_k b_n e^{i(k+n)\theta} \\ &= \sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} a_k b_{m-k} e^{im\theta} \quad (m = n + k) \end{aligned}$$

Thus

$$(\hat{fg})(n) = \frac{1}{2\pi}\int_0^{2\pi}\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} a_k b_{m-k} e^{i(m-n)\theta} d\theta.$$

If $m \neq n$, we get 0, and so

$$\begin{aligned}(\hat{fg})(n) &= \frac{1}{2\pi}\int_0^{2\pi}\sum_{k\in\mathbb{Z}} a_k b_{n-k} e^{i0} d\theta \\ &= \sum_{k\in\mathbb{Z}} a_k b_{n-k} \\ &= (ab)_n,\end{aligned}$$

where $a = (a_n)$ and $b = (b_n)$ lie in $\ell^1(\mathbb{Z})$. It follow that the map

$$\begin{array}{rccl}\tau: & \ell^1(\mathbb{Z}) & \to & \mathcal{AC}(\mathbb{T}) \\ & (a_n) & \mapsto & \sum_{n\in\mathbb{Z}} a_n e^{in\theta}\end{array}$$

is an isometric algebra isomorphism.

Suppose that $\phi$ is a non-zero multiplicative linear functional on $\mathcal{AC}(\mathbb{T})$. If $\phi(e^{i\theta}) = \lambda$, then $|\lambda| = \phi(e^{i\theta})| \leq \|\phi\|\,\|e^{i\theta}\|_1 = 1$. Also, $\phi(e^{-i\theta}) = \phi((e^{i\theta})^{-1}) = \frac{1}{\lambda}$, and $|\frac{1}{\lambda}| = |\phi(e^{-i\theta})| \leq \|\phi\|\,\|e^{-i\theta}\|_1 = 1$. Thus $|\lambda| = 1$.

Conversely, if $|\lambda| = 1$, then

$$\phi(\sum_{n\in\mathbb{Z}} a_n e^{in\theta}) = \sum_{n\in\mathbb{Z}} a_n \lambda^n$$

is an absolutely convergent, multiplicative evaluation functional, and $\phi(1) = 1$.

We conclude again that $\sum_{\mathcal{AC}(\mathbb{T})} = \sum_{\mathbb{Z}} = \mathbb{T}$. The argument with regards to the topology follows as in Theorem 3.5. Namely, let $\{\lambda_\alpha\}_\alpha$ be a net in $\mathbb{T}$ with $\lim_\alpha \lambda_\alpha = \lambda \in \mathbb{T}$. Let $\phi_{\lambda_\alpha}$, $\phi_\alpha$ be the associated multiplicative linear functionals with $\phi_{\alpha_\lambda}(e^{i\theta}) = \lambda_\alpha$, $\phi_\lambda(e^{i\theta}) = \lambda$. Then $\lim_\alpha \lambda_\alpha = \lambda$ implies $\lim_\alpha f(\lambda_\alpha) = f(\lambda)$ for all $f \in \mathcal{C}(\mathbb{T})$, hence $\lim_\alpha \phi_{\lambda_\alpha}(f) = \phi_\lambda(f)$ for all $f \in \mathcal{AC}(\mathbb{T})$. Thus $\lim_\alpha \phi_{\lambda_\alpha} = \phi_\lambda$ in the weak*-topology on $\sum_{\mathcal{AC}(\mathbb{T})}$.

As an application of this result, we obtain the following:

**3.16. Theorem.** **[Wiener's Tauberian Theorem]** *If $f \in \mathcal{AC}(\mathbb{T})$ and $f(z) \neq 0$ for all $z \in \mathbb{T}$, then $\dfrac{1}{f}$ has an absolutely convergent Fourier series.*

**Proof.** By Theorem 1.15,

$$\sigma(f) = \sigma(\hat{f}) = \operatorname{ran}\hat{f}.$$

But if $\phi \in \sum_{\mathcal{AC}(\mathbb{T})}$, then $\phi = \phi_\lambda$ for some $\lambda \in \mathbb{T}$, where $\phi_\lambda(f) = f(\lambda)$ is the evaluation functional corresponding to $\lambda$. Thus

$$\begin{aligned} \operatorname{ran} \hat{f} &= \{\hat{f}(\phi_\lambda) : \phi_\lambda \in \sum_{\mathcal{AC}(\mathbb{T})}\} \\ &= \{\hat{f}(\phi_\lambda) : \lambda \in \mathbb{T}\} \\ &= \{\phi_\lambda(f) : \lambda \in \mathbb{T}\} \\ &= \{f(\lambda) : \lambda \in \mathbb{T}\} \\ &= \operatorname{ran} f. \end{aligned}$$

Since $0 \notin \operatorname{ran} f$, we get $0 \notin \sigma_{\mathcal{AC}(\mathbb{T})}(f)$, so $\frac{1}{f}$ has an absolutely convergent Fourier series.

□

**3.17. Example.** Let $V \in \mathcal{B}(L^2([0,1], dx))$ denote the classical Volterra operator as defined in Example 3.4.5. Let $\mathcal{A} = \{p(V) : p \text{ a polynomial }\}^{-\|\cdot\|}$. Then $\mathcal{A}$ is an abelian Banach algebra, $\operatorname{rad} \mathcal{A} = \{R \in \mathcal{A} : \operatorname{spr}(R) = 0\}$, and thus $\operatorname{rad} \mathcal{A} = \{p(V) : p \text{ a polynomial with } p(0) = 0\}^{-\|\cdot\|}$.

*Bulletin: Curiosity pleads guilty*
*to lesser charge of manslaughtering cat.*

CHAPTER 5

# C*-Algebras

*There are only two truly infinite things, the universe and stupidity. And I am unsure about the universe.*

**Albert Einstein**

## 1. Definitions and Basic Theory.

**1.1.** In this chapter we turn our attention to an important class of Banach algebras known as *C*-algebras*.

**1.2. Definition.** *Let $\mathcal{A}$ be an algebra. Then an **involution** on $\mathcal{A}$ is a map*

$$\begin{array}{rccc} * : & \mathcal{A} & \rightarrow & \mathcal{A} \\ & a & \mapsto & a^* \end{array}$$

*satisfying*

(i) $(a^*)^* = a$ *for all* $a \in \mathcal{A}$;

(ii) $(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*$ *for all* $a, b \in \mathcal{A}$, $\alpha, \beta \in \mathbb{C}$;

(iii) $(ab)^* = b^* a^*$ *for all* $a, b \in \mathcal{A}$.

*If $\mathcal{A}$ carries an involution, we say that $\mathcal{A}$ is an **involutive algebra**, or a ***-algebra**. A subset $\mathcal{F}$ of $\mathcal{A}$ is said to be **self-adjoint** if $x \in \mathcal{F}$ implies $x^* \in \mathcal{F}$.*

*A homomorphism $\tau : \mathcal{A} \rightarrow \mathcal{B}$ between involutive algebras is said to be a ***-homomorphism** if $\tau$ respects the involution. That is, $\tau(a^*) = (\tau(a))^*$ for all $a \in \mathcal{A}$.*

*Finally, a **Banach *-algebra** is an involutive Banach algebra $\mathcal{A}$ whose involution satisfies $\|a^*\| = \|a\|$ for all $a \in \mathcal{A}$.*

Observe that if $\mathcal{A}$ is a unital involutive algebra with unit $e_{\mathcal{A}}$, then for all $a \in \mathcal{A}$ we have $(e_{\mathcal{A}}^*\, a) = (a^*\, e_{\mathcal{A}})^* = (a^*)^* = a = (e_{\mathcal{A}}\, a^*)^* = (a\, e_{\mathcal{A}}^*)$. Thus $e_{\mathcal{A}} = e_{\mathcal{A}}^*$, since the unit must be unique.

**1.3. Remark.** The condition that a homomorphism $\tau$ from an involutive Banach algebra $\mathcal{A}$ to an involutive Banach algebra $\mathcal{B}$ be a *-homomorphism is equivalent to the condition $\tau(h) = \tau(h)^*$ whenever $h = h^*$. To see this, note that if this condition is met, then given $a \in \mathcal{A}$, we may write $a = h + i\,k$, where $h = (a + a^*)/2$ and $k = (a - a^*)/2i$. Then $h = h^*$,

$k = k^*$, and $\tau(a^*) = \tau(h - i\,k) = (\tau(h) + i\tau(k))^* = \tau(a^*)$, implying that $\tau$ is a *-homorphism. The other direction is clear.

**1.4. Example.** Let $\mathcal{A} = (\mathbb{C}, |\cdot|)$. Then $* : \lambda \mapsto \overline{\lambda}$ defines an involution on $\mathbb{C}$.

**1.5. Example.** Consider the disk algebra $\mathcal{A}(\mathbb{D})$. For each $f \in \mathcal{A}(\mathbb{D})$, define $f^*(z) = \overline{f(\overline{z})}$ for each $z \in \mathbb{D}$. Then the map $* : f \mapsto f^*$ defines an involution on $\mathcal{A}(\mathbb{D})$, under which it becomes a Banach *-algebra.

**1.6. Example.** Recall from Remark 3.6 that the map that if $\mathcal{H}$ is a Hilbert space, then the map that sends a continuous linear operator $T$ to its Hilbert space adjoint $T^*$ is an involution. Thus $\mathcal{B}(\mathcal{H})$ is an involutive Banach algebra.

Suppose $\dim \mathcal{H} = 2$, and identify $\mathcal{B}(\mathcal{H})$ with $\mathbb{M}_2$. Let $S \in \mathcal{B}(\mathcal{H})$ be the invertible operator $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Consider the map

$$\begin{array}{rccc} \mathrm{Ad}_S : & \mathcal{B}(\mathcal{H}) & \to & \mathcal{B}(\mathcal{H}) \\ & T & \mapsto & S^{-1}\,T\,S \end{array}.$$

Then $\mathrm{Ad}_S$ is a multiplicative homomorphism of $\mathcal{B}(\mathcal{H})$, but it is not a *-homomorphism.

For example,

$$\mathrm{Ad}_S \begin{bmatrix} 1-i & 2-i \\ 3-i & 4-i \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ 3-i & 7-2i \end{bmatrix},$$

while

$$\left(\mathrm{Ad}_S \begin{bmatrix} 1+i & 3+i \\ 2+i & 4+i \end{bmatrix}\right)^* = \begin{bmatrix} -1 & 2-i \\ -2 & 6-2i \end{bmatrix}.$$

On the other hand, if $U \in \mathcal{B}(\mathcal{H})$ is unitary, then it is not hard to verify that $\mathrm{Ad}_U$ *does* define a *-automorphism.

**1.7. Example.** Let $\mathcal{T}_n$ denote the algebra of $n \times n$ upper triangular matrices, viewed as a Banach subalgebra of $\mathcal{B}(\mathbb{C}^n)$ equipped with the operator norm. We can define an involution on $\mathcal{T}_n$ via the map: $[t_{ij}]^* = [\overline{t_{(n+1)-j\,(n+1)-i}}]$.

**1.8. Definition.** *A* ***C*-algebra*** $\mathcal{A}$ *is an involutive Banach algebra which satisfies the* C*-equation*:*

$$\|a^* a\| = \|a\|^2 \text{ for all } a \in \mathcal{A}.$$

*A norm on an involutive Banach algebra which satisfies this equation will be called a* ***C*-norm****.*

**1.9. Remark.** First observe that if $\mathcal{B}$ is an involutive Banach algebra and $\|b^* b\| \ge \|b\|^2$ for all $b \in \mathcal{B}$, then $\|b\|^2 \le \|b^*\| \, \|b\|$, which implies that $\|b\| \le \|b^*\|$. But then $\|b^*\| \le \|(b^*)^*\| = \|b\|$, so that $\|b\| = \|b^*\|$. Moreover, $\|b\|^2 \le \|b^* b\| \le \|b^*\| \, \|b\| = \|b\|^2$, showing that the norm on $\mathcal{B}$ is a C*-norm.

Secondly, if $\mathcal{B}$ is a non-zero unital $C^*$-algebra with unit $e_{\mathcal{B}}$, then

$$\|e_{\mathcal{B}}\| = \|e_{\mathcal{B}}^2\| = \|e_{\mathcal{B}}^* \, e_{\mathcal{B}}\| = \|e_{\mathcal{B}}\|^2,$$

and hence $\|e_{\mathcal{B}}\| = 1$.

**1.10. Example.** It is always useful to have counterexamples as well as examples. To that end, consider the following:

(i) The disk algebra $\mathcal{A}(\mathbb{D})$ is not a $C^*$-algebra with the involution $f^*(z) = \overline{f(\overline{z})}$. Indeed, if $f(z) = i\,z + z^2$, then $f^*(z) = -i\,z + z^2$. Thus $\|f^* f\| = \sup_{|z|=1} |z^4 + z^2| = 2$, while $\|f\|^2 \ge |f(i)|^2 = 4$.

(ii) $\mathcal{T}_n$ is not a $C^*$-algebra with the involution defined in Example 1.7. To see this, note that if $E_{1\,n}$ denotes the standard $(1, n)$ matrix unit, then $E_{1\,n}^* = E_{1\,n}$, so that $\|E_{1\,n}^* \, E_{1\,n}\| = \|E_{1\,n}^2\| = \|0\| = 0$, while $\|E_{1\,n}\| = 1$, as is readily verified.

(iii) Recall that $\ell^1(\mathbb{Z})$ is a Banach algebra, where for $f, g \in \ell^1(\mathbb{Z})$, we defined the product via convolution:

$$(f * g)(n) = \sum_{k \in \mathbb{Z}} f(n-k)\, g(k)$$

and

$$\|f\|_1 = \sum_{k \in \mathbb{Z}} |f(k)|.$$

Consider the involution $f^*(n) = \overline{f(-n)}$. Let $g \in \ell^1(\mathbb{Z})$ be the element defined by: $g(n) = 0$ if $n \notin \{0, 1, 2\}$; $g(0) = -i = g(2)$, and $g(1) = 1$. We leave it to the reader to verify that $\|g^* \, g\|_1 = 5$, while $\|g\|_1 = 3$. Again, this is not a $C^*$-norm.

**1.11. Example.** Let $X$ be a locally compact, Hausdorff space. Consider $(\mathcal{C}_0(X), \|\cdot\|_\infty)$. For $f \in \mathcal{C}_0(X)$, define $f^*(x) = \overline{f(x)}$ for each $x \in X$. Then $\mathcal{C}(X)$ is a C*-algebra. The details are left to the reader.

This C*-algebra is unital precisely when $X$ is compact.

**1.12. Example.** Let $\mathcal{H}$ be a Hilbert space. As we have just recalled, $\mathcal{B}(\mathcal{H})$ is an involutive Banach algebra using the Hilbert space adjoint as our involution. We now check that equipped with this involution, $\mathcal{B}(\mathcal{H})$ verifies the C*-equation.

Let $T \in \mathcal{B}(\mathcal{H})$. Then $\|T^*T\| \leq \|T\|^2$ from above. For the reverse inequality, observe that

$$\begin{aligned} \|T\|^2 &= \sup_{\|x\|=1} \|Tx\|^2 \\ &= \sup_{\|x\|=1} < Tx, Tx > \\ &= \sup_{\|x\|=1} < T^*Tx, x > \\ &\leq \sup_{\|x\|=1} \|T^*T\| \, \|x\|^2 \\ &= \|T^*T\|. \end{aligned}$$

Thus $\mathcal{B}(\mathcal{H})$ is a C*-algebra. By considering the case where $\mathcal{H} = \mathbb{C}^n$ is finite dimensional, we find that $\mathbb{M}_n$ equipped with the operator norm and Hilbert space adjoint is a C*-algebra.

**1.13. Remark.** Suppose that $\mathcal{A}$ is a C*-algebra and that $\mathcal{B}$ is a closed, self-adjoint subalgebra of $\mathcal{A}$. (For a subset $\mathcal{D}$ of an involutive Banach algebra $\mathcal{Q}$ to be *self-adjoint* means that if $d \in \mathcal{D}$, then $d^* \in \mathcal{D}$.) Then the C*-equation is trivially satisfied for all $b \in \mathcal{B}$, because it is already satisfied in $\mathcal{A}$, and the norm is inherited from $\mathcal{A}$. It follows that $\mathcal{B}$ is also a C*-algebra.

In particular, if $a \in \mathcal{A}$ and $\mathcal{A}$ is unital, then we denote by $C^*(a)$ the $C^*-$**algebra generated by** $a$. It is the smallest unital subalgebra of $\mathcal{A}$ containing $a$, that is, it is the intersection of all $C^*$-subalgebras of $\mathcal{A}$ containing $a$, and it is easily seen to coincide with the closure of $\{p(a, a^*) :$ $p$ a polynomial in two non-commuting variables $\}$.

When $\mathcal{A}$ is non-unital, $C^*(a)$ is understood to mean the smallest $C^*$-algebra of $\mathcal{A}$ containing $a$. It coincides with the closure of $\{p(a, a^*) :$ $p$ a polynomial in two-noncommuting variables satisfying $p(0, 0) = 0\}$.

When we wish to emphasize the fact that $C^*(a)$ is non-unital, or when we wish to consider the non-unital $C^*$-algebra generated by $a$ in a unital $C^*$-algebra $\mathcal{A}$, we shall denote it by $C_0^*(a)$.

**1.14. Example.** More generally, if $\mathcal{A}$ is any $C^*$-algebra , and if $\mathcal{F} \subseteq \mathcal{A}$, we denote by $C^*(\mathcal{F})$ the $C^*$-subalgebra of $\mathcal{A}$ generated by $\mathcal{F}$. As before, it is the intersection of all $C^*$-algebras of $\mathcal{A}$ containing $\mathcal{F}$, it being understood that it is unital when the algebra $\mathcal{A}$ is unital.

**1.15. Example.** Let $\mathcal{H}$ be a Hilbert space. Then $\mathcal{K}(\mathcal{H})$ is a closed, self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$, and thus $\mathcal{K}(\mathcal{H})$ is a C*-algebra. $\mathcal{K}(\mathcal{H})$ is not unital unless $\mathcal{H}$ is finite-dimensional.

**1.16. Example.** Let $\{\mathcal{A}_\alpha\}_{\alpha \in \Lambda}$ be a family of C*-algebras indexed by a set $\Lambda$. It is elementary to verify that

$$\mathcal{A} = \{(a_\alpha)_{\alpha \in \Lambda} : a_\alpha \in \mathcal{A}_\alpha, \alpha \in \Lambda, \sup_\alpha \|a_\alpha\| < \infty\}$$

is a C*-algebra, where the involution is given by $(a_\alpha)^* = (a_\alpha^*)$, and the norm is given by $\|(a_\alpha)\| = \sup_\alpha \|a_\alpha\|$.

Let $\mathcal{K} = \{(a_\alpha) \in \mathcal{A} : \text{ for all } \varepsilon > 0, \ \{\alpha \in \Lambda : \|a_\alpha\| \geq \varepsilon\} \text{ is finite}\}$. Then $\mathcal{K}$ is a C*-algebra; in fact, $\mathcal{K}$ is a closed, self-adjoint ideal of $\mathcal{A}$.

In particular, if $\{k_n\}_{n=1}^\infty \subseteq \mathbb{N}$, then $\mathcal{A} = \oplus_{n=1}^\infty \mathbb{M}_{k_n}$ is a C*-algebra under this norm. Setting $k_n = 1$ for all $n \geq 1$ shows that $\ell^\infty$ is a C*-algebra, and that $c_0$ is a closed, self-adjoint ideal in $\ell^\infty$.

**1.17. Example.** More generally, let $\mu$ be a finite regular Borel measure on the measure space $X$. Then $L^\infty(X, \mu)$ is a C*-algebra with the standard norm. As in the case of $\mathcal{C}(X)$, the involution here is $f^*(x) = \overline{f(x)}$ for all $x \in X$.

In fact, we can think of $L^\infty(X, \mu)$ as a commutative C*-subalgebra of $\mathcal{B}(L^2(X, \mu))$ as follows. Recall from Example 3.3.14, for each $\phi \in L^\infty(X, \mu)$, we define the multiplication operator

$$\begin{array}{rlcl} M_\phi : & L^2(X, \mu) & \to & L^2(X, \mu) \\ & f & \mapsto & \phi\, f. \end{array}$$

Our goal is to show that the map $\Theta : \phi \mapsto M_\phi$ is an isometric *-monomorphism of $L^\infty(X, \mu)$ into $\mathcal{B}(L^2(X, \mu))$. We then identify $L^\infty(X, \mu)$ with its image under this map $\Theta$, and use the same notation for both algebras. Since $\Theta$ preserves products, the image algebra is clearly also abelian.

That $\Theta$ is a homomorphism is readily verified. Furthermore, notice that for $f \in L^2(X, \mu)$,

$$\|M_\phi\, f\| = (\int_X |\phi\, f|^2 d\mu)^{\frac{1}{2}} \leq \|\phi\|_\infty \, \|f\|_2,$$

so that $\|M_\phi\| \leq \|\phi\|_\infty$. Meanwhile, if for each $n \geq 1$ we set $E_n = \{x \in X : |\phi(x)| \geq \|\phi\|_\infty - \frac{1}{n}\}$, then for $\chi_{E_n}$ equal to the characteristic function of $E_n$, we have

$$\|M_\phi \chi_{E_n}\| = (\int_X |\phi\, \chi_{E_n}|^2 d\mu)^{\frac{1}{2}} \geq (\|\phi\|_\infty - \frac{1}{n})\, \|\chi_{E_n}\|_2,$$

so that $\|M_\phi\| = \|\phi\|_\infty$, and thus $\Theta$ is isometric..

Finally, for $f$, $g \in L^2(X, \mu)$ and $\phi \in L^\infty(X, \mu)$, we have

$$< M_\phi f, g >= \int_X (\phi f)\overline{g} d\mu = \int_X f\overline{(\overline{\phi} g)} d\mu =< f, M_{\overline{\phi}} g >,$$

so that $M_\phi^* = M_{\overline{\phi}}$. Hence $\Theta$ is a *-monomorphism.

**1.18. Example.** Let $\mathcal{H}$ be a Hilbert space and $\mathcal{F} \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint family of operators on $\mathcal{H}$. Consider the *commutant* $\mathcal{F}^{'}$ of $\mathcal{F}$ defined as:

$$\mathcal{F}^{'} = \{T \in \mathcal{B}(\mathcal{H}) : TF = FT \text{ for all } F \in \mathcal{F}\}.$$

We claim that $\mathcal{F}^{'}$ is a C*-algebra.

That it is an algebra is an easy exercise. If $\{T_n\}_{n=1}^\infty \subseteq \mathcal{F}'$ and if $\lim_{n\to\infty} T_n = T \in \mathcal{B}(\mathcal{H})$, then for any $F \in \mathcal{F}$, we have $T\,F = \lim_{n\to\infty} T_n\, F = \lim_{n\to\infty} F\, T_n = F\,T$. Hence $T \in \mathcal{F}'$, and so $\mathcal{F}'$ is closed in $\mathcal{B}(\mathcal{H})$. Finally, if $T \in \mathcal{F}'$ and $F \in \mathcal{F}$, then $F^* \in \mathcal{F}$ by assumption. Thus $T\,F^* = F^*\,T$. Taking adjoints, we obtain $T^*\,F = F\,T^*$, and therefore $T^* \in \mathcal{F}'$, proving that $\mathcal{F}'$ is a closed, self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. By Remark 1.13. it is a C*-algebra.

**1.19.** It is difficult to overstate the importance of the C*-equation. It allows us to relate analytic information to algebraic information. For example, consider the following Lemma, which relates the norm of an element of a C*-algebra to its spectral radius, and its consequence, Theorem 1.21.

**1.20. Lemma.** *Let $\mathcal{A}$ be a C*-algebra, and suppose $h = h^* \in \mathcal{A}$. Then $\|h\| = \operatorname{spr}(h)$. More generally, if $a \in \mathcal{A}$, then $\|a\| = (\operatorname{spr}(a^*a))^{1/2}$.*

**Proof.** Now $\|h\|^2 = \|h^*h\| = \|h^2\|$. By induction, we find that $\|h\|^{2^n} = \|h^{2^n}\|$ for all $n \geq 1$. Using Beurling's Spectral Radius Formula, $\operatorname{spr}(h) = \lim_{n\to\infty} \|h^{2^n}\|^{1/2^n} = \lim_{n\to\infty}(\|h\|^{2^n})^{1/2^n} = \|h\|$.

In general, $a^*a$ is self-adjoint, and hence $\|a\|^2 = \|a^*a\| = \operatorname{spr}(a^*a)$.

□

**1.21. Theorem.** *Let $\alpha : \mathcal{A} \to \mathcal{B}$ be a *-isomorphism from a C*-algebra $\mathcal{A}$ to a C*-algebra $\mathcal{B}$. Then $\alpha$ is isometric. In particular, each C*-algebra possesses a unique C*-norm.*

**Proof.** First note that since $\alpha$ is a *-isomorphism, $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(\alpha(a))$ for all $a \in \mathcal{A}$. As such,

$$\begin{aligned} \|a\|_{\mathcal{A}} &= [\operatorname{spr}_{\mathcal{A}}(a^*a)]^{1/2} \\ &= [\operatorname{spr}_{\mathcal{B}}(\alpha(a)^*\alpha(a))]^{1/2} \\ &= \|\alpha(a)\|_{\mathcal{B}}. \end{aligned}$$

Thus $\alpha$ is isometric.

If $\mathcal{A}$ has two C*-norms $\|\cdot\|_1$ and $\|\cdot\|_2$, then the identity map $id(a) = a$ is clearly a *-isomorphism of $\mathcal{A}$ onto itself, and thus is isometric from above, implying that the two norms coincide.

□

**1.22. Definition.** *Let $\mathcal{K}$ be an ideal of a C*-algebra $\mathcal{A}$. The **annihilator** of $\mathcal{K}$ in $\mathcal{A}$ is the set*

$$\mathcal{K}^\perp = \{a \in \mathcal{A} : a\,k = 0 \text{ for all } k \in \mathcal{K}\}.$$

*$\mathcal{K}$ is said to be **essential** in $\mathcal{A}$ if its annihilator $\mathcal{K} = \{0\}$.*

**1.23.** The apparent asymmetry of this definition is illusory. Suppose $\mathcal{A}$ and $\mathcal{K}$ are as above. Let $a \in \mathcal{K}^\perp$. Given $k \in \mathcal{K}$, $a^* k \in \mathcal{K}$ and hence $a\, a^* k = 0$. But then $\|a^* k\|^2 = \|k^* a\, a^* k\| = 0$, forcing $a^* \in \mathcal{K}^\perp$. But then $k\, a = (a^* k^*)^* = 0$ for all $k \in \mathcal{K}$. As such, $\mathcal{K}^\perp = \{a \in \mathcal{A} : k\, a = 0 \text{ for all } k \in \mathcal{K}\}$.

It is routine to verify that $\mathcal{K}^\perp$ is a closed subalgebra of $\mathcal{A}$, and from above, we see that $\mathcal{K}^\perp$ is self-adjoint, implying that $\mathcal{K}^\perp$ is a C*-subalgebra of $\mathcal{A}$.

**1.24. Example.** Let $\mathcal{H}$ be a complex, infinite dimensional Hilbert space. Then $\mathcal{K}(\mathcal{H})$ is essential in $\mathcal{B}(\mathcal{H})$.

For if $0 \neq T \in \mathcal{B}(\mathcal{H})$, choose a non-zero vector $x \in \mathcal{H}$ such that $y = Tx \neq 0$. Then $0 \neq T(x \otimes x^*)$, and hence $T \notin \mathcal{K}(\mathcal{H})^\perp$.

**1.25. Example.** Recall that if $X$ is a compact, Hausdorff space, then there is a bijective correspondence between the closed subsets $Y$ of $X$ and the closed ideals $\mathcal{K}$ of $\mathcal{C}(X)$. Given $Y \subseteq X$ closed, the associated ideal $\mathcal{K}_Y = \{f \in \mathcal{C}(X) : f(x) = 0 \text{ for all } x \in Y\}$, while given an ideal $\mathcal{K}$ in $\mathcal{C}(X)$, the correspoding closed subset of $X$ is $Y_\mathcal{K} = \{x \in X : f(x) = 0 \text{ for all } f \in \mathcal{K}\}$.

Let $Y \subseteq X$ be closed. We claim that $\mathcal{K}_Y$ is an essential ideal of $\mathcal{C}(X)$ if and only if $Y$ is nowhere dense in $X$.

Suppose first that $Y$ is nowhere dense. Let $f \in \mathcal{K}^\perp$. If $x \in X\backslash\overline{Y} = X\backslash Y$, then by Urysohn's Lemma we can find $g_x \in \mathcal{K}_Y$ such that $g_x(x) \neq 0$. Since $f\, g_x = 0$, we have $f(x) = 0$. But $X\backslash Y$ is dense in $X$ and $f$ is continuous, and so $f = 0$ and $\mathcal{K}_Y$ is essential.

To prove the converse, suppose $Y$ is *not* nowhere dense. Then we can find an open set $G \subseteq \overline{Y} = Y$. Choose $y_0 \in G$. Again, by Urysohn's Lemma, we can find $f \in \mathcal{C}(X)$ such that $f(y_0) = 1$ and $f(x) = 0$ for all $x \in X\backslash G$. It is routine to verify that $f \in \mathcal{K}_Y^\perp$, and hence $\mathcal{K}_Y$ is not essential.

**1.26. Example.** Let $X$ be a locally compact, Hausdorff space. Then $\mathcal{C}_0(X)$ is an essential ideal in $\mathcal{C}_b(X)$, the space of bounded continuous functions on $X$ with the supremum norm.

**1.27. Definition.** *Let $\mathcal{A}$ be a C*-algebra. A C*-algebra $\mathcal{B}$ is said to be a **unitization** of $\mathcal{A}$ if $\mathcal{B}$ is unital and $\mathcal{A}$ is *-isomorphic to an essential ideal in $\mathcal{B}$.*

**1.28. Example.** Let $\mathcal{H}$ be an infinite dimensional Hilbert space and let $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ be any unital C*-algebra containing $\mathcal{K}(\mathcal{H})$. Then $\mathcal{B}$ is a unitization of $\mathcal{K}(\mathcal{H})$.

**1.29. Example.** Let $\mathcal{A}$ be a unital C*-algebra and suppose $\mathcal{B}$ is a unitization of $\mathcal{A}$. Let $\rho : \mathcal{A} \to \mathcal{B}$ be the *-monomorphic embedding of $\mathcal{A}$ into $\mathcal{B}$ as an essential ideal. Then for each $a \in \mathcal{A}$,

$$(e_{\mathcal{B}} - \rho(e_{\mathcal{A}}))\,(\rho(a)) = 0,$$

and hence $e_{\mathcal{B}} = \rho(e_{\mathcal{A}})$. But $\rho(\mathcal{A})$ is an ideal in $\mathcal{B}$, and hence $\rho(\mathcal{A}) = \mathcal{B}$. Thus any unitization of $\mathcal{A}$ is *-isomorphic to $\mathcal{A}$ itself.

**1.30. Theorem.** *Every C*-algebra $\mathcal{A}$ possesses a unitization $\tilde{\mathcal{A}}$.*
**Proof.** If $\mathcal{A}$ is unital, then it serves as its own unitization. Suppose, therefore, that $\mathcal{A}$ is not unital. Consider the map:

$$\begin{array}{rccc} \kappa : & \mathcal{A} & \to & \mathcal{B}(\mathcal{A}) \\ & a & \mapsto & L_a \end{array}$$

where $L_a(x) = a\,x$ for all $x \in \mathcal{A}$. Then $\kappa$ is clearly a homomorphism. Denote by $\tilde{\mathcal{A}}$ the subalgebra of $\mathcal{B}(\mathcal{A})$ generated by $\kappa(\mathcal{A})$ and $I$, the identity operator. While there is no obvious candidate for an involution on $\mathcal{B}(\mathcal{A})$, nevertheless we may define one on $\tilde{\mathcal{A}}$ via $(L_a + \lambda I)^* = L_{a^*} + \overline{\lambda} I$.

Now $\|L_a\| = \sup_{\|x\|=1} \|L_a x\| = \sup_{\|x\|=1} \|a\,x\| \leq \|a\|$, so that $\kappa$ is continuous. In fact, $\|L_a\| \geq \|L_a \left(\dfrac{a^*}{\|a\|}\right)\| = \|a\|$, so that $\kappa$ is an isometric *-monomorphism. In particular, therefore, $\kappa(\mathcal{A})$ is closed in $\mathcal{B}(\mathcal{A})$. Since $\tilde{\mathcal{A}}$ is a finite dimensional extension of $\kappa(\mathcal{A})$, $\tilde{\mathcal{A}}$ is closed as well.

Next,

$$\begin{aligned} \|(L_a + \lambda I)^*(L_a + \lambda I)\| &= \sup_{\|x\|=1} \|(a^* + \overline{\lambda})(a + \lambda)x\| \\ &\geq \sup_{\|x\|=1} \|x^*(a^* + \overline{\lambda})(a + \lambda)x\| \\ &= \sup_{\|x\|=1} \|ax + \lambda x\|^2 \\ &= \|L_a + \lambda\|^2. \end{aligned}$$

By Remark 1.9, $\tilde{\mathcal{A}}$ is a C*-algebra.

That $\kappa(\mathcal{A})$ is an ideal in $\tilde{\mathcal{A}}$ is easily checked. Suppose $(L_a + \lambda I)\,L_b = 0$ for all $b \in \mathcal{A}$. Then for all $b, x \in \mathcal{A}$, we have $(a\,b + \lambda\,b)x = 0$. Letting $x = (a\,b + \lambda\,b)^*$, we find that $a\,b = -\lambda\,b$. Since $b$ is arbitrary, this implies that $-\lambda^{-1}a$ is a unit for $\mathcal{A}$, a contradiction. This implies that $\kappa(\mathcal{A})$ is essential in $\tilde{\mathcal{A}}$, completing the proof.

□

**1.31.** Two observations are in order. First, it will be useful to keep in mind that for any $x \in \tilde{\mathcal{A}}$, $\|x\|_{\tilde{\mathcal{A}}} = \sup\{\|x\,a\|_{\mathcal{A}} : \|a\|_{\mathcal{A}} = 1\}$. Second, the unitization of $\mathcal{A}$ above is unique in the following sense:

If $\mathcal{B}$ is any unital C*-algebra containing $\mathcal{A}$, then $\mathcal{B}$ contains an isometric *-isomorphic copy of $\tilde{\mathcal{A}}$. Indeed, if $\mathcal{B}_0$ is the algebra generated by $\mathcal{A}$ and $e_{\mathcal{B}}$, then either $e_{\mathcal{B}} \in \mathcal{A}$, in which case $\mathcal{A} = \tilde{\mathcal{A}} \subseteq \mathcal{B}$, or $\mathcal{B}_0$ is a 1-dimensional extension of $\mathcal{A}$, and hence is closed in $\mathcal{B}$. Since $\mathcal{B}_0$ is clearly self-adjoint, it is a C*-algebra. The map:

$$\begin{array}{rlcl} \Phi: & \tilde{\mathcal{A}} & \rightarrow & \mathcal{B}_0 \\ & L_a + \lambda I & \mapsto & a + \lambda\, e_{\mathcal{B}} \end{array}$$

is easily seen to be a *-isomorphism, and thus is isometric, by Theorem 1.21. $\mathcal{B}_0$ is our desired copy of $\tilde{\mathcal{A}}$.

**1.32. Example.** Let $X$ be a locally compact, Hausdorff space, and denote by $X_0$ the one point compactification of $X$. Then $\mathcal{C}(X_0)$ is the minimal unitization of $\mathcal{C}(X)$.

**1.33.** We mention that there is also a notion of a *largest* unitization for a $C^*$-algebra $\mathcal{A}$, called the **multiplier algebra** of $\mathcal{A}$. It plays an analogous rôle for abstract $C^*$-algebras that $\mathcal{B}(\mathcal{H})$ plays for $\mathcal{K}(\mathcal{H})$.

*The love of honey is the root of all beehives.*

## 2. Elements of $C^*$-algebras.

**2.1.** In this section, we study the internal structure of $C^*$-algebras. Using the involution and $C^*$-equation, we are able to show that the Gelfand Transform for abelian $C^*$-algebras is injective, and hence that the only abelian $C^*$-algebras are of the form $\mathcal{C}_0(X)$ for some locally compact Hausdorff space $X$. We also develop a partial order on the set of self-adjoint elements.

**2.2.** Recall that if $\mathcal{A}$ is a unital Banach algebra and $a \in \mathcal{A}$, then the *spectrum* of $a$ relative to $\mathcal{A}$ is

$$\sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} : (a - \lambda\, e_{\mathcal{A}}) \text{ is not invertible in } \mathcal{A}\}.$$

The *resolvent* of $a$ relative to $\mathcal{A}$ is $\rho_{\mathcal{A}}(a) = \mathbb{C}\backslash\sigma_{\mathcal{A}}(a)$. When $\mathcal{A}$ is not unital, we set $\sigma_{\mathcal{A}}(a) = \sigma_{\tilde{\mathcal{A}}}(a)$, where $\tilde{\mathcal{A}}$ is the unitization of $\mathcal{A}$ described in the last section. It is clear that in this case, $0 \in \sigma_{\mathcal{A}}(a)$, since $\mathcal{A}$ is an ideal in its unitization. (We shall show below that in fact, *any* unitization and more generally any $C^*$-algebra $\mathcal{B}$ containing $\mathcal{A}$ will yield the same spectrum.)

When only one $C^*$-algebra is under consideration, we suppress the subscripts to simplify the notation.

It is easy to verify that $\sigma(x^*) = \sigma(x)^* = \{\overline{\lambda} : \lambda \in \sigma(x)\}$. Moreover, $x$ is invertible if and only if both $x^*x$ and $x\,x^*$ are invertible. Indeed, if $x \in \mathcal{A}^{-1}$, then so is $x^*$. Thus $x^*\,x$ and $x\,x^*$ lie in $\mathcal{A}^{-1}$, since this latter is a group. Conversely, if $x^*\,x$ is invertible with inverse $z$, then $z\,x^*\,x = e_{\mathcal{A}}$ and so $x$ is left invertible. But $(x\,x^*)r = e_{\mathcal{A}}$ for some $r \in \mathcal{A}$, and so $x$ is right invertible.

Finally, we remark that the invertibility of *both* $x\,x^*$ and of $x^*\,x$ is required. Indeed, if $S$ denotes the unilateral backward shift operator from Example 3.9, then $S\,S^* = I$, but $S$ is not invertible, as we have seen.

**2.3. Definition.** *For each $a$ in a $C^*$-algebra $\mathcal{A}$, we define the* ***real part*** $\mathrm{Re}\, a = (a + a^*)/2$ *and the* ***imaginary part*** $\mathrm{Im}\, a = (a - a^*)/2i$ *of $a$.*

The terminology is of course borrowed from $\mathbb{C}$.

**2.4. Definition.** *An element $x$ of a $C^*$-algebra $\mathcal{A}$ is called*

- ***hermitian*** *if $x = x^*$;*
- ***normal*** *if $x\,x^* = x^*x$;*
- ***positive*** *if $x = x^*$ and $\sigma(x) \subseteq [0, \infty)$;*
- ***unitary*** *if $x^* = x^{-1}$;*
- ***idempotent*** *if $x = x^2$;*
- *a* ***projection*** *if $x = x^* = x^2$*
- *a* ***partial isometry*** *if $x\,x^*$ and $x^*\,x$ are projections (called the* range projection *and the* initial projection *of $x$, respectively).*

**2.5. Example.** Consider the $C^*$-algebra $c_0$. A sequence $\mathbf{x} = (x_n)$ is

- hermitian if and only if $x_n \in \mathbb{R}$ for all $n \geq 1$;
- always normal;
- positive if and only if $x_n \geq 0$ for all $n \geq 1$;
- unitary if and only if $|x_n| = 1$ for all $n \geq 1$;
- idempotent (or a projection, or a partial isometry) if and only if $x_n \in \{0, 1\}$ for all $n \geq 1$;

**2.6. Proposition.** *Let $\mathcal{A}$ be a $C^*$-algebra.*

(i) *If $u \in \mathcal{A}$ is unitary, then $\sigma(u) \subseteq \mathbb{T}$.*
(ii) *If $h \in \mathcal{A}$ is hermitian, then $\sigma(h) \subseteq \mathbb{R}$.*

**Proof.**

(i) First observe that $1 = \|e_{\mathcal{A}}\| = \|u^*u\| = \|u\|^2$. Thus $\operatorname{spr}(u) \leq \|u\| = 1$ implies $\sigma(u) \subseteq \overline{\mathbb{D}}$. But $\|u^{-1}\| \geq 1/\operatorname{dist}(0, \sigma(u))$ implies that $\operatorname{dist}(0, \sigma(u)) \geq \|u^*\| = 1$, and so $\sigma(u) \subseteq \mathbb{T}$.
(ii) Suppose $h = h^* \in \mathcal{A}$. Consider $u = \exp(ih)$. Using the uniform convergence of the power series expansion of $\exp(ih)$ we see that $u^* = \exp(-ih^*) = \exp(-ih)$. Since $(ih)$ and $(-ih)$ obviously commute, we obtain:

$$\begin{aligned} u^*u &= \exp(-ih)\, exp\,(ih) \\ &= \exp(-ih + ih) \\ &= \exp(0) \\ &= 1 \\ &= uu^*. \end{aligned}$$

Thus $u$ is unitary. By (i) and the holomorphic functional calculus, $\sigma(u) = \exp(i\sigma(h)) \subseteq \mathbb{T}$, from which we conclude that $\sigma(h) \subseteq \mathbb{R}$.

□

**2.7.** Suppose $\mathbb{S}$ is a unital, self-adjoint linear manifold in a $C^*$-algebra $\mathcal{A}$. If $h = h^* \in \mathbb{S}$, then $\operatorname{spr}(h) \leq \|h\|$, and hence $\sigma(h) \subseteq [-\|h\|, \|h\|]$. Letting $p_1 = h + \|h\|e_{\mathcal{A}}$ and $p_2 = \|h\|e_{\mathcal{A}}$, we find that both $p_1$ and $p_2$ are positive and $h = p_1 - p_2$. Thus for any $s \in \mathbb{S}$, we may apply this to the real and imaginary parts of $s$ to see that $s$ is a linear combination of four positive elements. This linear combination is *far* from unique. (Another such linear combination is obtained by simply letting $q_1 = p_1 + e_{\mathcal{A}}$, $q_2 = p_2 + e_{\mathcal{A}}$.)

Such linear manifolds $\mathbb{S}$ as above are referred to as **operator systems**. For example, $\{\alpha_{-1}\overline{z} + \alpha_0 + \alpha_1 z : \alpha_{-1}, \alpha_0, \alpha_1 \in \mathbb{C}\}$ is an operator system in $\mathcal{C}(\mathbb{T})$. Many results stated for $C^*$-algebras carry over to operator systems. We refer the reader to [Pau86] for an excellent treatment of this vast topic.

**2.8. Theorem.** *Suppose that $\mathcal{A} \subseteq \mathcal{B}$ are $C^*$-algebras and $x \in \mathcal{A}$. Then $\sigma_{\mathcal{A}}(x) = \sigma_{\mathcal{B}}(x)$.*

**Proof.** By considering $\tilde{\mathcal{A}}$ instead of $\mathcal{A}$, we may assume that $\mathcal{A}$ is unital. Clearly it suffices to prove that $\sigma_{\mathcal{A}}(x) \subseteq \sigma_{\mathcal{B}}(x)$.

First consider the case where $h = h^* \in \mathcal{A}$. Then $\sigma_{\mathcal{A}}(h) \subseteq \mathbb{R}$, and as such $\sigma_{\mathcal{A}}(h) = \partial\sigma_{\mathcal{A}}(h)$. By Proposition 3.7 $\partial\sigma_{\mathcal{A}}(h) \subseteq \sigma_{\mathcal{B}}(h)$ for any $C^*$-algebra $\mathcal{B}$ containing $\mathcal{A}$.

In general, if $x \in \mathcal{A}$ is not invertible in $\mathcal{A}$, then either $h_1 = x^*x$ or $h_2 = xx^*$ is not invertible. As $h_1$ and $h_2$ are self-adjoint, from above we have either $0 \in \sigma_{\mathcal{B}}(h_1)$ or $0 \in \sigma_{\mathcal{B}}(h_2)$. Either way, it follows that $x$ is not invertible in $\mathcal{B}$.

□

**2.9. Theorem.** ***The Gelfand-Naimark Theorem.*** *Let $\mathcal{A}$ be an abelian $C^*$-algebra. Then the Gelfand Transform $\Gamma : \mathcal{A} \to \mathcal{C}_0(\Sigma_{\mathcal{A}})$ is an isometric *-isomorphism.*

**Proof.** We have seen that the Gelfand Transform is a norm decreasing homomorphism from $\mathcal{A}$ into $\mathcal{C}_0(\Sigma_{\mathcal{A}})$. By Theorem 1.15, $\sigma(a) \cup \{0\} = \operatorname{ran}\Gamma(a) \cup \{0\}$ in both the unital and non-unital cases. In particular, if $h = h^* \in \mathcal{A}$, then $\operatorname{ran}\Gamma(a) \subseteq \mathbb{R}$, and so $\Gamma(h) = \Gamma(h)^*$. Thus $\Gamma$ is a *-homomorphism.

Also, $\|\Gamma(a)\|^2 = \|\Gamma(a^*a)\| = \operatorname{spr}(\Gamma(a^*a)) = \operatorname{spr}(a^*a) = \|a\|^2$, and so $\Gamma$ is isometric. Finally, $\Gamma(\mathcal{A})$ is a closed, self-adjoint subalgebra of $\mathcal{C}_0(\Sigma_{\mathcal{A}})$ which (by Theorem 1.14) separates the points of $\Sigma_{\mathcal{A}}$. By the Stone-Weierstraß Theorem, $\Gamma(\mathcal{A}) = \mathcal{C}_0(\Sigma_{\mathcal{A}})$.

□

**2.10. Theorem.** ***The Abstract Spectral Theorem.*** *Let $\mathcal{A}$ be a unital $C^*$-algebra and $n \in \mathcal{A}$ be normal. Then $\Sigma_{C^*(n)}$ is homeomorphic to $\sigma(n)$. As such, $C^*(n)$ is isometrically *-isomorphic to $(\mathcal{C}(\sigma(n), \|\cdot\|)$.*

**Proof.** We claim that $\Gamma(n)$ implements the homeomorphism between $\Sigma_{C^*(n)}$ and $\sigma(n)$. Since $\Sigma_{C^*(n)}$ is compact, $\sigma(n)$ is Hausdorff, and $\Gamma(n)$ is continuous, it suffices to show that $\Gamma(n)$ is a bijection. By Theorem 1.15, $\operatorname{ran}\Gamma(n) = \sigma(n)$, and so $\Gamma(n)$ is onto. Suppose $\phi_1, \phi_2 \in \Sigma_{C^*(n)}$ and $\phi_1(n) = \Gamma(n)(\phi_1) = \Gamma(n)(\phi_2) = \phi_2(n)$. Since $\Gamma$ is a *-homomorphism, $\phi_1(n^*) = \Gamma(n^*)(\phi_1) = \overline{\Gamma(n)(\phi_1)} = \overline{\Gamma(n)(\phi_2)} = \Gamma(n^*)(\phi_2) = \phi_2(n^*)$. Then $\phi_1(p(n, n^*)) = \phi_2(n, n^*))$ for all polynomials $p$ in two non-commuting variables, as both $\phi_1$ and $\phi_2$ are multiplicative. By the continuity of $\phi_1$ and $\phi_2$ and the density of $\{p(n, n^*) : p \text{ a polynomial in two non-commuting variables }\}$ in $C^*(n)$, we find that $\phi_1 = \phi_2$ and $\Gamma(n)$ is injective. By the Gelfand-Naimark Theorem 2.9, $C^*(n) \simeq^* \mathcal{C}(\Sigma_{C^*(n)})$. It follows immediately that $C^*(n) \simeq^* \mathcal{C}(\sigma(n))$.

□

It is worth drawing attention to the fact that if $\Gamma : C^*(n) \to \mathcal{C}(\sum_{C^*(n)})$ is the Gelfand Transform and for $x \in C^*(n)$ we set $\Gamma'(x) = \Gamma(x) \circ \Gamma^{-1}(n)$, then $\Gamma'$ implements the *-isomorphism between $C^*(n)$ and $\mathcal{C}(\sigma(n))$. Furthermore $\Gamma'(n)(z) = z$ for all $z \in \sigma(n)$; that is, $\Gamma'(n) = q$, where $q(z) = z$. In practice, we usually identify $\mathcal{C}(\Sigma_{C^*(n)})$ and $\mathcal{C}(\sigma(n))$, and still refer to the induced map $\Gamma'$ as the Gelfand Transform, relabelling it as $\Gamma$.

When $\mathcal{A}$ is non-unital, the $\mathcal{A} \subseteq \tilde{\mathcal{A}}$ and $C^*(1_{\tilde{\mathcal{A}}}, n) \simeq^* \mathcal{C}(\sigma(n))$. But then $C^*_0(n) \subseteq \tilde{\mathcal{A}}$ corresponds to the functions in $\mathcal{C}(\sigma(n))$ which vanish at 0, namely $\mathcal{C}_0(\sigma(n)\backslash\{0\})$.

As an immediate Corollary to the above theorem, we are able to extend the holomorphic functional Calculus developed in Chapter Two to a broader class of functions.

**2.11. Theorem.** ***The Continuous Functional Calculus.*** *Let $\mathcal{A}$ be a unital $C^*$-algebra and $n \in \mathcal{A}$ be normal. Then $\Gamma^{-1} : \mathcal{C}(\sigma(n)) \to C^*(n)$ is an isometric *-isomorphism and for all $f, g \in \mathcal{C}(\sigma(n))$, $\lambda \in \mathbb{C}$, we have*

(i) $(\lambda f + g)(n) = \lambda f(n) + g(n)$;
(ii) $(f\, g)(n) = f(n)\, g(n)$;
(iii) *the Spectral Mapping Theorem:* $\sigma(f(n)) = f(\sigma(n))$;
(iv) $\|f(n)\| = \mathrm{spr}\,(f(n)) = \mathrm{spr}\,(f) = \|f\|$.

*In particular, if $q(z) = z,\ z \in \sigma(n)$, then $n = \Gamma^{-1}(q)$.*

**Remark.** When $\mathcal{A}$ is non-unital, the Gelfand Transform induces a functional calculus for continuous functions vanishing at 0.

**2.12. Corollary.** *Let $\mathcal{A}$ be a unital $C^*$-algebra and $n \in \mathcal{A}$ be normal. Then*

(i) $n = n^*$ *if and only if* $\sigma(n) \subseteq \mathbb{R}$;
(ii) $n \geq 0$ *if and only if* $\sigma(n) \subseteq [0, \infty)$;
(iii) $n^* = n^{-1}$ *if and only if* $\sigma(n) \subseteq \mathbb{T}$;
(iv) $n = n^* = n^2$ *if and only if* $\sigma(n) \subseteq \{0, 1\}$.

**Proof.** This is an immediate consequence of identifying $C^*(n)$ with $\mathcal{C}(\sigma(n))$.

□

It is worth observing that all of the above notions are $C^*$-notions; that is, if $\phi : \mathcal{A} \to \mathcal{B}$ is a *-isomorphism of $C^*$-algebras, then each of the above notions is preserved by $\phi$.

**2.13. Proposition.** *Let $\mathcal{A}$ be a $C^*$-algebra and $0 \leq r \in \mathcal{A}$. Then there exists a unique element $q \in \mathcal{A}$ such that $0 \leq q$ and $q^2 = r$. Moreover, if $a \in \mathcal{A}$ and $a\, r = r\, a$, then $a$ commutes with $q$.*

**Proof.** Then function $f(z) = z^{\frac{1}{2}} \in \mathcal{C}(\sigma(r))$, and $f(0) = 0$. Thus $q := f(r) \in C^*_0(r)$ and thus is a normal element of $\mathcal{A}$. In fact, $\sigma(q) = f(\sigma(r)) \subseteq [0, \infty)$, and so $q \geq 0$. Next, $q^2 = (f(r))^2 = f^2(r) = j(r) = r$, where $j(z) = z,\ z \in \sigma(r)$.

Suppose $0 \leq s \in \mathcal{A}$ and $s^2 = r$. Then $s\,r = s(s^2) = (s^2)s = r\,s$, so that $C_0^*(r,s)$ is abelian. The Gelfand Map $\Gamma_1 : C_0^*(r,s) \to \mathcal{C}_0(\Sigma_{C_0^*(r,s)})$ is an isometric *-isomorphism and $\Gamma_1(q)$, $\Gamma_1(s)$ are two positive functions whose square is $\Gamma_1(r)$. Thus $\Gamma_1(q) = \Gamma_1(s)$. Since $\Gamma_1$ is injective, $q = s$. This shows that $q$ is unique.

Finally, if $a\,r = r\,a$, then $a$ commutes with every polynomial in $r$. Since $q = f(r)$ is a limit of polynomials in $r$, and since multiplication is jointly continuous, $a\,q = q\,a$.

□

For obvious reasons, we write $q = r^{\frac{1}{2}}$ and refer to $q$ as the (positive) **square root** of $r$.

Let us momentarily pause to address a natural question which arises. For $\mathcal{H}$ a Hilbert space and $R \in \mathcal{B}(\mathcal{H})$, we currently have two apparently different notions of positivity. That is, we have the operator notion (1): $R = R^*$ and $< Rx, x > \geq 0$ for all $x \in \mathcal{H}$, and the $C^*$-algebra notion (2): $R$ is normal and $\sigma(R) \subseteq [0, \infty)$. The following proposition reconciles these two notions.

**2.14. Proposition.** *Let $\mathcal{H}$ be a complex Hilbert space and $R \in \mathcal{B}(\mathcal{H})$. The following are equivalent:*

(i) *$R = R^*$ and $< Rx, x > \geq 0$ for all $x \in \mathcal{H}$;*
(ii) *$R$ is normal and $\sigma(R) \subseteq [0, \infty)$.*

**Proof.**

(i) $\Rightarrow$ (ii) Clearly $R = R^*$ implies $R$ is normal, and $\sigma(R) \subseteq \mathbb{R}$. Let $\lambda \in \mathbb{R}$ with $\lambda < 0$. Then

$$\begin{aligned} \|(R - \lambda I)x\|^2 &= < (R - \lambda I)x, (R - \lambda I)x > \\ &= < Rx, Rx > -2\lambda < Rx, x > +\lambda^2 < x, x > \\ &\geq \lambda^2 < x, x > . \end{aligned}$$

Thus $(R - \lambda I)$ is bounded below. Since $R$ is normal, $\sigma(R) = \sigma_a(R)$ by Proposition 3.17, and therefore $\lambda \notin \sigma(R)$. Hence $\sigma(R) \in [0, \infty)$.

(ii) $\Rightarrow$ (i) Suppose $R$ is normal and $\sigma(R) \subseteq [0, \infty)$. Then by Proposition 2.13, the operator $Q = R^{\frac{1}{2}}$ is positive. Let $x \in \mathcal{H}$. Then

$$\begin{aligned} < Rx, x > &= < Q^2 x, x > \\ &= < Qx, Qx > \\ &= \|Qx\|^2 \geq 0. \end{aligned}$$

□

**2.15. Remark.** Of course, the above Proposition fails spectacularly when $R$ is not normal. For example, if $V$ is the classical Volterra operator from Example 4.5, then $\sigma(V) = \{0\} \subseteq [0, \infty)$. But $V$ is not positive, or even normal, for the only normal quasinilpotent operator is 0.

**2.16. Definition.** *Let $\mathcal{A}$ be a $C^*$-algebra and $h = h^* \in \mathcal{A}$. Consider the function $f_+ : \mathbb{R} \to \mathbb{R}$, $f_+(x) = \max\{x, 0\}$. We define the* ***positive*** *part $h_+$ of $h$ to be $h_+ = f_+(h)$, and the* ***negative*** *part $h_-$ of $h$ to be $h_- = h_+ - h$. It follows easily from the continuous functional calculus that $h_- = f_-(h)$, where $f_-(x) = -\min\{x, 0\}$ for all $x \in \mathbb{R}$. Both $h_+, h_- \geq 0$, as $h_+, h_-$ are normal and $\sigma(h_+) = \sigma(f_+(h)) = f_+(\sigma(h)) \subseteq [0, \infty)$ (with a parallel proof holding for $h_-$). We therefore have $h = h_+ - h_-$.*

Clearly, given $x \in \mathcal{A}$, we can write $x$ in terms of its real and imaginary parts, $x = y + iz$, and $y = y_+ - y_-$, $z = z_+ - z_-$. Thus every element of $\mathcal{A}$ is a linear combination of (at most 4) positive elements.

A useful result that follows from the functional calculus is:

**2.17. Proposition.** *Let $\mathcal{A}$ be a $C^*$-algebra and $h = h^* \in \mathcal{A}$. Then $\|h\| = \max(\|h_+\|, \|h_-\|)$.*

**Proof.** Consider $\|h_+\| = \operatorname{spr}(h_+) = \operatorname{spr}(f_+(h)) = \|f_+|_{\sigma(h)}\| = \max(\{0\}, \{\lambda : \lambda \in \sigma(h), \lambda \geq 0\})$, while $\|h_-\| = \operatorname{spr}(h_-) = \operatorname{spr}(f_-(h)) = \|f_-|_{\sigma(h)}\| = \max(\{0\}, \{-\lambda : \lambda \in \sigma(h), \lambda \leq 0\})$. A moment's reflection shows that $\max(\|h_+\|, \|h_-\|) = \operatorname{spr}(h) = \|h\|$.

□

**2.18. Lemma.** *Let $\mathcal{A}$ be a unital $C^*$-algebra and $h = h^* \in \mathcal{A}$. The following are equivalent:*

(i) *$h \geq 0$;*
(ii) *$\|t1 - h\| \leq t$ for some $t \geq \|h\|$;*
(iii) *$\|t1 - h\| \leq t$ for all $t \geq \|h\|$.*

**Proof.** First let us identify $C^*(h)$ with $\mathcal{C}(\sigma(h))$ via the Gelfand Transform $\Gamma$. Let $\hat{h} = \Gamma(h)$ so that $\hat{h}(z) = z$ for all $z \in \sigma(h)$. The equivalence of the above three conditions is a result of their equivalence in $\mathcal{C}(\sigma(h))$, combined with the fact that positivity is a $C^*$-notion, as noted in the comments following Corollary 2.12. Thus we have

(i) $\Rightarrow$ (iii)

$$\begin{aligned} \|t1 - h\| &= \|\Gamma(t1 - h)\| \\ &= \|t\mathbf{1} - \hat{h}\| \\ &\leq t \qquad \text{for all } t \geq \|\hat{h}\| = \|h\|. \end{aligned}$$

(iii) $\Rightarrow$ (ii) Obvious.

(ii) $\Rightarrow$ (i) If $\|t1 - h\| = \|t\mathbf{1} - \hat{h}\| \leq t$, then $\hat{h} \geq 0$, and so $h \geq 0$.

□

**2.19. Definition.** *Let $\mathcal{A}$ be a Banach space. A* ***real cone*** *in $\mathcal{A}$ is a subset $\mathcal{F}$ of $\mathcal{A}$ satisfying:*

(a) *$0 \in \mathcal{F}$;*
(b) *if $x, y \in \mathcal{F}$ and $\lambda \geq 0$ in $\mathbb{R}$, then $\lambda x + y \in \mathcal{F}$;*
(iii) *$\mathcal{F} \cap \{-x : x \in \mathcal{F}\} = \{0\}$.*

For the sake of convenience, we shall write $-\mathcal{F}$ for $\{-x : x \in \mathcal{F}\}$.

**2.20. Example.** Let $\mathcal{A} = \mathbb{C}$, the complex numbers viewed as a 1-dimensional Banach space over itself. The set $\mathcal{F} = \{z \in \mathbb{C} : \mathrm{Re}(z) \in [0,\infty),\ \mathrm{Im}(z) \in [0,\infty)\}$ is a real cone in $\mathcal{A}$. More generally, any of the four "quadrants" in $\mathbb{C}$ determined by two lines passing through the origin forms a real cone.

**2.21. Example.** Let $\mathcal{A} = \mathcal{C}(X)$, $X$ a compact Hausdorff space. The set $\mathcal{F} = \{f \in \mathcal{C}(X) : f \geq 0\}$ is a real cone in $\mathcal{A}$.

**2.22. Proposition.** *Let $\mathcal{A}$ be a $C^*$-algebra. Then $\mathcal{A}_+ = \{p \in \mathcal{A} : p \geq 0\}$ is a norm-closed, real cone in $\mathcal{A}$, called* ***the positive cone*** *of $\mathcal{A}$.*
**Proof.** We may assume without loss of generality that $1 \in \mathcal{A}$. Clearly $0 \in \mathcal{A}_+$, and if $p \in \mathcal{A}_+$ and $0 \leq \lambda \in \mathbb{R}$, then $(\lambda p)^* = \overline{\lambda} p^* = \lambda p$ and $\sigma(\lambda p) = \lambda\sigma(p) \subseteq [0,\infty)$, so that $\lambda p \in \mathcal{A}_+$.

Next suppose that $x, y \in \mathcal{A}_+$. By Lemma 2.18, we obtain:

$$\begin{aligned} \|\,(\|x\| + \|y\|)1 - (x+y)\,\| &\leq \|\,\|x\|1 - x\,\| + \|\,\|y\|1 - y\,\| \\ &\leq \|x\| + \|y\|, \end{aligned}$$

imply by the same Lemma that $x + y \geq 0$. Suppose $x \in \mathcal{F} \cap (-\mathcal{F})$. Then $x = x^*$ and $\sigma(x) \subseteq [0,\infty) \cap (-\infty, 0] = \{0\}$. Since $\|x\| = \mathrm{spr}\,(x) = 0$, we have $x = 0$. So far we have shown that $\mathcal{F}$ is a real cone.

Finally, suppose that we have $\{x_n\}_{n=1}^\infty \subseteq \mathcal{F}$ and $x = \lim_{n\to\infty} x_n$. Then $x^* = \lim_{n\to\infty} x_n^* = \lim_{n\to\infty} x_n = x$, so that $x$ is self-adjoint. By dropping to a subsequence if necessary, we may assume that $\|x\| \geq \|x_n\|/2$ for all $n \geq 1$. Then

$$\begin{aligned} \|\,(2\|x\|)1 - x\,\| &= \lim_{n\to\infty} \|\,(2\|x\|)1 - x_n\,\| \\ &\leq \lim_{n\to\infty} 2\|x\| \\ &= 2\|x\|. \end{aligned}$$

By Lemma 2.18, $x \geq 0$ and so $\mathcal{F}$ is norm-closed, completing the proof.

□

Let $\mathcal{H}$ be a Hilbert space and $Z \in \mathcal{B}(\mathcal{H})$. If $R = Z^*Z$, then for any $x \in \mathcal{H}$, $< Rx, x >=< Z^*Zx, x >= \|Zx\|^2 \geq 0$, and so $R \geq 0$. Our next goal is to show that in any $C^*$-algebra, $r \in \mathcal{A}$ is positive precisely if $r$ factors as $z^*z$ for some $z \in \mathcal{A}$. The proof is rather more delicate than in the $\mathcal{B}(\mathcal{H})$ setting. The next Lemma comes in handy.

**2.23. Lemma.** *Let $\mathcal{A}$ be a unital Banach algebra and $a, b \in \mathcal{A}$. Then $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$.*
**Proof.** Clearly it suffices to consider the case where $\mathcal{A}$ is unital. The proof, while completely unmotivated, is a simple algebraic calculation.

Suppose $0 \neq \lambda \in \rho(ab)$. Let $c = \lambda^{-1}(\lambda - ab)^{-1}$, and verify that

$$\begin{aligned}(\lambda - ba)^{-1} &= (\lambda^{-1} + bca) \\ &= \lambda^{-1} + \lambda^{-1} b(\lambda - ab)^{-1} a \\ &= \lambda^{-1}(1 + b(\lambda - ab)^{-1} a).\end{aligned}$$

□

**2.24. Theorem.** *Let $\mathcal{A}$ be a $C^*$-algebra and $r \in \mathcal{A}$. Then $r \geq 0$ if and only if $r = z^* z$ for some $z \in \mathcal{A}$.*

**Proof.** First suppose that $r \geq 0$. By Proposition 2.13, there exists a unique $z \geq 0$ so that $r = z^2 = z^* z$.

Next, suppose $r = z^* z$ for some $z \in \mathcal{A}$. Clearly $r = r^*$. Let us write $r$ as the difference of its positive and negative parts, namely $r = r_+ - r_-$. Our goal is to show that $r_- = 0$.

Now $r_- \geq 0$ and so $r_-$ has a positive square root. Consider $y = z r_-^{\frac{1}{2}}$. Then $y^* y$ is self-adjoint and

$$\begin{aligned}y^* y &= (z r_-^{\frac{1}{2}})^* (z r_-^{\frac{1}{2}}) \\ &= r_-^{\frac{1}{2}} z^* z r_-^{\frac{1}{2}} \\ &= r_-(r_+ - r_-) r_-^{\frac{1}{2}} \\ &= r_-^{\frac{1}{2}} r_+ r_-^{\frac{1}{2}} - r_-^2 \\ &= -r_-^2 \\ &\leq 0.\end{aligned}$$

(Note that the last equality follows from the fact that $f_-^{\frac{1}{2}} f_+ = 0$.) Thus $\sigma(y^* y) \subseteq (-\infty, 0]$. Writing $y = h + ik$, where $h = \mathrm{Re}\, y$, $k = \mathrm{Im}\, y$, we have

$$\begin{aligned}yy^* &= h^2 + i\,k\,h - i\,h\,k + k^2 \\ y^* y &= h^2 - i\,h\,h + i\,h\,k + k^2\end{aligned}$$

so that

$$yy^* = (yy^* + y^* y) - (y^* y) = 2(h^2 + k^2) - (y^* y).$$

Since $h^2 + k^2 \geq 0$ and $y^* y \leq 0$ from above, the fact that $\mathcal{A}_+$ is a positive cone implies that $yy^* \geq 0$. Thus $\sigma(yy^*) \subseteq [0, \infty)$.

By the previous Lemma, $\sigma(yy^*) \cup \{0\} = \sigma(y^* y) \cup \{0\}$, from which we deduce that $\sigma(yy^*) = \{0\} = \sigma(y^* y)$. But then $\|y\|^2 = \|y^* y\| = \mathrm{spr}\,(y^* y) = 0$, and so $y = 0$. That is, $\| - r_-^2 \| = \|y^* y\| = 0$, so that $r_- = 0$ and $r = r_+ \geq 0$, as claimed.

□

**2.25. Remark.**

(a) Given $a \in \mathcal{A}$, a $C^*$-algebra, we can now define $|a| = (a^*a)^{\frac{1}{2}}$, and we call this the **absolute value** of $a$.

(b) The above theorem has a partial extension to involutive unital Banach algebras. Suppose $\mathcal{B}$ is such an algebra with unit $e$, and $x \in \mathcal{B}$ satisfies $\|x - e\| < 1$. Then $x = y^2$ for some $y \in \mathcal{B}$. Indeed, when $\|e - x\| < 1$, we have $\sigma(x) \subseteq \{z \in \mathbb{C} : |z - 1| < 1\}$. As such the function $f(z) = z^{\frac{1}{2}}$ is analytic on $\sigma(x)$, and so we set $y = f(x)$.

**2.26. A partial order on $\mathcal{A}_{sa}$.** Given two self-adjoint elements $x, y \in \mathcal{A}$, a $C^*$-algebra, we set $x \leq y$ if $y - x \geq 0$. It is easy to check that this defines a partial order. Certain, but not all properties of the order on $\mathbb{R}$ carry over to this setting. Consider the following:

**2.27. Proposition.** *Let $\mathcal{A}$ be a $C^*$-algebra.*

(i) *If $a, b \in \mathcal{A}_{sa}$ and $c \in \mathcal{A}$, then $a \leq b$ implies $c^*ac \leq c^*bc$.*

(ii) *If $0 \leq a \leq b$, then $\|a\| \leq \|b\|$.*

(iii) *If $1 \in \mathcal{A}$, $a, b \in \mathcal{A}_+$ are invertible and $a \leq b$, then $b^{-1} \leq a^{-1}$.*

**Proof.**

(i) Since $a \leq b$, $b - a$ is positive, and so we can find $z \in \mathcal{A}$ so that $b - a = z^*z$. Then $c^*z^*zc = (zc)^*(zc) \leq 0$ by Theorem 2.24. That is, $c^*bc - c^*ac \geq 0$, which is equivalent to our claim.

(ii) It suffices to consider the case where $1 \in \mathcal{A}$. Then the unital $C^*$-algebra generated by $b$, namely $C^*(b) \simeq^* \mathcal{C}(\sigma(b))$. Then $\Gamma(b) \leq \|\Gamma(b)\|\mathbf{1} = \|b\|\mathbf{1}$, and since positivity is a $C^*$-property, $b \leq \|b\|1$.

But then $a \leq b$ and $b \leq \|b\|1$ implies $a \leq \|b\|1$. Again, by the Gelfand-Naimark Theorem, $\Gamma(a) \leq \|b\|\mathbf{1}$, and so $\|a\| = \|\Gamma(a)\| \leq \|b\|$.

(iii) First suppose $c \geq 1$. Then $\Gamma(c) \geq \mathbf{1}$, and so $\Gamma(c)$ is invertible and $\Gamma(c)^{-1} \leq \mathbf{1}$. This in turn implies that $c$ is invertible and that $c^{-1} \leq 1$.

More generally, given $a \leq b$, $1 = a^{-\frac{1}{2}}aa^{-\frac{1}{2}} \leq a^{-\frac{1}{2}}ba^{-\frac{1}{2}}$, and so the above argument implies that $1 \geq (a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{-1} = a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}}$.

Finally, $a^{-1} = a^{-\frac{1}{2}}1a^{-\frac{1}{2}} \geq b^{-1}$, by (i) above.

□

*Where there's a will, there's a wake. [Old Irish Proverb]*

## 3. Ideals in $C^*$-algebras.

**3.1.** In Section 5.1, we briefly discussed ideals of $C^*$-algebras in connection with unitizations. Now we return for a more detailed and structured look at ideals and their elements. At times it is not desirable to adjoin a unit to a $C^*$-algebra . For many purposes, it suffices to consider "approximate identities", also called "approximate units", which we now define.

**3.2. Definition.** *Let $\mathbb{A}$ be a $C^*$-algebra and suppose $\mathbb{K}$ is a linear manifold in $\mathbb{A}$. Then a **right approximate identity** for $\mathbb{K}$ is a net $(u_\lambda)$ of positive elements in $\mathbb{K}$ such that $\|u_\lambda\| \leq 1$ for all $\lambda$, and such that*

$$\lim_\lambda \|k - k\, u_\lambda\| = 0$$

*for all $k \in \overline{\mathbb{K}}$.*

Analogously, one can define a **left approximate identity** for a linear manifold $\mathbb{K}$ of $\mathbb{A}$.

By an **algebraic** (**left**, **right**, or **two-sided**) **ideal** of a C*-algebra $\mathbb{A}$, we shall simply mean a linear manifold $\mathbb{K}$ which is invariant under multiplication (on the left, right, or both sides) by elements of $\mathbb{A}$. The notion of a (left, right or two-sided) ideal differs only in that ideals are assumed to be norm-closed. Unless otherwise specified, *algebraic ideals and ideals are assumed to be two-sided.*

**3.3. Example.** Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then $\mathcal{F}(\mathcal{H}) = \{F \in \mathcal{B}(\mathcal{H}) : \operatorname{rank} F < \infty\}$ is an algebraic ideal whose closure is the set $\mathcal{K}(\mathcal{H})$ of compact operators on $\mathcal{H}$, by Theorem 4.2.

**3.4. Example.** Let $\mathcal{C}_{00}(\mathbb{R}) = \{f \in \mathcal{C}(\mathbb{R}) : \ \operatorname{supp}(f) \text{ is compact }\}$. Then $\mathcal{C}_{00}(\mathbb{R})$ is an algebraic ideal of $\mathcal{C}(\mathbb{R})$ whose norm closure is $\mathcal{C}_0(\mathbb{R})$, the set of continuous functions which vanish at infinity.

**3.5. Proposition.** *Let $\mathbb{A}$ be a $C^*$-algebra and suppose $\mathbb{K}$ is an **algebraic** left ideal in $\mathbb{A}$. Then $\mathbb{K}$ has a right approximate identity.*

**Proof.** We may assume without loss of generality that $\mathbb{A}$ is unital. Given a finite subset $F = \{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{K}$, we define

$$h_F = \sum_{i=1}^n a_i^* a_i$$

and

$$v_F = h_F \left(h_F + \frac{1}{n} 1\right)^{-1} = \left(h_F + \frac{1}{n} 1\right)^{-1} h_F.$$

Note that $v_F \in \mathbb{K}$, since $h_F \in \mathbb{K}$.

Let $\mathcal{F} = \{F : F \subseteq \mathbb{K},\ F \text{ finite }\}$ be directed by inclusion.

Claim: the set $(v_F : F \in \mathcal{F}, \supseteq)$ is an approximate identity for $\mathbb{K}$.

To see this, note first that $h_F \geq 0$ and that $0 \leq t(t + \frac{1}{n})^{-1} \leq 1$ for all $t \in \mathbb{R}^+$. Thus $0 \leq v_F \leq 1$ by the functional calculus. Suppose $F, G \in \mathcal{F}$ and $F \supseteq G$. We may assume that $F = \{a_1, a_2, \ldots, a_m\}$ and that $G = \{a_1, a_2, \ldots, a_n\}$, where $n \leq m$.

Now $h_F \geq h_G$ since $h_F - h_G = \sum_{i=n+1}^m a_i^* a_i \geq 0$. Thus $h_F + \frac{1}{n}1 \geq h_G + \frac{1}{n}1$ and hence

$$(h_F + \frac{1}{n}1)^{-1} \leq (h_G + \frac{1}{n}1)^{-1}.$$

From this, and since $\frac{1}{m}(t + \frac{1}{m})^{-1} \leq \frac{1}{n}(t + \frac{1}{n})^{-1}$ for all $t \in \mathbb{R}^+$, we have

$$\frac{1}{m}(h_F + \frac{1}{m}1)^{-1} \leq \frac{1}{n}(h_F + \frac{1}{n}1)^{-1} \leq \frac{1}{n}(h_G + \frac{1}{n}1)^{-1}.$$

But

$$\frac{1}{m}(h_F + \frac{1}{m}1)^{-1} = 1 - v_F$$

and

$$\frac{1}{n}(h_G + \frac{1}{n}1)^{-1} = 1 - v_G,$$

and so $1 - v_F \leq 1 - v_G$, implying that $v_G \leq v_F$ when $F \supseteq G$.

Suppose $k \in \mathbb{K}$. Given $n \in \mathbb{N}$, choose $F_0 \in \mathcal{F}$ such that $F_0$ has $n$ elements and $k \in F_0$. If $F \in \mathcal{F}$ and $F_0 \subseteq F$, then $F$ has $m\,(\geq n)$ elements, including $k$. Thus $k^*k \leq h_F$, and

$$\begin{aligned}(k - kv_F)^*(k - kv_F) &= (1 - v_F)k^*k(1 - v_F) \\ &\leq (1 - v_F)h_F(1 - v_F) \\ &= \tfrac{1}{m^2}(h_F + \tfrac{1}{m}1)^{-2}h_F.\end{aligned}$$

Since $\frac{1}{m^2}(t + \frac{1}{m}1)^{-2}t \leq \frac{1}{4m}$ for all $t \in \mathbb{R}^+$, we have

$$\begin{aligned}\|k - kv_F\|^2 &= \|(k - kv_F)^*(k - kv_F)\| \\ &\leq \|\tfrac{1}{m^2}(h_F + \tfrac{1}{m}1)^{-2}h_F\| \\ &\leq \tfrac{1}{4m} \\ &\leq \tfrac{1}{4n}.\end{aligned}$$

Thus $\|k - kv_F\| \leq \frac{1}{2\sqrt{n}}$ for all $F \supseteq F_0$. By definition, $\lim_{F \in \mathcal{F}} \|k - kv_F\| = 0$.

This concludes the proof.

□

If $\mathbb{K}$ is an algebraic right ideal of $\mathbb{A}$, then $\mathbb{K}^*$ is an algebraic left ideal of $\mathbb{A}$. By applying the above Proposition to $\mathbb{K}^*$ and interpreting it in terms of $\mathbb{K}$ itself we obtain:

**3.6. Corollary.** *Let $\mathbb{A}$ be a $C^*$-algebra and suppose $\mathbb{K}$ is an* ***algebraic*** *right ideal in $\mathbb{A}$. Then $\mathbb{K}$ has a left approximate identity.*

**3.7. Remark.** We have shown that an algebraic two sided ideal of a C*-algebra $\mathbb{A}$ possesses both a left and a right approximate identity. We now wish to show that these two identities can be chosen to coincide. First we require a lemma.

**3.8. Lemma.** *Let $\mathbb{A}$ be a C*-algebra and suppose that $\mathbb{K}$ is an algebraic, self-adjoint ideal in $\mathbb{A}$. Then any left approximate identity for $\mathbb{K}$ is also a right approximate identity for $\mathbb{K}$, and vice-versa.*
**Proof.** Suppose $(u_\lambda)$ is a right approximate unit for $\mathbb{K}$. Then $\lim_\lambda \|k - ku_\lambda\| = 0$ for all $k \in \overline{\mathbb{K}}$. But then $\lim_\lambda \|k^* - k^* u_\lambda\| = 0 = \lim_\lambda \|k - u_\lambda k\|$, so that $(u_\lambda)$ is also a left approximate unit for $\mathbb{K}$.

$\square$

**3.9. Theorem.** *Every C*-algebra has an approximate identity. If the C*-algebra is separable, then a countable approximate identity may be chosen*
**Proof.** Let $\mathbb{A}$ be the C*-algebra. It is clearly a self-adjoint left ideal in itself, and therefore has a right approximate identity by Proposition 3.5, which is an approximate identity by Lemma 3.8.

Next, suppose $\mathbb{A}$ is separable, and let $\{a_n\}_{n=1}^\infty$ be a countable dense subset of $\mathbb{A}$. Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate identity for $\mathbb{A}$. Choose $\lambda_0 \in \Lambda$ arbitrarily. For each $k \geq 1$, we can find $\lambda_k \in \Lambda$ such that $\lambda_k \geq \lambda_{k-1}$ and $\max(\|u_{\lambda_k} a_n - a_n\|, \|a_n u_{\lambda_k} - a_n\|) < \varepsilon$ for each $1 \leq n \leq k$. A relatively routine approximation argument then implies that $\{u_{\lambda_k}\}_{k=1}^\infty$ is the desired countable approximate identity.

$\square$

**3.10. Corollary.** *Every closed ideal $\mathbb{K}$ in a C*-algebra $\mathbb{A}$ is self-adjoint.*
**Proof.** Let $k \in \mathbb{K}$, and let $(u_\lambda)$ denote the approximate identity for $\mathbb{K}$. Then $k^* = \lim_\lambda k^* u_\lambda$, but $u_\lambda \in \mathbb{K}$ for all $\lambda$, implying that each $k^* u_\lambda$ and therefore $k^*$ lies in $\overline{\mathbb{K}} = \mathbb{K}$.

$\square$

The above result is, in general, false if the ideal is not closed. For example, if $\mathbb{A} = \mathcal{C}(\overline{\mathbb{D}})$, the continuous functions on the closed unit disk, and if $\mathbb{K} = q\,\mathbb{A}$, where $q \in \mathbb{A}$ is the identity function $q(z) = z$, then $\mathbb{K}$ is an algebraic ideal in $\mathbb{A}$, but $q^*$ does not lie in $\mathbb{K}$.

**3.11. Corollary.** *Every algebraic ideal $\mathbb{K}$ in a C*-algebra $\mathbb{A}$ has an approximate identity.*
**Proof.** Since $\overline{\mathbb{K}}$ is a *closed* ideal in $\mathbb{A}$, it must be self-adjoint, by the previous Corollary. The left approximate identity $(u_\lambda)$ for $\mathbb{K}$ is again a left approximate identity for $\overline{\mathbb{K}}$. By Lemma 3.8, $(u_\lambda)$ is a right approximate identity for $\overline{\mathbb{K}}$, and since it already lies in $\mathbb{K}$, it is therefore an approximate unit for $\mathbb{K}$.

□

**3.12. Example.** Let $\mathbb{A}$ be a unital C*-algebra. For each $n \geq 1$, set $u_n = e_{\mathbb{A}}$. Then $\{u_n\}_{n=1}^{\infty}$ is an approximate unit for $\mathbb{A}$.

**3.13. Example.** Let $\mathcal{H}$ be an infinite dimensional, separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$. For each $k \geq 1$, let $P_k$ denote the orthogonal projection of $\mathcal{H}$ onto the span of $\{e_1, e_2, \ldots, e_k\}$. Then from the arguments of Theorem 4.2, $\{P_k\}_{k=1}^{\infty}$ is an approximate identity for $\mathcal{K}(\mathcal{H})$.

**3.14. Example.** Consider the ideal $\mathcal{C}_0(\mathbb{R})$ of $\mathcal{C}(\mathbb{R})$. For each $n \geq 1$, let

$$u_n(x) = \begin{cases} 1 & \text{if } |x| \leq n, \\ (n+1) - |x| & \text{if } |x| \in (n, n+1), \\ 0 & \text{if } |x| \geq n. \end{cases}$$

Then $\{u_n\}_{n=1}^{\infty}$ is an approximate identity for $\mathcal{C}_0(\mathbb{R})$.

**3.15. Remark.** For most purposes where we do not have a unit in the C*-algebra, an approximate unit will do. In certain circumstances, we need something a bit stronger, namely a *quasicentral approximate unit.* (defn)

**3.16. Lemma.** *Let $(u_\lambda)$ be an approximate unit in a C*-algebra $\mathbb{A}$. Then the convex hull of $(u_\lambda)$ is again an approximate unit.*

**3.17. Theorem.** *Every C*-algebra admits a quasicentral approximate unit.*

**3.18. Proposition.** *Suppose $\mathbb{A}$ is a C*-algebra and $\mathbb{L}$ is an ideal in $\mathbb{A}$. If $\mathbb{K}$ is an ideal in $\mathbb{L}$, then $\mathbb{K}$ is also an ideal in $\mathbb{A}$.*

**Proof.** Since $\mathbb{K}$ is the linear span of its positive elements, it suffices to prove that $ak$ and $ka$ lie in $\mathbb{K}$ for all $a \in \mathbb{A}$ and $0 \leq k \in \mathbb{K}$. Since $ak = (a\,k^{\frac{1}{2}})k^{\frac{1}{2}}$, and since $k^{\frac{1}{2}} \in \mathbb{K}$, we have $ak \in \mathbb{K}\,\mathbb{L} \subseteq \mathbb{K}$.

□

**3.19. Definition.** *Let $\mathbb{B}$ be a C*-algebra. A C*-algebra $\mathbb{A}$ of $\mathbb{B}$ is said to be **hereditary** if $b \in \mathbb{B}_+$, $a \in \mathbb{A}_+$ with $0 \leq b \leq a$ implies $b \in \mathbb{A}$.*

**3.20. Example.** Let $\mathbb{B} = \mathcal{C}([0,1])$ and $\mathbb{A} = \{f \in \mathcal{C}([0,1]) : f(x) = 0 \text{ for all } x \in [\frac{1}{4}, \frac{3}{4}]\}$. If $g \in \mathbb{B}_+$, $f \in \mathbb{A}_+$ and $0 \leq g \leq f$, then $0 \leq g(x) \leq f(x) = 0$ for all $x \in [\frac{1}{4}, \frac{3}{4}]$, and hence $g \in \mathbb{A}$. Thus $\mathbb{A}$ is a hereditary C*-subalgebra of $\mathbb{B}$.

**3.21. Proposition.** *Let $\mathbb{B}$ be a $C^*$-algebra and $0 \neq p \neq 1$ be a projection in $\mathbb{B}$. Then $\mathbb{A} = p\mathbb{B}p$ is hereditary.*
**Proof.** That $\mathbb{A} = p\mathbb{B}p$ is a C*-subalgebra of $\mathbb{B}$ is routine. Suppose $0 \leq b \leq a$ for some $a \in \mathbb{A}$, $b \in \mathbb{B}$. Then by Proposition 2.27, $0 \leq (1-p)b(1-p) \leq (1-p)a(1-p) = 0$, and so $(1-p)b(1-p) = 0$.

Next, $\|b^{\frac{1}{2}}(1-p)\|^2 = \|(1-p)b(1-p)\| = 0$, so that $b(1-p) = b^{\frac{1}{2}}(b^{\frac{1}{2}}(1-p)) = 0$. Finally, since $b = b^*$, $(1-p)b = (b(1-p))^* = 0$, and so $b = pbp \in \mathbb{A}$, as required.

□

**3.22. Lemma.** *Let $\mathbb{A}$ be a $C^*$-algebra and $a, b \in \mathbb{A}$. Suppose $0 \leq b$, $\|b\| \leq 1$ and $aa^* \leq b^4$. Then there exists $c \in \mathbb{A}$, $\|c\| \leq 1$ such that $a = b\,c$.*
**Proof.** Let $\tilde{\mathbb{A}}$ denote the minimal unitization of $\mathbb{A}$, and denote by $\mathbf{1}$ the identity in $\tilde{\mathbb{A}}$. For $0 < \lambda < 1$, let $c_\lambda = (b + \lambda 1)^{-1} a$, which lies in $\mathbb{A}$ because $\mathbb{A}$ is an ideal of $\tilde{\mathbb{A}}$. Our goal is to prove that $c = \lim_{\lambda \to 0} c_\lambda$ exists, and that this is the element we want. Now

$$\begin{aligned} c_\lambda c_\lambda^* &= (b+\lambda 1)^{-1} aa^* (b + \lambda 1)^{-1} \\ &\leq (b+\lambda 1)^{-1} b^4 (b+\lambda 1)^{-1} \\ &\leq b^2, \end{aligned}$$

and hence $\|c_\lambda\|^2 = \|c_\lambda^*\|^2 \leq \|b^2\| \leq 1$. Next we prove that $\{c_\lambda\}_{\lambda \in (0,1)}$ is Cauchy. If $\lambda, \beta \in (0,1)$, then

$$\begin{aligned} \|c_\lambda - c_\beta\|^2 &= \|(c_\lambda - c_\beta)^*\|^2 \\ &= \|(c_\lambda - c_\beta)(c_\lambda - c_\beta)^*\| \\ &= \|((b+\lambda 1)^{-1} - (b+\beta 1)^{-1}) aa^* (b+\lambda 1)^{-1} - (b+\beta 1)^{-1})\| \\ &= |\lambda - \beta|^2 \|(b+\lambda 1)^{-1}(b+\beta 1)^{-1} aa^* (b+\lambda 1)^{-1}(b+\beta 1)^{-1}\| \\ &\leq |\lambda - \beta|^2 \|(b+\lambda 1)^{-1}(b+\beta 1)^{-1} b^4 (b+\lambda 1)^{-1}(b+\beta 1)^{-1}\| \\ &\leq |\lambda - \beta|^2, \end{aligned}$$

as $b^4(b+\lambda 1)^{-2}(b+\beta 1)^{-2} \leq 1$. Let $c = \lim_{\lambda \to 0} c_\lambda$. Then $bc = \lim_{\lambda \to 0} bc_\lambda = a$.

□

**3.23. Proposition.** *Let $\mathbb{A}$ be a $C^*$-algebra. Then every ideal in $\mathbb{A}$ is hereditary.*
**Proof.** Suppose that $0 \neq \mathbb{K}$ is an ideal in $\mathbb{A}$, $a \in \mathbb{A}, k \in \mathbb{K}$ and $0 \leq a \leq k$. Then we can write $a = zz^*$ for some $z \in \mathbb{A}$ and $k = (k^{\frac{1}{4}})^4$, where $k^{\frac{1}{4}} \in \mathbb{K}$ by the continuous functional calculus. Then

$$0 \leq zz^* \leq (k^{\frac{1}{4}})^4.$$

By Lemma 3.22, $z = (k^{\frac{1}{4}})c$ for some $c \in \mathbb{A}$. In particular, $z \in \mathbb{K}$ and hence $a = zz^* \in \mathbb{K}$, as required.

□

**3.24. Theorem.** *Let $\mathbb{A}$ be a $C^*$-algebra and $\mathbb{K}$ be an ideal in $\mathbb{A}$. Let $(u_\lambda)$ be any approximate unit for $\mathbb{K}$. Then $\mathbb{A}/\mathbb{K}$ is a $C^*$-algebra, and for $a \in \mathbb{A}$, we have*

$$\|\pi_{\mathbb{K}}(a)\| = \lim_\lambda \|a - au_\lambda\|.$$

**Proof.** Fix $a \in \mathbb{A}$. Clearly

$$\begin{aligned} \|\pi_{\mathbb{K}}(a)\| &= \inf\{\|a+k\| : k \in \mathbb{K}\} \\ &\leq \inf\{\|a - au_\lambda\| : \lambda \in \Lambda\}, \end{aligned}$$

as each $u_\lambda$ and hence $au_\lambda$ lies in $\mathbb{K}$.

Now, given $\epsilon > 0$, choose $k \in \mathbb{K}$ so that $\|\pi_{\mathbb{K}}(a)\| + \epsilon > \|a + k\|$. Then

$$\begin{aligned} \|a - au_\lambda\| &= \|(a+k) - k - au_\lambda\| \\ &= \|(a+k) - (k - ku_\lambda) - (a+k)u_\lambda\| \\ &\leq \|(a+k) - (a+k)u_\lambda\| + \|k - ku_\lambda\|. \end{aligned}$$

We shall work in the unitization $\tilde{\mathbb{A}}$ of $\mathbb{A}$ in order to obtain our desired norm estimates.

$$\begin{aligned} \|a - au_\lambda\| &\leq \|(a+k)(1-u_\lambda)\| + \|k - ku_\lambda\| \\ &\leq \|a+k\|\ \|1 - u_\lambda\| + \|k - ku_\lambda\|. \end{aligned}$$

Hence $\lim_\lambda \|a - au_\lambda\| \leq (\|\pi_{\mathbb{K}}(a)\| + \epsilon)1 + 0$. Since $\epsilon > 0$ was arbitrary, $\|pi_{\mathbb{K}}(a)\| = \lim_\lambda \|a - au_\lambda\|$.

We saw in Proposition 1.16 that $\mathbb{A}/\mathbb{K}$ is a Banach algebra. Since $\mathbb{K}$ is self-adjoint, we can set $\pi_{\mathbb{K}}(a)^* = \pi_{\mathbb{K}}(a^*)$, and this is a well-defined involution on $\mathbb{A}/\mathbb{K}$. There remains only to verify the $C^*$-equation.

Given $a \in \mathbb{A}$,

$$\begin{aligned} \|\pi_{\mathbb{K}}(a)^*\, \pi_{\mathbb{K}}(a)\| &= \|\pi_{\mathbb{K}}(a^*a)\| \\ &= \inf_\lambda \|a^*a - a^*au_\lambda\| \\ &\geq \inf_\lambda \|(1-u_\lambda)(a^*a)(1-u_\lambda)\| \\ &= \inf_\lambda \|a(1-u_\lambda)\|^2 \\ &= \|\pi_{\mathbb{K}}(a)\|^2. \end{aligned}$$

By Remark 1.9, we see that the quotient norm is a $C^*$-norm, and thus $\mathbb{A}/\mathbb{K}$ is a $C^*$-algebra .

□

A subspace of a Banach space is said to be **proximinal** if the distance from an arbitrary vector to that subspace is always attained. Although we shall not prove it here, it can be shown that ideals of $C^*$-algebras are proximinal - that is, the quotient norm is attained.

**3.25. Theorem.** *Let $\tau : \mathbb{A} \to \mathbb{B}$ be a *-homomorphism between $C^*$-algebras $\mathbb{A}$ and $\mathbb{B}$. Then $\|\tau\| \le 1$, and $\tau$ is isometric if and only if $\tau$ is injective.*

**Proof.** First suppose $0 \le r \in \mathbb{A}$. Then $r = z^*z$ for some $z \in \mathbb{A}$, and hence $\tau(r) = \tau(z)^*\tau(z) \ge 0$. In particular, $\tau(r)$ is normal and so $C_0^*(\tau(r))$ is abelian. Let $\varphi \in \sum_{C_0^*(\tau(r))}$. Then $\varphi \circ \tau \in \sum_{C_0^*(r)}$ and hence $\|\varphi \circ \tau\| \le 1$. But

$$\begin{aligned} \|\tau(r)\| &= \sup\{\varphi(\tau(r)) : \varphi \in \textstyle\sum_{C_0^*(r)}\} \\ &\le \|r\|. \end{aligned}$$

More generally, if $a \in \mathbb{A}$, then $a^*a \ge 0$, and hence from above,

$$\|\tau(a)\|^2 = \|\tau(a^*a)\| \le \|a^*a\| = \|a\|^2.$$

Thus we have shown that $\tau$ is continuous, with $\|\tau\| \le 1$.

Clearly if $\tau$ is isometric, it must be injective.

Next, suppose that $\tau$ is not isometric, and choose $a \in \mathbb{A}$ such that $\|a\| = 1$, but $\|\tau(a)\| < 1$. Let $r = a^*a$. Then $\|r\| = \|a\|^2 = 1$, but $\|\tau(r)\| = \|\tau(a)\|^2 = 1 - \delta < 1$ for some $\delta > 0$. We shall work with $r \ge 0$ instead of $a$. Choose $f \in \mathcal{C}([0,1])$ such that $f(x) = 0$ for all $x \in [0, 1-\delta]$, but $f(1) = 1$. By the Stone-Weierstraß Theorem, $f$ is a limit of polynomials $p_n$ in one variable with $p_n(0) = 0$ for each $n \ge 1$. For any such polynomial,

$$\tau(p_n(r)) = p_n(\tau(r)),$$

since $\tau$ is a *-homomorphism. Since $\tau$ is continuous from above,

$$\begin{aligned} \tau(f(r)) &= \tau(\lim_{n\to\infty} p_n(r)) \\ &= \lim_{n\to\infty} \tau(p_n(r)) \\ &= \lim_{n\to\infty} p_n(\tau(r)) \\ &= f(\tau(r)). \end{aligned}$$

Now $\operatorname{spr}(r) = \|r\| = 1$, and since $0 \le r$, we conclude that $1 \in \sigma(r)$. Thus $1 = f(1) \in f(\sigma(r)) = \sigma(f(r))$, so that $f(r) \ne 0$. Finally, $\tau(r) \ge 0$ and $\operatorname{spr}(\tau(r)) \le \|\tau(r)\| \le 1 - \delta$. Since $f|_{[0,1-\delta]} = 0$, we have $f(\tau(r)) = 0 = \tau(f(r))$, implying that $\tau$ is not injective.

□

**3.26. Corollary.** *Let $\tau : \mathbb{A} \to \mathbb{B}$ be a *-homomorphism between $C^*$-algebras $\mathbb{A}$ and $\mathbb{B}$. Then $\tau$ can be factored as $\tau = \overline{\tau} \circ \pi$, where $\pi : \mathbb{A} \to \mathbb{A}/\ker \tau$ is the canonical map, and $\overline{\tau} : \mathbb{A}/\ker \tau \to \operatorname{ran} \tau$ is an isometric *-isomorphism. In particular, $\tau(\mathbb{A})$ is a $C^*$-algebra.*

**Proof.** Since $\tau$ is continuous, $\ker \tau$ is a norm-closed ideal of $\mathbb{A}$, and hence is self-adjoint by Corollary 3.10. By Theorem 3.24, $\mathbb{A}/\ker \tau$ is a $C^*$-algebra and from elementary algebra arguments, $\tau$ factors as $\tau = \overline{\tau} \circ \pi$, where $\pi : \mathbb{A} \to \mathbb{A}/\ker\, \tau$ is the canonical map and $\overline{\tau}$ is the *-homomorphism

$$\begin{array}{rccc} \overline{\tau}: & \mathbb{A}/\ker\, \tau & \to & \operatorname{ran}\tau \\ & a + \ker\, \tau & \mapsto & \tau(a). \end{array}$$

Since $\ker\, \overline{\tau} = 0$, $\overline{\tau}$ is an isometric map onto its range, and thus $\operatorname{ran}\overline{\tau} = \operatorname{ran}\tau$ is a $C^*$-subalgebra of $\mathbb{B}$.

□

**3.27. Proposition.** *Let $\mathbb{A}$ be a C*-subalgebra of a C*-algebra $\mathbb{B}$, and let $\mathbb{K}$ be an ideal of $\mathbb{B}$. Then $\mathbb{A} \cap \mathbb{K}$ is an ideal in $\mathbb{A}$, and*

$$\frac{\mathbb{A} + \mathbb{K}}{\mathbb{K}} \simeq \frac{\mathbb{A}}{\mathbb{A} \cap \mathbb{K}}.$$

*In particular, $\mathbb{A} + \mathbb{K}$ is a C*-subalgebra of $\mathbb{B}$.*

**Proof.** The first statement is a routine exercise. Consider the map

$$\begin{array}{rccc} \beta: & \mathbb{A} & \to & \mathbb{B}/\mathbb{K} \\ & a & \mapsto & a + \mathbb{K}. \end{array}$$

It is readily seen to be a *-homomorphism. Moreover, $\ker\, \beta = \mathbb{A} \cap \mathbb{K}$. By Corollary 3.26, $\operatorname{ran}\beta = \mathbb{A} + \mathbb{K}/\mathbb{K}$ is isometrically *-isomorphic to $\mathbb{A}/(\mathbb{A} \cap \mathbb{K})$, and so $\mathbb{A} + \mathbb{K}/\mathbb{K}$ is a $C^*$-algebra. Thus it is complete. Since $\mathbb{K}$ is also complete, we conclude that $\mathbb{A} + \mathbb{K}$ is complete as well. Hence $\mathbb{A} + \mathbb{K}$ is a closed, self-adjoint subalgebra of $\mathbb{B}$, as was required to prove.

□

*"Finnish guys last nice": Mae West*

## 4. Linear Functionals and States on $C^*$-algebras.

**4.1.** Let us now turn our attention to the dual space of a $C^*$-algebra. As we shall see, the linear functionals generalize the notion of measures on spaces of continuous functions, and are crucial to the representation theory of $C^*$-algebras.

**4.2. Definition.** *Let $\mathbb{A}$ and $\mathbb{B}$ be $C^*$-algebras, and let $\varphi$ denote a linear map from $\mathbb{A}$ to $\mathbb{B}$. We define the **adjoint** of $\varphi$ as $\varphi^* : \mathbb{A} \to \mathbb{B}$ via $\varphi^*(a) = (\varphi(a^*))^*$ for all $a \in \mathbb{A}$. Then $\varphi$ is said to be **self-adjoint** if $\varphi = \varphi^*$.*

*The map $\varphi$ is said to be **positive** if $\varphi(x^*x) \geq 0$ for all $x \in \mathbb{A}$. We write $\varphi \geq 0$ when this is the case.*

*If $\mathbb{B} = \mathbb{C}$, the complex numbers, and $\varphi \in \mathbb{A}^*$, then $\varphi$ is called a **state** if $\varphi$ is a positive linear functional of norm one. We denote by $\mathcal{S}(\mathbb{A})$ the set of all states on $\mathbb{A}$, and refer to this as the **state space** of $\mathbb{A}$.*

**4.3. Remarks.** A few comments are in order. By definition, a linear map $\varphi$ between $C^*$-algebras is self-adjoint if and only if $\varphi(x^*) = (\varphi(x))^*$ for all $x \in \mathbb{A}$. It is routine to verify that this is equivalent to asking that $\varphi$ send hermitian elements of $\mathbb{A}$ to hermitian elements of $\mathbb{B}$.

If $\varphi \geq 0$, then $\varphi$ preserves order. That is, if $x \leq y$ in $\mathbb{A}$, then $y - x \geq 0$, and hence $\varphi(y - x) = \varphi(y) - \varphi(x) \geq 0$ in $\mathbb{B}$.

Finally, it is easy to see that every positive linear map $\varphi$ is automatically self-adjoint. Indeed, given $h = h^* \in \mathbb{A}$, write $h = h_+ - h_-$, and observe that $\varphi(h) = \varphi(h_+) - \varphi(h_-)$ is self-adjoint.

**4.4. Example.** Let $\varphi : \mathbb{A} \to \mathbb{B}$ be any $^*$-homomorphism between $C^*$-algebras $\mathbb{A}$ and $\mathbb{B}$. Then

$$\varphi(x^*x) = \varphi(x)^* \ \varphi(x) \geq 0$$

for all $x \in \mathbb{A}$, and hence $\varphi \geq 0$.

In particular, every multiplicative linear functional on $\mathbb{A}$ is positive.

**4.5. Example.** Let $X$ be a compact, Hausdorff space. The Riesz-Markov Theorem [reference] asserts that $\mathcal{C}(X)^* \simeq \mathcal{M}(X)$, the space of complex-valued regular Borel measures on $X$. The action of a measure $\mu$ on $f \in \mathcal{C}(X)$ is through integration, that is: $\mu(f) := \int_X f \, \mathrm{d}\mu$.

When $X = [0, 1]$, we can identify $\mathcal{M}(X)$ with the space $BV[0, 1]$ of functions of bounded variation on $[0, 1]$. Now, given $F \in BV[0, 1]$, we define $\mu_F \in \mathcal{M}(X)$ via

$$\mu_F(f) = \int_X f \, \mathrm{d}F,$$

the quantity on the right being a Riemann-Stieltjes integral. For example, the evalution functional $\delta_x(f) = f(x)$ for some $x \in X$ corresponds to the point mass at $x$.

Observe that $\mu$ is a self-adjoint (resp. positive) linear functional precisely when the measure $\mathrm{d}\mu$ is real-valued (resp. positive).

**4.6. Example.** Let $n, m \geq 1$ be integers, and consider the $C^*$-algebra $\mathbb{A} = \mathbb{M}_n \oplus \mathbb{M}_m \subseteq \mathcal{B}(\mathbb{C}^{n+m})$. For each $k \geq 1$, let $\mathrm{tr} : \mathbb{M}_k \to \mathbb{C}$ denote the **normalized trace functional**

$$\mathrm{tr}\,([a_{ij}]) = \frac{1}{k} \sum_{i=1}^{k} a_{ii}.$$

For $a = (a_1, a_2) \in \mathbb{A}$ and $\lambda \in [0,1]$, we can define $\varphi_\lambda(a) = \lambda \mathrm{tr}_n(a_1) + (1 - \lambda)\mathrm{tr}_m(a_2)$. Then $\{\varphi_\lambda\}_{\lambda \in [0,1]}$ is a family of states on $\mathbb{A}$.

**4.7. Example.** Let $\mathcal{H}$ be an infinite dimensional Hilbert space and let $P$ be a non-trivial projection on $\mathcal{H}$. The map

$$\begin{array}{rccc} \varphi : & \mathcal{B}(\mathcal{H}) & \to & \mathcal{B}(\mathcal{H}) \\ & T & \mapsto & PTP \end{array}$$

is a positive linear map.

Indeed, if $T \geq 0$, the $\varphi(T^*T) = PT^*TP = (TP)^*(TP) \geq 0$. Observe that $\varphi$ is *not* a *-homomorphism!

**4.8. Remark.** We have shown that every element of a $C^*$-algebra is, in a natural way, a linear combination of four positive elements. Of course, this is a generalization of the corresponding fact for complex numbers.

in a similar vein, every complex measure possesses a *Jordan decomposition* [reference] as a linear combination of four positive measures. Because of the association between linear functionals on commutative $C^*$-algebras and measures as outlined above, we shall think of linear functionals on $C^*$-algebras as abstract measures, and obtain a corresponding Jordan decomposition for these as well. This will imply that the state space of a $C^*$-algebra $\mathbb{A}$ is in some sense "large", a fact which we shall exploit in the proof of the Gelfand-Naimark Theorem below.

We have seen that every multiplicative linear functional on an abelian $C^*$-algebra is automatically continuous of norm one. Furthermore, every such functional is also positive. In fact, more is true.

**4.9. Theorem.** *Let $\mathbb{A}$ be a $C^*$-algebra and $\varphi : \mathbb{A} \to \mathbb{C}$ be a positive linear map. Then $\varphi$ is continuous.*

**Proof.** First observe that $\varphi$ is bounded if and only if there exists $K > 0$ so that $0 \leq r \in \mathbb{A}_+$ with $\|r\| \leq 1$ implies $\varphi(r) \leq K$. Indeed, if $\varphi$ is bounded, we can trivially choose $K = \|\varphi\|$.

Conversely, if $0 \leq r \in \mathbb{A}_+$ with $\|r\| \leq 1$ implies $\varphi(r) \leq K$, then given any $x \in \mathbb{A}$, we can write $x = y + iz$, where $y = \mathrm{Re}x$, $z = \mathrm{Im}x$. Then we set $y = y_+ - y_-$ and $z = z_+ - z_-$, and recall that $\max(\|y_+\|, \|y_-\|, \|z_+\|, \|z_-\|) \leq \|x\|$. From this we obtain

$$\begin{aligned} |\varphi(x)| &= |\varphi(y_+) - \varphi(y_-) + i\varphi(z_+) - i\varphi(z_-)| \\ &\leq \varphi(y_+) + \varphi(y_-) + \varphi(z_+) + \varphi(z_-) \\ &\leq 4K\|x\|, \end{aligned}$$

and so $\|\varphi\| \leq 4K < \infty$.

Now we argue by contradiction. Suppose, to the contrary, that for every $n \geq 1$ we can find $0 \leq r_n$ in $\mathbb{A}$ so that $\|r_n\| \leq \frac{1}{2^n}$ and $\varphi(r_n) \geq 1$. Then for each $k \geq 1$, $s_k = \sum_{n=1}^k r_n \in \mathbb{A}_+$, and $s_k \leq s = \sum_{n=1}^\infty r_n \in \mathbb{A}_+$. From Remark **??**, we see that $k \leq \varphi(s_k) \leq \varphi(s)$ for all $k \geq 1$, which is absurd. It follows that $\varphi$ must be bounded on $\mathbb{A}_+$, and hence on $\mathbb{A}$.

□

**4.10.** Given a positive linear functional $\varphi$ on a $C^*$-algebra $\mathbb{A}$, we can construct a pseudo-inner product on $\mathbb{A}$ by setting

$$[a, b] := \varphi(b^* a)$$

for $a, b \in \mathbb{A}$. Then we have

(i) $[a, b]$ is clearly a sesquilinear function, linear in $a$ and conjugate linear in $b$;
(ii) $[a, a] \geq 0$ for all $a \in \mathbb{A}$, as $\varphi \geq 0$ and $a^* a \geq 0$;
(iii) Since $\varphi$ is self-adjoint, $[a, b] = \varphi(b^* a) = \overline{\varphi(a^* b)} = \overline{[b, a]}$;
(iv) If $x \in \mathbb{A}$, then $[xa, b] = \varphi(b^*(xa)) = \varphi((x^* b)^* a) = [a, x^* b]$.

The following will also prove useful in the GNS construction.

**4.11. Lemma.** *Let $[\cdot, \cdot]$ be a positive sesquilinear function on a $C^*$-algebra $\mathbb{A}$. Then $[\cdot, \cdot]$ satisfies the Cauchy-Schwarz Inequality:*

$$|[a, b]|^2 \leq [a, a]\ [b, b].$$

**Proof.**

(a) If $[a, b] = 0$, there is nothing to prove.
(b) If $[a, a] = 0$, then we claim that $[a, b] = 0$ for all $b \in \mathbb{A}$. To see this, note that for all $\beta \in \mathbb{C}$,

$$\begin{aligned} 0 &\leq [a + \beta b, a + \beta b] \\ &= [a, a] + |\beta|^2 [b, b] + 2\mathrm{Re}(\beta [a, b]). \end{aligned}$$

Suppose there exists $b \in \mathbb{A}$ such that $[a, b] \neq 0$. We may then scale $b$ so that $[a, b] = -1$. Now choose $\beta > 0$. The above equation then becomes:

$$\begin{aligned} 0 &\leq [a, a] - 2\beta + \beta^2 [b, b] \\ &= -2\beta + \beta^2 [b, b], \end{aligned}$$

which implies $0 \leq \beta [b, b] - 2$. This yields a contradiction when $\beta$ is chosen sufficiently small and positive. Thus $[a, a] = 0$ implies $|[a, b]|^2 = 0 \leq [a, a]\,[b, b]$, which is clearly true.

(c) If $[a,b] \neq 0$ and $[a,a] \neq 0$, we may choose $\beta = -[a,a]/[a,b]$. Then, as above,

$$\begin{aligned} 0 &\leq [a,a] + |\beta|^2 [b,b] + 2\mathrm{Re}(\beta[a,b]) \\ &= [a,a] - 2[a,a] + \frac{[a,a]^2[b,b]}{|[a,b]|^2}, \end{aligned}$$

which implies

$$|[a,b]|^2 \leq [a,a]\,[b,b],$$

as claimed.

□

**4.12. Lemma.** *Let $\mathbb{A}$ be a C*-algebra, and $0 \leq \varphi \in \mathbb{A}^*$. Then*

(i) $|\varphi(b^*a)| \leq \varphi(a^*a)^{\frac{1}{2}}\, \varphi(b^*b)^{\frac{1}{2}}$;
(ii) $|\varphi(a)|^2 \leq \|\varphi\| \varphi(a^*a)$.

**Proof.**

(i) This is just a reformulation of the Cauchy-Schwarz Inequality which we deduced for the pseudo-inner product associated to $\varphi$ in the previous Lemma.
(ii) Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for $\mathbb{A}$. Then

$$\begin{aligned} |\varphi(a)|^2 &= \lim_\lambda |\varphi(a u_\lambda)|^2 \\ &\leq \sup_\lambda \varphi(a^*a)\, \varphi(u_\lambda^* u_\lambda) \\ &\leq \sup_\lambda \varphi(a^*a) \|\varphi\|\, \|u_\lambda\|^2 \\ &\leq \varphi(a^*a)\|\varphi\|. \end{aligned}$$

□

**4.13. Theorem.** *Let $\mathbb{A}$ be a $C^*$-algebra and $\varphi \in \mathbb{A}^*$. The following are then equivalent:*

(i) $0 \leq \varphi$;
(ii) $\|\varphi\| = \lim_\lambda \varphi(u_\lambda)$ *for some approximate unit* $(u_\lambda)_{\lambda\in\Lambda}$ *of* $\mathbb{A}$;
(iii) $\|\varphi\| = \lim_\lambda \varphi(u_\lambda)$ *for every approximate unit* $(u_\lambda)_{\lambda\in\Lambda}$ *of* $\mathbb{A}$.

**Proof.**

(i) implies (iii) Consider $\{\varphi(u_\lambda)\}_{\lambda\in\Lambda}$, which is an increasing net of positive real numbers, bounded above by $\|\varphi\|$. Then $\lim_\lambda \varphi(u_\lambda)$ exists. Clearly $\lim_\lambda \varphi(u_\lambda) \leq \sup_\lambda \|\varphi\|\,\|u_\lambda\| \leq \|\varphi\|$.

For the other inequality, first observe that if $0 \leq r$ and $\|r\| \leq 1$, then $0 \leq r^2 \leq r$. This follows from the Gelfand-Naimark Theorem by indentifying $r$ with the identity function $q(z) = z$ on $\sigma(r) \subseteq [0,1]$. Then, given $a \in \mathbb{A}$,

$$\begin{aligned} |\varphi(a)| &= \lim_\lambda |\varphi(u_\lambda a)| \\ &\leq \lim_\lambda \varphi(u_\lambda^* u_\lambda)^{\frac{1}{2}} \varphi(a^* a)^{\frac{1}{2}} \\ &\leq \lim_\lambda \varphi(u_\lambda)^{\frac{1}{2}} \varphi(a^* a)^{\frac{1}{2}} \\ &\leq \lim_\lambda \varphi(u_\lambda)^{\frac{1}{2}} \|\varphi\|^{\frac{1}{2}} \|a^* a\|^{\frac{1}{2}} \\ &\leq (\lim_\lambda \varphi(u_\lambda)^{\frac{1}{2}}) \|\varphi\|^{\frac{1}{2}} \|a\|. \end{aligned}$$

By taking the supremum over $a \in \mathbb{A}$, $\|a\| = 1$, we find that $\|\varphi\|^{\frac{1}{2}} \leq \lim_\lambda \varphi(u_\lambda)^{\frac{1}{2}}$, and hence $\|\varphi\| = \lim_\lambda \varphi(u_\lambda)$.

(iii) implies (ii) Obvious.

(ii) implies (i) Let us scale $\varphi$ so that $\|\varphi\| = 1$. Consider $h = h^* \in \mathbb{A}$ with $\|h\| = 1$. Let $\varphi(h) = s + it \in \mathbb{C}$ where $s, t \in \mathbb{R}$. Our first goal is to show that $t = 0$. By considering $-h$ instead of $h$, we may assume that $t \geq 0$. Fix an integer $n \geq 1$, and consider $x_{n,\lambda} = h + inu_\lambda$. Now

$$\begin{aligned} \|x_{n,\lambda}\|^2 &= \|x_{n,\lambda}^* x_{n,\lambda}\| \\ &= \|h^2 + in(hu_\lambda - u_\lambda h) - n^2 u_\lambda\| \\ &\leq \|h\|^2 + n\|hu_\lambda - u_\lambda h\| + n^2 \\ &= 1 + n^2 + n\|hu_\lambda - u_\lambda h\|. \end{aligned}$$

Now $\lim_\lambda \varphi(x_{n,\lambda}) = \lim_\lambda(\varphi(h) + in\varphi(u_\lambda)) = \varphi(h) + in = s + i(t+n)$. Furthermore, $|\varphi(x_{n,\lambda})|^2 \leq \|x_{n,\lambda}\|^2$, and so

$$s^2 + (t+n)^2 \leq \lim_\lambda (1 + n^2 + n\|hu_\lambda - u_\lambda h\|) = 1 + n^2.$$

Thus $s^2 + t^2 + 2tn + n^2 \leq 1 + n^2$. Since $t > 0$, we obtain a contradiction by choosing $n$ sufficiently large.

So far we have shown that $\varphi$ is self-adjoint. We still want $0 \leq \varphi(r)$. Suppose $0 \leq r \leq 1$. Let $h_\lambda = r - u_\lambda$. By Lemma **??**, $\|h\| \leq 1$. Now $\lim_\lambda \varphi(h) = \varphi(r) - 1$, and since $|\varphi(h)| \leq 1$, we have $\varphi(r) - 1| \leq 1$, from which we conclude that $0 \leq \varphi(r) \leq 1$, which completes the proof.

□

**4.14. Corollary.** *Suppose $\mathbb{A}$ is a $C^*$-algebra, and $\varphi, \alpha, \beta \in \mathbb{A}^*$.*

(i) *If $\alpha, \beta \geq 0$, then $\|\alpha + \beta\| = \|\alpha\| + \|\beta\|$.*

(ii) *Suppose $\mathbb{A}$ is unital. Then $\varphi \geq 0$ if and only if $\|\varphi\| = \varphi(e_{\mathbb{A}})$. In particular, $\varphi$ is a state on $\mathbb{A}$ if and only if $\varphi(e_{\mathbb{A}}) = 1 = \|\varphi\|$.*

**Proof.**

(i) Since $\alpha, \beta \geq 0$, so is $\alpha + \beta$. But then if $(u_\lambda)_{\lambda \in \Lambda}$ is any approximate unit for $\mathbb{A}$,

$$\begin{aligned} \|\alpha + \beta\| &= \lim_\lambda (\alpha + \beta)(u_\lambda) \\ &= \lim_\lambda \alpha(u_\lambda) + \lim_\lambda \beta(u_\lambda) \\ &= \|\alpha\| + \|\beta\|. \end{aligned}$$

(ii) This is an immediate consequence of Theorem 4.13, after observing that $u_\lambda = e_{\mathbb{A}}$ is an approximate identity for $\mathbb{A}$.

□

**4.15. Proposition.** *Let $\mathbb{A}$ be a $C^*$-algebra and $0 \leq \varphi \in \mathbb{A}^*$. Then for all $a, b \in \mathbb{A}$,*

$$\varphi(b^* a^* a b) \leq \|a^* a\| \varphi(b^* b).$$

**Proof.** We claim that $b^* a^* a b \leq \|a^* a\| b^* b$, from which the above equation clearly follows. For the sake of convenience, we shall work in $\tilde{\mathbb{A}}$.

We know that $a^* a \leq \|a^* a\| e_{\mathbb{A}}$ in $\tilde{\mathbb{A}}$, and thus

$$b^* a^* a b \leq b^* (\|a^* a\| e_{\mathbb{A}}) b = \|a^* a\| b^* b.$$

Since $\varphi$ is positive, it preserves order, and we are done.

□

**4.16. Theorem.** *Let $\mathbb{A}$ be a unital $C^*$-algebra. Then the state space $\mathcal{S}(\mathbb{A})$ is a weak*-compact, convex subset of the unit ball $\mathbb{A}_1^*$ of $\mathbb{A}^*$.*

**Proof.** Clearly $\mathcal{S}(\mathbb{A}) \subseteq \mathbb{A}_1^*$. Since $\mathbb{A}_1^*$ is weak*-compact by the Banach-Alaoglu Theorem, it suffices to show that $\mathcal{S}(\mathbb{A})$ is weak*-closed.

Suppose $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ is a net in $\mathcal{S}(\mathbb{A})$ converging in the weak*-topology to $\varphi \in \mathbb{A}^*$. Again, the weak*-compactness of $\mathbb{A}_1^*$ implies that $\|\varphi\| \leq 1$. Moreover,

$$\varphi(1) = \lim_\lambda \varphi_\lambda(1) = 1,$$

and so by Corollary 4.14, $\varphi \in \mathcal{S}(\mathbb{A})$. Thus $\mathcal{S}(\mathbb{A})$ is weak*-closed, as required.

If $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{A})$ and $0 < t < 1$, then clearly $\varphi = t\varphi_1 + (1-t)\varphi_2$ is positive, and $\varphi(1) = 1$. Since $\|\varphi\| \leq t\|\varphi_1\| + (1-t)\|\varphi_2\| = 1$, $\varphi \in \mathcal{S}(\mathbb{A})$, which is therefore convex.

□

**4.17.** Our next goal is to prove that if $\mathbb{A}$ and $\mathbb{B}$ are $C^*$-algebras with $\mathbb{A} \subseteq \mathbb{B}$, then every state on $\mathbb{A}$ can be extended to a state on $\mathbb{B}$. Before doing that, let us observe that the restriction of a state on $\mathbb{B}$ is not necessarily a state on $\mathbb{A}$, although it is still clearly a positive linear functional.

For example, let $c$ denote the $C^*$-subalgebra of $\ell^\infty(\mathbb{N})$ consisting of *convergent* sequences. Then $c_0 = \{(a_n)_{n=1}^\infty \in c : \lim_{n\to\infty} a_n = 0\}$ is a non-unital $C^*$-subalgebra of $c$. Consider the states $\beta_1$ and $\beta_2$ on $c$, where

$$\beta_1(a_n) = \lim_{n\to\infty} a_n \quad \text{and} \quad \beta_2(a_n) = a_1.$$

Then $\beta = \frac{1}{2}(\beta_1 + \beta_2)$ is again a state on $c$, by Theorem 4.16. The restriction of $\beta$ to $c_0$ is $\frac{1}{2}\beta_2$, which is *not* a state on $c_0$.

**4.18. Theorem.** *Let $\mathbb{A}$ and $\mathbb{B}$ be $C^*$-algebras with $\mathbb{A} \subseteq \mathbb{B}$. Suppose $\varphi \in \mathcal{S}(\mathbb{A})$. Then there exists $\beta \in \mathcal{S}(\mathbb{B})$ whose restriction to $\mathbb{A}$ coincides with $\varphi$.*

**Proof.** Consider first the case where $\mathbb{B} = \tilde{\mathbb{A}}$, the unitization of $\mathbb{A}$.

Here we have no choice as to the definition of $\beta$ since $\beta \in \mathcal{S}(\mathbb{B})$ implies $\beta(e_{\mathbb{B}}) = 1$. In other words, we must have $\beta(a + \alpha e_{\mathbb{B}}) = \varphi(a) + \alpha$. It remains only to verify that this $\beta$ is in fact a state, which reduces to verifying that $\|\beta\| = 1$. Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for $\mathbb{A}$. Now

$$\begin{aligned}
|\beta(a + \alpha e_{\mathbb{B}})| &= |\varphi(a) + \alpha| \\
&= \lim_\lambda \varphi(a u_\lambda) + \alpha\varphi(u_\lambda)| \\
&= \lim_\lambda |\varphi(a u_\lambda + \alpha u_\lambda)| \\
&\leq \liminf_\lambda \|\varphi\| \, \|a + \alpha e_{\mathbb{B}}\| \, \|u_\lambda\| \\
&\leq \|a + \alpha e_{\mathbb{B}}\|.
\end{aligned}$$

It follows that $\|\beta\| \leq 1$. Since $\beta$ is an extension of $\varphi$, it has norm at least 1, i.e. $\beta \in \mathcal{S}(\mathbb{B})$.

Consider next the case where $\mathbb{B}$ is any unital $C^*$-algebra containing $\mathbb{A}$. Then we can assume, using the above paragraph, that $\mathbb{A}$ is unital as well. If $\varphi \in \mathcal{S}(\mathbb{A})$ and $\beta$ is any extension of $\varphi$ to $\mathbb{B}$ given us by the Hahn-Banach Theorem (with $\|\beta\| = \|\varphi\| = 1$), then $\|\beta\| = 1 = \varphi(e_{\mathbb{B}}) = \beta(e_{\mathbb{B}})$, and so $\beta \in \mathcal{S}(\mathbb{B})$.

Finally, suppose $\mathbb{B}$ is not unital. First we extend $\varphi$ to a state $\tilde{\varphi}$ on $\tilde{\mathbb{A}}$ by the first paragraph. From the second paragraph, $\tilde{\varphi}$ extends to a state $\tilde{\beta}$ on $\tilde{\mathbb{B}}$. Let $\beta$ be the restriction of $\tilde{\beta}$ to $\mathbb{B}$. Clearly $\beta$ is positive, and $1 = \|\tilde{\beta}\| \geq \|\beta\| \geq \|\varphi\| = 1$, since $\beta$ is an extension of $\varphi$. Thus $\beta \in \mathcal{S}(\mathbb{B})$.

$\square$

**4.19. Corollary.** *Suppose that $\mathbb{A}$ and $\mathbb{B}$ are $C^*$-algebras and that $\mathbb{A} \subseteq \mathbb{B}$. Then every positive linear functional on $\mathbb{A}$ extends to a positive linear functional on $\mathbb{B}$ with the same norm.*

**Proof.** If $0 < \varphi$ is a positive linear functional on $\mathbb{A}$, then $\alpha = \varphi / \|\varphi\|$ is a state on $\mathbb{A}$, which extends to a state $\beta$ on $\mathbb{B}$ by Theorem 4.18 above. Hence $\|\varphi\| \, \beta$ extends $\varphi$.

□

**4.20. Proposition.** *If $\mathbb{A}$ is an ideal of a $C^*$-algebra $\mathbb{B}$, then any positive linear functional $\varphi$ on $\mathbb{A}$ extends in a* unique *way to a positive linear functional $\beta$ on $\mathbb{B}$ with $\|\beta\| = \|\varphi\|$.*

**Proof.** Suppose $\mathbb{A} \subseteq \mathbb{B}$ is an ideal. From Corollary 4.19, given $0 \le \varphi \in \mathbb{A}^*$, we can find $0 \le \gamma_1 \in \mathbb{B}^*$ so that $\|\gamma_1\| = \|\varphi\|$ and $\gamma_1|_{\mathbb{A}} = \varphi$. Let $\gamma_2$ be any positive extension of $\varphi$ to $\mathbb{B}$ with $\|\gamma_2\| = \|\varphi\|$. Let $(u_\lambda)_\lambda$ be an approximate identity for $\mathbb{A}$.

Then $\lim_\lambda \gamma_2(1 - u_\lambda) = 0$. Moreover, $(1 - u_\lambda)^2 \le (1 - u_\lambda)$, and so $\lim_\lambda \gamma_2((1 - u_\lambda)^2) = 0$.

For all $b \in \mathbb{B}$,

$$\begin{aligned} |\gamma_2(b) - \gamma_2(u_\lambda b)|^2 &= |\gamma_2((1 - u_\lambda)b)|^2 \\ &\le \gamma_2((1 - u_\lambda)^2)^{1/2} \gamma_2(b^* b)^{1/2} \end{aligned}$$

by the Cauchy-Schwarz inequality. It follows that $\lim_\lambda |\gamma_2(b) - \gamma_2(u_\lambda b)| = 0$, so that $\gamma_2(b) = \lim_\lambda \gamma_2(u_\lambda b)$.

Since $\mathbb{A}$ is an ideal, we have $u_\lambda b \in \mathbb{A}$, and hence $\gamma_2(b) = \lim_\lambda \varphi(u_\lambda b)$. In particular, the values of $\gamma$ on $\mathbb{B}$ are completely determined by the values of $\varphi$ on $\mathbb{A}$, and so $\gamma_1$ is unique.

□

**4.21. Proposition.** *Let $\mathbb{A}$ be a $C^*$-algebra and $0 \ne n \in \mathbb{A}$ be normal.*

(a) *If $\tau \in \mathcal{S}(\mathbb{A})$, then $\tau(n) \in \overline{\mathrm{co}}(\sigma(n))$, the closed convex hull of the spectrum of $n$.*
(b) *There exists $\tau \in \mathcal{S}(\mathbb{A})$ such that $|\tau(n)| = \|n\|$.*

**Proof.**

(a) First recall that the closed convex hull of a compact subset subset $\Omega \subseteq \mathbb{C}$ is the intersection of all closed disks which contain $\Omega$.

Suppose that $\tau \in \mathcal{S}(\mathbb{A})$ and that $\tau(n) \notin \overline{\mathrm{co}}(\sigma(n))$. Then there exists $z_0 \in \mathbb{C}$ and $r > 0$ so that $\sigma(n) \in \overline{D}(z_0, r) := \{\lambda \in \mathbb{C} : |z_0 - \lambda| \le r\}$, but $|\tau(n) - z_0| > r$. Let $\tilde{\tau}$ denote the positive extension of $\tau$ to $\tilde{\mathbb{A}}$, with $\|\tilde{\tau}\| = \|\tau\| = 1$. Let $e$ denote the identity in $\tilde{\mathbb{A}}$. Now $n - z_0 e$ is normal and $\sigma(n - z_0 e) = \sigma(n) - z_0 \subseteq \overline{D}(0, r)$, so that

$$\|n - z_0 e\| = \mathrm{spr}(n - z_0 e) \le r,$$

while $|\tilde{\tau}(n - z_0 e)| = |\tilde{\tau}(n) - z_0| > r \leq \|n - z_0 e\|$, implying that the extension $\tilde{\tau}$ has norm greater than one, a contradiction since $\|\tilde{\tau}\| = \|\tau\| = 1$.

(b) We may assume that $n \neq 0$. Now $C_0^*(n) \simeq^* \mathcal{C}_0(\sigma(n)\backslash\{0\})$. Let $\lambda \in \sigma(n)\backslash\{0\}$ such that $|\lambda| = \mathrm{spr}(n) = \|n\|$. Let $\tau \in \Sigma_{C_0^*(n)}$ be the corresponding multiplicative linear functional, so that

$$\tau(m) = [\Gamma(m)](\lambda), \quad m \in C_0^*(n).$$

Then $\tau \in \mathcal{S}(\mathbb{A})$ and

$$|\tau(n)| = |[\Gamma(n)](\lambda)| = \|n\|.$$

$\square$

**4.22.** Let $\mathbb{A}$ be a $C^*$-algebra and $\varphi \in \mathbb{A}^*$. Recall from Definition 4.2 that $\varphi^* : \mathbb{A} \to \mathbb{C}$ is the map $\varphi^*(a) = \overline{\varphi(a^*)}$. Note that

$$\begin{aligned} \|\varphi^*\| & \\ &= \sup\{\overline{|\varphi(a^*)|} : \|a\| \leq 1\} \\ &= \sup\{\overline{|\varphi(b)|} : \|b\| \leq 1\} \\ &= \|\varphi\|. \end{aligned}$$

Moreover, $(\lambda\varphi_1 + \varphi_2)^* = \overline{\lambda}\varphi_1^* + \varphi_2^*$ and $(\varphi_1^*)^* = \varphi$. The map $\varphi \mapsto \varphi^*$ fails to be an involution only because $\mathbb{A}^*$ is not algebra.

Given $\varphi \in \mathbb{A}^*$, we can define

$$\varphi_r = (\varphi + \varphi^*)/2 \qquad \varphi_i = (\varphi - \varphi^*)/2i.$$

Clearly $\varphi_r = \varphi_r^*$ and $\varphi_i = \varphi_i^*$ and $\varphi = \varphi_r + i\varphi_i$.

We are now in a position to extend the Jordan decomposition for real-valued measures on a commutative $C^*$-algebra to selfadjoint functionals on a general $C^*$-algebra.

**4.23. Theorem.** [***Jordan Decomposition***] *Let $\mathbb{A}$ be a $C^*$-algebra and $\varphi = \varphi^* \in \mathbb{A}^*$. Then there exist $0 \leq \varphi_+, \varphi_- \in \mathbb{A}^*$ so that*

$$\varphi = \varphi_+ - \varphi_-$$

*and $\|\varphi\| = \|\varphi_+\| - \|\varphi_-\|$.*

*Time waits for no man. No man is an island. So... time waits for an island... I don't get it.*

## 5. The GNS Construction.

**5.1.** In this section we prove that every $C^*$-algebra of operators is isometrically $*$-isomorphic to a $C^*$-algebra of operators on a Hilbert space.

**5.2. Definition.** *A **representation** of a $C^*$-algebra $\mathbb{A}$ is a pair $(\mathcal{H}, \rho)$ where*

$$\rho : \mathbb{A} \to \mathcal{B}(\mathcal{H})$$

*is a $^*$-homomorphism. The representation is said to be **faithful** if $\rho$ is injective.*

*A **cyclic** vector for the representation is a vector $\nu \in \mathcal{H}$ for which $\rho(\mathbb{A})\nu = \{\rho(a)\nu : a \in \mathbb{A}\}$ is dense in $\mathcal{H}$. The representation $(\mathcal{H}, \rho)$ is said to be **cyclic** if it admits a cyclic vector $\nu$, in which case we shall often write $(\mathcal{H}, \rho, \nu)$ to emphasize the fact the $\nu$ is cyclic for $(\mathcal{H}, \rho)$.*

We note that it is common to refer to $\rho$ as the representation, and to apply adjectives such as "faithful" or "cyclic" to $\rho$.

**5.3. Example.**

(a) Let $\mathbb{A} = \mathcal{C}([0,1])$, and $\mathcal{H} = L^2([0,1], dx)$, where $dx$ denotes Lebesgue measure on the interval $[0,1]$. Then $(\mathcal{H}, \rho)$ is a representation of $\mathbb{A}$, where

$$\rho(f) = M_f, \quad f \in \mathcal{C}([0,1])$$

and $M_f g = fg$, $g \in \mathcal{H}$. Since $\|M_f\| = \|fnorm_\infty$ by ??, $\rho$ is injective, and hence $(\mathcal{H}, \rho)$ is faithful.

Consider the constant function $\nu(x) = 1, x \in [0,1]$ as an element of $\mathcal{H}$. (Strictly speaking, of course, $\nu$ is an equivalence class of this function in $L^2([0,1], dx)$.) For any $a \in \mathbb{A}$, $\rho(a)\nu = a$, and so $\rho(\mathbb{A})\nu = \mathcal{C}([0,1])$, which is dense in $L^2([0,1], dx)$. Thus $\nu$ is cyclic for $(\mathcal{H}, \rho)$.

(b) Let $\mathcal{H}$ be a separable complex Hilbert space with orthonormal basis $\{e_n\}_{n=1}^\infty$. Let $\mathbb{A} = \mathcal{K}(\mathcal{H})$, and consider the representation

$$\begin{array}{rccl} \rho : & \mathbb{A} & \to & \mathcal{B}(\mathcal{H}^{(2)}) \\ & K & \mapsto & K \oplus K. \end{array}$$

Let $\nu = e_1 \oplus e_2 \in \mathcal{H}^{(2)}$. For each $y, z \in \mathcal{H}$, $y \otimes e_1^*$ and $z \otimes e_2^* \in \mathcal{K}(\mathcal{H})$, being rank one operators. Then

$$\begin{array}{rcl} \rho(y \otimes e_1^*)(\nu) & = & y \oplus 0 \\ \rho(z \otimes e_2^*)(\nu) & = & 0 \oplus z \end{array}$$

and so $\mathcal{H}^{(2)} = \rho(\mathbb{A})\nu$, i.e. $\nu$ is cyclic for $(\mathcal{H}^{(2)}, \rho)$.

(c) Let $\mathbb{A} = \mathcal{C}([0,1])$ once again and let $\mathcal{H} = \mathbb{C}$. Then $(\mathbb{C}, \rho)$ is a representation, where $\rho(f) = f(1)$, $f \in \mathcal{C}([0,1])$. Note that $(\mathbb{C}, \rho)$ is not faithful, since, for example, if $g(x) = 1 - x$, $x \in [0,1]$, then $g \neq 0$, but $\rho(g) = g(1) = 0$.

(d) With $\mathbb{A}$ as above, consider $\mathcal{H} = \mathbb{C}^3$ along with the representation

$$\begin{array}{rccc} \rho: & \mathcal{C}([0,1]) & \mapsto & \mathcal{B}(\mathbb{C}^3) \\ & f & \mapsto & f(0) \oplus f(0) \oplus f(1). \end{array}$$

We leave it as an exercise for the reader to verify that $(\mathcal{H}, \rho)$ is not cyclic.

**5.4.** Let $\mathbb{A}$ be a $C^*$-algebra. We now describe a process that allows us to identify a certain quotient of $\mathbb{A}$ by a closed left ideal with a pre-Hilbert space.

Suppose $0 \le \varphi \in \mathbb{A}^*$. Recall from paragraph 4.10 that we obtain a pseudo-inner product on $\mathbb{A}$ via

$$[a, b] := \varphi(b^* a).$$

Let $\mathbb{L} = \{m \in \mathbb{A} : [m, m] = 0\}$. It follows from the Cauchy-Schwarz Inequality (Lemma 4.11) that $m \in \mathbb{L}$ if and only if $[m, b] = 0$ for all $b \in \mathbb{A}$. In particular, if $m_1, m_2 \in \mathbb{L}$ and $\lambda \in \mathbb{C}$, $b \in \mathbb{A}$, then $[\lambda m_1 + m_2, b] = \lambda[m_1, b] + [m_2, b] = 0 + 0$, so that $\mathbb{L}$ is easily seen to be a subspace of $\mathbb{A}$. Moreover, by paragraph **??** (iv), if $m \in \mathbb{L}$ and $a \in \mathbb{A}$, then $[am, am] = [m, a^* am] = 0$ from above, and so $am \in \mathbb{L}$. Thus $\mathbb{L}$ is in fact a left ideal of $\mathbb{A}$.

It is routine to verify that $\mathbb{A}/\mathbb{L}$ is a pre-Hilbert space when equipped with the inner product $\langle a + \mathbb{L}, b + \mathbb{L}\rangle := [a, b] := \varphi(b^* a)$. Furthermore, we can define a left module action of $\mathbb{A}$ upon $\mathbb{A}/\mathbb{L}$ via

$$a \circ (x + \mathbb{L}) = ax + \mathbb{L}, \qquad a \in \mathbb{A}, x + \mathbb{L} \in \mathbb{A}/\mathbb{L}.$$

This map is well-defined because if $x + \mathbb{L} = y + \mathbb{L}$, then $x - y \in \mathbb{L}$. Since this latter is a left ideal of $\mathbb{A}$, $ax - ay \in \mathbb{L}$, and so $ax + \mathbb{L} = ay + \mathbb{L}$.

**5.5. Theorem.** [ ***The GNS Construction*** ] *Let $\mathbb{A}$ be a $C^*$-algebra and $0 \le \varphi \in \mathbb{A}^*$. Then there exists a cyclic representation $(\mathcal{H}, \rho, \nu)$ of $\mathbb{A}$ where $\nu$ is a cyclic vector satisfying $\|\nu\| = \|\varphi\|^{\frac{1}{2}}$ and*

$$\langle \rho(a)\nu, \nu\rangle = \varphi(a), \qquad a \in \mathbb{A}.$$

**Proof.** Using the notation above, let $\mathcal{H}$ denote the completion of the pre-Hilbert space $\mathbb{A}/\mathbb{L}$, where $\mathbb{L} = \{m \in \mathbb{A} : \varphi(m^* m) = 0\}$. For $a, x \in \mathbb{A}$,

$$\begin{aligned} \|a \circ (x + \mathbb{L})\|^2 &= \|ax + \mathbb{L}\|^2 \\ &= \langle ax + \mathbb{L}, ax + \mathbb{L}\rangle \\ &= [ax, ax] \\ &= \varphi(x^* a^* ax) \\ &\le \|a^* a\| \ \varphi(x^* x) \\ &= \|a\|^2 \ [x, x] \\ &= \|a\|^2 \ \|x + \mathbb{L}\|^2, \end{aligned}$$

and so if we define

$$\begin{array}{rccc} \rho_0 : & \mathbb{A}/\mathbb{L} & \to & \mathbb{A}/\mathbb{L} \\ & x+\mathbb{L} & \mapsto & ax+\mathbb{L} \end{array}$$

then $\|\rho_0(a)\| \le \|a\|$ and therefore $\rho_0(a)$ extends to a bounded linear map $\rho(a)$ on $\mathcal{H}$. It is now routine to verify that $a \mapsto \rho(a)$ is a linear homomorphism of $\mathbb{A}$ into $\mathcal{B}(\mathcal{H})$.

Also,

$$\langle \rho(a^*)x+\mathbb{L}, y+\mathbb{L}\rangle = [a^*x, y] = [x, ay] = \langle x+\mathbb{L}, \rho(a)y+\mathbb{L}\rangle$$

for all $x, y, a \in \mathbb{A}$, and so by the density of $\mathbb{A}/\mathbb{L}$ in $\mathcal{H}$, we see that

$$\langle \rho(a^*)\xi_1, \xi_2\rangle = \langle \xi_1, \rho(a)\xi_2\rangle \quad \text{for all } \xi_1, \xi_2 \in \mathcal{H}.$$

Hence $\rho(a^*) = \rho(a)^*$ for all $a \in \mathbb{A}$, which implies that $(\mathcal{H}, \rho)$ is a representation of $\mathbb{A}$.

Let $(u_\lambda)_\lambda$ be an approximate identity for $\mathbb{A}$. Then $(u_\lambda + \mathbb{L})_\lambda$ is a net of vectors in the unit ball of $\mathcal{H}$. Furthermore, since $(u_\lambda)_\lambda$ is increasing, so is $(\varphi(u_\lambda))_\lambda$ in $[0,1]$. Given $0 < \varepsilon < 1$, choose $\lambda_0$ so that $\lambda \ge \lambda_0$ implies that $0 \le \|\varphi\| - \varphi(u_\lambda) < \varepsilon/2$. If $\beta \ge \alpha \ge \lambda_0$, then

$$\begin{aligned} \|(u_\beta + \mathbb{L}) - (u_\alpha + \mathbb{L})\|^2 &= [(u_\beta - u_\alpha), (u_\beta - u_\alpha)] \\ &= \varphi((u_\beta - u_\alpha)^2) \\ &\le \varphi(u_\beta - u_\alpha) \\ &< |\|\varphi\| - \varphi(u_\beta)| + |\|\varphi\| - \varphi(u_\alpha)| \\ &< \varepsilon. \end{aligned}$$

Thus $(u_\lambda)_\lambda$ is Cauchy in the complete space $\mathcal{H}$, and therefore it converges to some vector $\nu$ in the unit ball of $\mathcal{H}$. Also, $\|\nu\|^2 = [u_\lambda + \mathbb{L}, u_\lambda + \mathbb{L}] = \varphi(u_\lambda^2) = \|\varphi\|$, since $(u_\lambda^2)_\lambda$ is also an approximate identity for $\mathbb{A}$.

For any $a \in \mathbb{A}$,

$$\begin{aligned} \rho(a)\nu &= \lim_\lambda \rho(a)(u_\lambda + \mathbb{L}) \\ &= \lim_\lambda a u_\lambda + \mathbb{L} \\ &= a + \mathbb{L}. \end{aligned}$$

Thus $\overline{\rho(\mathbb{A})\nu} = \overline{\mathbb{A}/\mathbb{L}} = \mathcal{H}$, and therefore $\nu$ is indeed a cyclic vector for $(\mathcal{H}, \rho)$.

Finally,

$$\begin{aligned} \langle \rho(a)\nu, \nu\rangle &= \langle a + \mathbb{L}, \nu\rangle \\ &= \lim_\lambda \langle a + \mathbb{L}, u_\lambda + \mathbb{L}\rangle \\ &= \lim_\lambda [a, u_\lambda] \\ &= \lim_\lambda \varphi(u_\lambda^* a) \\ &= \varphi(a) \end{aligned}$$

for all $a \in \mathbb{A}$, completing the proof.

□

**5.6.** Let $(\mathcal{H}_\lambda, \rho_\lambda)_\lambda$ be a family of representations of a fixed $C^*$-algebra $\mathbb{A}$. Let $\mathcal{H} = \oplus_\lambda \mathcal{H}_\lambda$ denote the Hilbert space direct sum of the family $(\mathcal{H}_\lambda)_\lambda$, and for $a \in \mathbb{A}$, define

$$\begin{array}{rccc} \rho: & \mathbb{A} & \to & \mathcal{B}(\mathcal{H}) \\ & a & \mapsto & \oplus_\lambda \rho_\lambda(a). \end{array}$$

Since each $\rho_\lambda$ is a representation, $\|\rho_\lambda\| \le 1$, and thus $\|\rho\| \le 1$. It is now routine to verify that $(\mathcal{H}, \rho)$ is a representation of $\mathbb{A}$, call the **direct sum** of $(\mathcal{H}_\lambda, \rho_\lambda)_\lambda$ and denoted by

$$(\mathcal{H}, \rho) = \oplus_\lambda (\mathcal{H}_\lambda, \rho_\lambda).$$

Clearly $\|\rho(a)\| = \sup_\lambda \|\rho_\lambda(a)\|$ for all $a \in \mathbb{A}$.

In particular, for each $\tau \in \mathcal{S}(\mathbb{A})$, the state space of $\mathbb{A}$, we have constructed a cyclic representation $(\mathcal{H}_\tau, \rho_\tau, \nu_\tau)$ via the GNS Construction (Theorem 5.5).

**5.7. Definition.** *The **universal representation** of a $C^*$-algebra $\mathbb{A}$ is the direct sum representation*

$$(\mathcal{H}, \rho) = \oplus\{(\mathcal{H}_\tau, \rho_\tau, \nu_\tau) : \tau \in \mathcal{S}(\mathbb{A})\}.$$

**5.8. Theorem.** [ ***Gelfand-Naimark*** ] *Let $\mathbb{A}$ be a $C^*$-algebra. The universal representation $(\mathcal{H}, \rho)$ is a faithful representation of $\mathbb{A}$, and hence $\mathbb{A}$ is isometrically $^*$-isomorphic to a $C^*$-algebra of operators on $\mathcal{H}$.*

**Proof.** Let $a \in \mathbb{A}$. Then $n = a^*a \ge 0$, and so by Proposition 4.21, there exists a state $\tau \in \mathcal{S}(\mathbb{A})$ with $|\tau(n)| = \|n\|$. Let $(\mathcal{H}_\tau, \rho_\tau, \nu_\tau)$ be the corresponding cyclic representation and observe that $\|\nu\| = \|\tau\|^{\frac{1}{2}} = 1$.

Now $\|n\| = |\tau(n)| = |\langle \rho_\tau(n)\nu_\tau, \nu_\tau\rangle| \le \|\rho_\tau(n)\| \le \|n\|$, and so $\|n\| = \rho_\tau(n)\|$.

It follows that

$$\begin{aligned} \|\rho(a)\|^2 &= \|\rho(a)^*\rho(a)\| = \|\rho(a^*a)\| \\ &= \|\rho(n)\| \ge \|\rho_\tau(n)\| \\ &= \|n\| = \|a^*a\| \\ &= \|a\|^2. \end{aligned}$$

Thus $\|a\| \le \|\rho(a)\|$. Since $\|\rho\| \le 1$, $\|\rho(a)\| \le \|a\|$, and thus $\rho$ is isometric.

□

*Time waits for no man. No man is an island. So... time waits for an island... I don't get it.*

CHAPTER 6

# Von Neumann algebras and the Spectral Theorem

*There are only two truly infinite things, the universe and stupidity. And I am unsure about the universe.*

**Albert Einstein**

## 1. Von Neumann algebras

**1.1.** In this Chapter, we study a class of concrete $C^*$-algebras which are closed in a second, weaker topology than the norm topology. These are the so-called *von Neumann algebras.* While various important and deep structure theorems for these algebras (based upon the projections which can be found in the algebra) exist, we shall restrict ourselves to that part of the theory necessary for us to prove the celebrated *Spectral Theorem for normal operators.*

**1.2. Definition.** *Let $\mathcal{H}$ be a Hilbert space. The* ***weak operator topology*** *- abbreviated $WOT$ - on $\mathcal{B}(\mathcal{H})$ is the weak topology generated by the functions*

$$\begin{array}{rccc} \phi_{x,y}: & \mathcal{B}(\mathcal{H}) & \to & \mathbb{C} \\ & T & \mapsto & \langle Tx, y\rangle \end{array}$$

*for all $x, y \in \mathcal{H}$. Equivalently, the weak operator topology is the locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by the family $\{T \mapsto |\langle Tx, y\rangle| : x, y \in \mathcal{H}\}$ of seminorms.*

*Thus a net $(T_\alpha)_\alpha$ converges in the $WOT$ and we write $WOT\text{-}\lim_\alpha T_\alpha = T$ if*

$$\lim_\alpha \langle T_\alpha x, y\rangle = \langle Tx, y\rangle$$

*for all $x, y \in \mathcal{H}$.*

*The family $\{\{A \in \mathcal{B}(\mathcal{H}) : |\langle Ax_k, y_k\rangle - \langle Tx_k, y_k\rangle| < \varepsilon\} : \ x_k, y_k \in \mathcal{H}, 1 \le k \le n, n \ge 1, \varepsilon > 0\}$ forms a neighbourhood base for the $WOT$.*

*The* ***strong operator topology*** *- abbreviated $SOT$ - on $\mathcal{B}(\mathcal{H})$ is the weak topology generated by the functions*

$$\begin{array}{rccc} \psi x: & \mathcal{B}(\mathcal{H}) & \to & \mathbb{C} \\ & T & \mapsto & Tx \end{array}$$

*for all $x \in \mathcal{H}$. Equivalently, the strong operator topology is the locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by the family $\{T \mapsto \|Tx\| : x \in \mathcal{H}\}$ of seminorms.*

*Thus a net $(T_\alpha)_\alpha$ converges in the SOT and we write $SOT\text{-}\lim_\alpha T_\alpha = T$ if*

$$\lim_\alpha T_\alpha x = Tx$$

*for all $x \in \mathcal{H}$.*

*The family $\{\{A \in \mathcal{B}(\mathcal{H}) : \|Ax_k - Tx_k\| < \varepsilon\} : \ x_k \in \mathcal{H}, 1 \le k \le n, \ n \ge 1, \varepsilon > 0\}$ forms a neighbourhood base for the SOT.*

It follows easily from these definitions that the $WOT$ is weaker than the $SOT$, while the $SOT$ is weaker than the norm topology.

**1.3. Example.** Let $\mathcal{H} = \mathbb{C}^n$ for some $n \ge 1$. We leave it as an exercise for the reader to verify that the $WOT$, $SOT$ and norm topologies on $\mathcal{B}(\mathcal{H})$ all coincide.

**1.4. Example.** Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^\infty$. Let $P_n$ denote the orthogonal projection onto the span of $\{e_1, e_2, ..., e_n\}$, $n \ge 1$. Then the sequence $\{P_n\}_{n=1}^\infty$ converges to the identity in the $SOT$.

Indeed, if $x \in \mathcal{H}$, say $x = \sum_{k=1}^\infty x_k e_k$, then $\|x - P_n x\| = \|\sum_{k=n+1}^\infty x_k e_k\| = (\sum_{k=n+1}^\infty |x_k|^2)^{\frac{1}{2}}$ and this tends to 0 as $n$ tends to infinity.

**1.5. Remark.** In infinite dimensional Hilbert spaces, the $SOT$, $WOT$ and norm topologies are all distinct. For example, if $\mathcal{H}$ is separable and infinite dimensional with orthonormal basis $\{e_n\}_{n=1}^\infty$, and if $F_n = e_1 \otimes e_n^*$, then it is easy to verify that $SOT\text{-}\lim_n F_n = 0$ but $\|F_n\| = 1$ for all $n \ge 1$, while if $G_n = e_n \otimes e_1^*$, then $WOT\text{-}\lim_n G_n = 0$, while $\|G_n e_1\| = 1$ for all $n \ge 1$, so that $SOT\text{-}\lim_n G_n \ne 0$.

These examples can easily be adapted to non-separable spaces.

**1.6. Proposition.** *Let $\mathcal{H}$ be a Hilbert space and let $A, B \in \mathcal{B}(\mathcal{H})$ be fixed. Then each of the functions*

(i) $\begin{array}{rccc} \sigma: & \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) & \to & \mathcal{B}(\mathcal{H}) \\ & (X, Y) & \mapsto & X + Y \end{array}$;

(ii) $\begin{array}{rccc} \mu: & \mathbb{C} \times \mathcal{B}(\mathcal{H}) & \to & \mathcal{B}(\mathcal{H}) \\ & (z, X) & \mapsto & zX \end{array}$;

(iii) $\begin{array}{rccc} \lambda_A: & \mathcal{B}(\mathcal{H}) & \to & \mathcal{B}(\mathcal{H}) \\ & X & \mapsto & AX \end{array}$;

(iv) $\begin{array}{rccc} \rho_B: & \mathcal{B}(\mathcal{H}) & \to & \mathcal{B}(\mathcal{H}) \\ & X & \mapsto & XB \end{array}$;

(v) $\begin{array}{rccc} \alpha: & \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) & \to & \mathcal{B}(\mathcal{H}) \\ & T & \mapsto & T^* \end{array}$.

*is continuous in the WOT. The first four are also SOT-continuous, while the adjoint operation $\alpha$ is not SOT continuous.*

**Proof.** Exercise.

□

**1.7. Definition.** *Let $\mathcal{H}$ be a Hilbert space. Then a* ***von Neumann algebra*** *(also called a $W^*$-algebra) $\mathfrak{M}$ is a $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed in the WOT.*

We remark that some authors require that the algebra $\mathfrak{M}$ contain the identity operator. As we shall see, every von Neumann algebra contains a maximal projection which serves as an identity for the algebra as a ring. By restricting our attention to the range of that projection, we can then assume that the identity operator lies in $\mathfrak{M}$.

**1.8. Example.** If $\mathcal{H}$ is a Hilbert space, then $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra.

**1.9. Proposition.** *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint subalgebra. Then $\overline{\mathcal{A}}^{\text{WOT}}$ is a von Neumann algebra. If $\mathcal{A}$ is abelian, then so is $\overline{\mathcal{A}}^{\text{WOT}}$.*

**Proof.** Suppose $(A_\alpha)_{\alpha\in\Lambda}$ and $(B_\beta)_{\beta\in\Gamma}$ are nets in $\mathcal{A}$ with WOT-$\lim_\alpha A_\alpha = A$ and WOT-$\lim_\beta B_\beta = B$. Now $\Lambda \times \Gamma$ is a directed set with the lexicographic order, so that $(\alpha_1, \beta_1) \le (\alpha_2, \beta_2)$ if $\alpha_1 < \alpha_2$, or $\alpha_1 = \alpha_2$ and $\beta_1 \le \beta_2$. If we set $A_{\alpha,\beta} = A_\alpha$, $B_{\alpha,\beta} = B_\beta$ for all $\alpha, \beta$, then $\lim_{\alpha,\beta} A_{\alpha,\beta} = A$ and $\lim_{\alpha,\beta} B_{\alpha,\beta} = B$. By Proposition 1.6, for all $z \in \mathbb{C}$, $zA + B = \text{WOT-}\lim_{\alpha,\beta} zA_{\alpha,\beta} + B_{\alpha,\beta} \in \overline{\mathcal{A}}^{\text{WOT}}$.

Next, for each $\beta \in \Gamma$, $AB_\beta = \text{WOT-}lim_\alpha A_\alpha B_\beta \overline{\mathcal{A}}^{\text{WOT}}$, and thus

$$\text{WOT-}\lim_\beta AB_\beta = AB \in \overline{\mathcal{A}}^{\text{WOT}}.$$

Thus $\overline{\mathcal{A}}^{\text{WOT}}$ is an algebra. Since the adjoint operation is continuous in the WOT, and since $\mathcal{A}$ is self-adjoint, $A_\alpha \mapsto_{\text{WOT}} A$ implies $A_\alpha^* \mapsto_{\text{WOT}} A^*$, and so $A^* \in \overline{\mathcal{A}}^{\text{WOT}}$. Hence $\overline{\mathcal{A}}^{\text{WOT}}$ is a von Neumann algebra.

Suppose $\mathcal{A}$ is abelian. For all $\beta \in \Gamma$ and $x, y \in \mathcal{H}$,

$$\begin{aligned} \langle AB_\beta x, y\rangle &= \text{WOT-}\lim_\alpha \langle A_\alpha B_\beta x, y\rangle \\ &= \text{WOT-}\lim_\alpha \langle B_\beta A_\alpha x, y\rangle \\ &= \text{WOT-}\lim_\alpha \langle A_\alpha x, B_\beta^* y\rangle \\ &= \langle Ax, B_\beta^* y\rangle = \langle B_\beta A x, y\rangle. \end{aligned}$$

Thus $AB_\beta = B_\beta A$ for all $\beta \in \Gamma$. The same argument then shows that $AB = BA$, and so $\overline{\mathcal{A}}^{\text{WOT}}$ is abelian.

□

**1.10. Definition.** *If $\mathcal{C} \subseteq \mathcal{B}(\mathcal{H})$ is any collection of operators, then*

$$\mathcal{C}' = \{T \in \mathcal{B}(\mathcal{H}) : TC = CT \text{ for all } C \in \mathcal{C}\}$$

*is called the* ***commutant*** *of $\mathcal{C}$.*

**1.11. Proposition.** *Let $\mathcal{H}$ be a Hilbert space. Then $\mathcal{K}(\mathcal{H})' = \mathbb{C}I$.*
**Proof.** Exercise.

□

**1.12. Proposition.** *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{C} \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint collection of operators. Then the commutant $\mathcal{C}'$ of $\mathcal{C}$ is a von Neumann algebra.*
**Proof.** Suppose $A, B \in \mathcal{C}'$, $z \in \mathbb{C}$ and $C \in \mathcal{C}$. Then $(zA + B)C = zAC + BC = zCA + CB = C(zA + B)$ and $(AB)C = A(BC) = A(CB) = (AC)B = (CA)B = C(AB)$, so that $\mathcal{C}'$ is an algebra. Also, $\mathcal{C}$ self-adjoint implies that $AC^* = C^*A$ and hence $CA^* = A^*C$ for all $C \in \mathcal{C}$. Thus $\mathcal{C}'$ is self-adjoint.

Finally, if $A_\alpha \in \mathcal{C}'$, $\alpha \in \Lambda$ and WOT-$\lim_\alpha A_\alpha = A$, then for all $x, y \in \mathcal{H}$,

$$\begin{aligned} \langle ACx, y\rangle &= \lim_\alpha \langle CA_\alpha x, y\rangle = \lim_\alpha \langle A_\alpha x, C^* y\rangle \\ &= \langle Ax, C^* y\rangle = \langle CAx, y\rangle, \end{aligned}$$

so that $A \in \mathcal{C}'$ and therefore $\mathcal{C}'$ is WOT-closed, which completes the proof.

□

**1.13. Definition.** *A* ***masa*** *$\mathbb{M}$ in a $C^*$-algebra $\mathbb{A}$ is a maximal abelian self-adjoint subalgebra. That is, $\mathbb{M}$ is a self-adjoint abelian subalgebra of $\mathbb{A}$, and is not properly contained in any abelian self-adjoint subalgebra of $\mathbb{A}$.*

**1.14. Example.** Let $\mathbb{A} = \mathbb{M}_n(\mathbb{C})$ for some $n \geq 1$. Then $\mathcal{D}_n = \{\mathrm{diag}(d_1, d_2, ..., d_n) : d_k \in \mathbb{C},\ 1 \leq k \leq n\}$ is a masa in $\mathbb{A}$. We leave the verification as an exercise, although this example will be covered by Proposition 1.16 below.

**1.15. Proposition.** *Let $\mathcal{H}$ be a Hilbert space and $\mathfrak{M} \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint algebra of operators. The following are equivalent:*

(a) $\mathfrak{M} = \mathfrak{M}'$;
(b) $\mathfrak{M}$ *is a masa.*

*In particular, every masa in $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra.*
**Proof.**

(a) implies (b): Since $\mathfrak{M} = \mathfrak{M}'$, $\mathfrak{M}$ is abelian. Suppose $\mathfrak{M} \subseteq \mathfrak{N}$, where $\mathfrak{N}$ is abelian and self-adjoint. Then $\mathfrak{N} \subseteq \mathfrak{M}'$, and so $\mathfrak{N} \subseteq \mathfrak{M}$. Thus $\mathfrak{M}$ is a masa.

(b) implies (a): Suppose that $\mathfrak{M}$ is a masa. Let $T \in \mathfrak{M}'$, $T = H + iK$, where $H = (T + T^*)/2$ and $K = (T - T^*)/2i$. If $M \in \mathfrak{M}$, then $M^* \in \mathfrak{M}$, so that $TM^* = M^*T$ and thus $T^*M = MT^*$ and $T^* \in \mathfrak{M}'$. But then $H, K \in \mathfrak{M}'$.

Now if $\mathfrak{N}$ is the WOT-closed algebra generated by $\mathfrak{M}$ and $H$, then $\mathfrak{N}$ is abelian and so $\mathfrak{N} = \mathfrak{M}$ by maximality. Thus $H \in \mathfrak{M}$. Similarly, $K \in \mathfrak{M}$ and therefore $T \in \mathfrak{M}$. That is, $\mathfrak{M}' \subseteq \mathfrak{M}$. Since $\mathfrak{M}$ is abelian, $\mathfrak{M} \subseteq \mathfrak{M}'$, from which equality follows.

□

Recall that a measure space $(X, \mu)$ is called a **probability space** if $\mu$ is a positive regular Borel measure on $X$ for which $\mu(X) = 1$. Recall that the map $f \mapsto M_f$ is an isometric embedding of $L^\infty(X, \mu)$ into $\mathcal{B}(L^2(X, \mu))$. Let us use $\mathcal{M}^\infty(X, \mu)$ to denote the image of $L^\infty(X, \mu)$ under this embedding.

**1.16. Proposition.** *Let $(X, \mu)$ be a probability space. Then $\mathcal{M}^\infty(X, \mu)$ is a masa in $\mathcal{B}(L^2(X, \mu))$, and as such is a von Neumann algebra.*

**Proof.** Since $\mathcal{M}^\infty(X, \mu)$ is self-adjoint, by Proposition 1.15, it suffices to show that $\mathcal{M}^\infty(X, \mu) = \mathcal{M}^\infty(X, \mu)'$. Observe that $\mathcal{M}^\infty(X, \mu)$ is abelian, and so $\mathcal{M}^\infty(X, \mu) \subseteq \mathcal{M}^\infty(X, \mu)'$.

Suppose $T \in \mathcal{B}(\mathcal{H})$ satisfies $TM_f = M_f T$ for all $f \in L^\infty(X, \mu)$. Let $e \in L^2(X, \mu)$ denote the constant function $e(x) = 1 a.e.$, and set $g = Te$.

Then $Tf = TM_f e = M_f Te = fg$ for all $f \in L^\infty(X, \mu)$. If we can show that $g \in L^\infty(X, \mu)$, then it will follow from the continuity of $T$ and the fact that $L^\infty(X, \mu)$ is dense in $L^2(X, \mu)$ that $T = M_g$.

Let $E = \{x \in X : |g(x)| \geq \|T\| + 1\}$, and let $f = \chi_E \in L^\infty(X, \mu)$. Then

$$\begin{aligned} \|Tf\|^2 &= \int_X |fg|^2 d\mu \\ &= \int_E |fg|^2 d\mu \\ &> \|T\|^2 \int_E |f|^2 d\mu \\ &= \|T\|^2 \; \|f\|_2^2, \end{aligned}$$

and so $\|f\|_2^2 = 0$, implying that $f = 0$ a.e.. Thus $|g(x)| \leq \|T\| + 1$ a.e., and hence $g \in L^\infty(X, \mu)$. From the argument above, $T = M_g \in \mathcal{M}^\infty(X, \mu)$, and hence $\mathcal{M}^\infty(X, \mu)' \subseteq \mathcal{M}^\infty(X, \mu)$.

□

**1.17. Lemma.** *Suppose $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a self-adjoint algebra and $x \in \mathcal{H}$. Let $P$ denote the orthogonal projection onto $[\mathcal{A}x]$, the closure of $\mathcal{A}x$ in $\mathcal{H}$. Then $P \in \mathcal{A}'$.*

**Proof.** We prove that $[\mathcal{A}x]$ is reducing for each element $A$ of $\mathcal{A}$. Indeed, if $z \in [\mathcal{A}x]$, then $z = \lim_{n\to\infty} A_n x$ for some sequence $\{A_n\}_n$ in $\mathcal{A}$. But then

$Az = \lim_{n\to\infty} AA_n x \in [\mathcal{A}x]$, and $A^*z = \lim_{n\to\infty} A^*A_n x \in [\mathcal{A}x]$, so that $[\mathcal{A}x]$ is reducing for $A$ by Proposition 3.**??**.

Thus $AP = PAP$ and $A^*P = PA^*P$, from which $PA = AP$, and $P \in \mathcal{A}'$, as claimed.

□

**1.18. Definition.** *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{C} \subseteq \mathcal{B}(\mathcal{H})$. The* ***kernel*** *of $\mathcal{C}$ is the set*

$$\ker \mathcal{C} = \{x \in \mathcal{H} : Cx = 0 \textit{ for all } C \in \mathcal{C}\}.$$

**1.19. Example.**

(a) We leave it as an exercise for the reader to verify that $\ker \mathcal{K}(\mathcal{H}) = \{0\}$.

(b) If $T \in \mathcal{B}(\mathcal{H})$, and $\mathcal{C}$ is the algebra generated by $T$, then $\ker \mathcal{C} = \ker T$.

**1.20. Lemma.** *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$. Set $\mathcal{A}^{(n)} = \{A \oplus A \oplus ... \oplus A : A \in \mathcal{A}\} \subseteq \mathcal{B}(\mathcal{H}^{(n)})$. Then $(\mathcal{A}^{(n)})'' = \{B \oplus B \oplus ... \oplus B : B \in \mathcal{A}''\}$.*

**Proof.** Exercise.

□

**1.21. Theorem.** [***The von Neumann Double Commutant Theorem*** ]. *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint algebra of operators and suppose that $\ker \mathcal{A} = \{0\}$. Then $\overline{\mathcal{A}}^{\text{WOT}} = \overline{\mathcal{A}}^{\text{SOT}} = \mathcal{A}''$. In particular, if $\mathcal{A}$ is a von Neumann algebra, then $\mathcal{A} = \mathcal{A}''$.*

**Remark:** Before proving the result, let us pause to observe what a truly remarkable Theorem this is. Indeed, the conclusion of this Theorem allows us to identify a topological concept, namely the closure of a given algebra in a certain topology, with a purely algebraic concept, the second commutant of the algebra. It is difficult to overstate the usefulness of this Theorem.

**Proof.** Observe that $\mathcal{A} \subseteq \mathcal{A}''$ and that this latter is a von Neumann algebra by Proposition 1.9. Thus $\overline{\mathcal{A}}^{\text{WOT}} \subseteq \mathcal{A}''$. Since the strong operator topology is stronger than the weak operator topology,

$$\overline{\mathcal{A}}^{\text{SOT}} \subseteq \overline{\mathcal{A}}^{\text{WOT}} \subseteq \mathcal{A}''.$$

It therefore suffices to prove that if $B \in \mathcal{A}''$, then $B \in \overline{\mathcal{A}}^{\text{SOT}}$. This amounts to proving that if $\varepsilon > 0$ and $x_1, x_2, ..., x_n \in \mathcal{H}$, then there exists $A \in \mathcal{A}$ so that $\|(A - B)x_k\| < \varepsilon$, $1 \le k \le n$.

Let $\varepsilon > 0$.

(1) Case One: $n = 1$ Let $x \in \mathcal{H}$. Then by Lemma 1.17, if $P$ is the orthogonal projection onto $[\mathcal{A}x]$, $P \in \mathcal{A}'$. Moreover, $x \in \operatorname{ran} P$, for if $C \in \mathcal{A}$, then $C(I-P)x = (I-P)x = 0$, and hence $(I-P)x \in \ker \mathcal{A} = \{0\}$. Since $P \in \mathcal{A}'$, we have $PB = BP$, and so $Bx = BPx = PBx \in \operatorname{ran} P$. That is, there exists $A \in \mathcal{A}$ so that $\|Bx - Ax\| < \varepsilon$.

(2) Case Two: $n > 1$ Let $x_1, x_2, ..., x_n \in \mathcal{H}$ and set $z = x_1 \oplus x_2 \oplus \cdots \oplus x_n \in \mathcal{H}^{(n)}$. By Lemma **??**, $\mathcal{A}^{(n)}$ is a self-adjoint algebra of operators and it is routine to check that $\ker \mathcal{A}^{(n)} = \{0\}$. By Case One above, we can find $A_0 \in (\mathcal{A}^{(n)})''$ so that

$$\|(A_0 - B^{(n)})z\| < \varepsilon.$$

Since $(\mathcal{A}^{(n)})'' = (\mathcal{A}'')^{(n)}$, $A_0 = A^{(n)}$ for some $A \in \mathcal{A}$, and so we have $(\sum_{k=1}^{n} \|(A-B)x_k\|^2)^{\frac{1}{2}} < \varepsilon$, which in turn implies that $\|(A-B)x_k\| < \varepsilon$ for all $1 \le k \le n$.

$\square$

**1.22. Proposition.** *Let $\mathcal{H}$ be a Hilbert space and suppose $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ is a linear map. The following are equivalent:*

(a) *$\varphi$ is SOT-continuous;*
(b) *$\varphi$ is WOT-continuous;*
(c) *there exist $\{x_k\}_{k=1}^n$, $\{y_k\}_{k=1}^n \in \mathcal{H}$ so that $\varphi(T) = \sum_{k=1}^n \langle Tx_k, y_k\rangle$ for all $T \in \mathcal{B}(\mathcal{H})$.*

**Proof.**

(c) implies (b): this is clear from the definition of the WOT.
(b) implies (a): this follows from the fact that the WOT is weaker than the SOT.
(a) implies (c): Let $\varepsilon > 0$. From the definition of a basic neighbourhood in the SOT, we can find vectors $x_1, x_2, ..., x_n \in \mathcal{H}$ such that $(\sum_{k=1}^n \|Tx_k - 0x_k\|^2)^{\frac{1}{2}} < \varepsilon$ implies $|\varphi(T) - \varphi(0)| < 1$. Consider $\begin{array}{rccc} \Psi : & \mathcal{B}(\mathcal{H}) & \to & \mathcal{H}^{(n)} \\ & T & \mapsto & (Tx_1, Tx_2, ..., Tx_n). \end{array}$ Then $\Psi$ is linear and so $R = \operatorname{ran} T$ is a linear manifold. Consider

$$\begin{array}{rccc} \beta_R : & R & \to & \mathbb{C} \\ & (Tx_1, Tx_2, ..., Tx_n) & \mapsto & \varphi(T). \end{array}$$

Then from above it follows that $\beta_R$ is well-defined, is continuous, and in fact $\|\beta_R\| \le 1/\varepsilon$. By the Hahn-Banach Theorem, $\beta_R$ extends to a continuous linear functional $\beta \in (\mathcal{H}^{(n)})^* \simeq \mathcal{H}^{(n)}$. By the Riesz Representation Theorem, $\beta(Z) = \langle Zx, x\rangle$ for some $y \in \mathcal{H}^{(n)}$, say

$y = (y_1, y_2, ..., y_n)$. In particular,

$$\begin{aligned}\varphi(T) &= \beta(Tx_1, Tx_2, ..., Tx_n) \\ &= \langle (Tx_1, Tx_2, ..., Tx_n), (y_1, y_2, ..., y_n)\rangle \\ &= \sum_{k=1}^{n} \langle Tx_k, y_k\rangle\end{aligned}$$

for all $T \in \mathcal{B}(\mathcal{H})$.

□

**1.23. Remark.** Suppose $\mathcal{H}$ is a separable, complex Hilbert space, $T \in \mathcal{B}(\mathcal{H})$ and $F \in \mathcal{F}(\mathcal{H})$ is a finite rank operator. Let $\{e_\alpha\}_\alpha$ be an orthonormal basis for $\mathcal{H}$. It can be shown that we can then define $\mathrm{tr}(TF) = \sum_{i=1}^{\infty} k_{\alpha\alpha}$, where $[TF] = [k_{\alpha,\beta}]$ with respect to the given basis. If $F = \sum_{i=1}^{n} y_{\alpha_i} \otimes x_{\alpha_i}^*$, then

$$\mathrm{tr}(TF) = \sum_{i=1}^{n} \langle Tx_{\alpha_i}, y_{\alpha_i}\rangle.$$

Thus the WOT-continuous (or SOT-continuous) linear functionals are those induced by $\varphi_F$, $F \in \mathcal{F}(\mathcal{H})$, where $\varphi_F(T) = \mathrm{tr}(TF)$.

**1.24. Corollary.** *$(\mathcal{B}(\mathcal{H}), SOT)$ and $(\mathcal{B}(\mathcal{H}), WOT)$ have the same closed, convex sets.*

**Proof.** By the Krein-Milman Theorem, the SOT-closed convex subsets are completely determined by the SOT-closed half-spaces which contain them. These in turn are determined by the SOT-continuous linear functionals on $\mathcal{B}(\mathcal{H})$. Since the SOT- and WOT-continuous linear functionals on $\mathcal{B}(\mathcal{H})$ coincide, every SOT-closed convex set is also WOT-closed.

Conversely, any WOT-closed set is automatically SOT-closed, and in particular, this applies to convex sets.

□

**1.25. Proposition.** *Let $\mathbb{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then the unit ball $\mathbb{M}_1$ of $\mathbb{M}$ is WOT-compact.*

**Proof.** First note that $\mathbb{M}_1$ is WOT-closed, since $(T_\alpha)_\alpha \subseteq \mathbb{M}_1$ and $T_\alpha \to T$ in the WOT implies that $T \in \overline{\mathbb{M}}^{\mathrm{WOT}}$ and $|\langle Tx, y\rangle = \lim_\alpha |\langle T_\alpha x, y\rangle| \leq \sup_\alpha |\|T_\alpha\| \, \|x\| \, \|y\|$ for all $x, y \in \mathcal{H}$. The remainder of the proof is similar to that of the Banach-Alaoglu Theorem.

For each $x, y \in \mathcal{H}$, consider $I_{x,y} = [-\|x\| \, \|y\|, \|x\| \, \|y\|]$. Let $B = \Pi_{x,y\in\mathcal{H}} I_{x,y}$, and suppose that $B$ carries the product topology so that $B$ is compact (since each $I_{x,y}$ clearly is). Now the map

$$\begin{array}{rccc} j: & \mathbb{M}_1 & \to & B \\ & T & \mapsto & \Pi_{x,y\in\mathcal{H}} \langle Tx, y\rangle \end{array}$$

is clearly an injective map from $\mathbb{M}_1$ into $B$. We clear that $j$ is a homeomorphism of $(\mathbb{M}_1, WOT)$ with its range.

Indeed, $T_\alpha \to_{\text{WOT}} T$ if and only if $\langle T_\alpha x, y\rangle \to \langle Tx, y\rangle$ for each $x, y \in \mathcal{H}$ if and only if $j(T_\alpha) \to j(T)$ in the product topology on $B$.

Moreover, $j(\mathbb{M}_1)$ is closed in $B$. To see this, suppose $(j(T_\alpha))_\alpha \subseteq j(\mathbb{M}_1)$. If $j(T_\alpha) \to (z_{x,y})_{x,y\in\mathcal{H}}$, then for each $y_0 \in \mathcal{H}$,

$$\phi_{y_0}(x) := z_{x,y_0}$$

defines a continuous linear functional on $\mathcal{H}$. By the Riesz Representation Theorem, there exists a vector $T^* y_0 \in \mathcal{H}$ so that $\phi_{y_0}(x) = \langle x, T^* y_0\rangle$. It is not difficult to verify that the function $y_0 \mapsto T^* y_0$ is linear. Moreover, $|z_{x,y}| \le \|x\| \, \|y\|$ for all $x, y \in \mathcal{H}$ and hence

$$\begin{aligned} \|T^* y_0\| &= \sup_{\|x\|=1} |\langle x, T^* y_0\rangle| \\ &= \sup_{\|x\|=1} |z_{x,y_0}| \\ &\le \|x\| \, \|y_0\| = \|y_0\|. \end{aligned}$$

Hence $\|T\| = \|T^*\| \le 1$.

Clearly $\langle T_\alpha x, y\rangle \mapsto z_{x,y} = \langle x, T^* y\rangle = \langle Tx, y\rangle$ for all $x, y \in \mathcal{H}$, and so $(z_{x,y})_{x,y\in\mathcal{H}} = \Pi_{x,y} = \Pi_{x,y}\langle Tx, y\rangle = j(T) \in \operatorname{ran} j$. Thus $\operatorname{ran} j$ is closed in the compact set $B$ and hence $\operatorname{ran} j$ is compact. But then $(\mathbb{M}_1, \text{WOT})$ is also compact, which is what we were trying to prove.

□

**1.26. Proposition.** *Let $(P_\beta)_{\beta\in\Gamma}$ be an increasing net of positive elements in the unit ball $\mathbb{M}_1$ of a unital von Neumann algebra $\mathbb{M}$. Then $P = \text{SOT} - \lim_\beta P_\beta$ exists, $P \in \mathbb{M}_1$ and $0 \le P \le I$.*

**Proof.** Fix $x \in \mathcal{H}$. Then $\langle P_\beta x, x\rangle_\beta$ is an increasing net of positive real numbers in $[0, 1]$ and hence $m_x := \lim_\beta \langle P_\beta x, x\rangle$ exists. Let $\varepsilon > 0$ and choose $\beta_0$ such that $\beta \ge \beta_0$ implies $|m_x - \langle P_\beta x, x\rangle| < \varepsilon$.

Since $(P_\beta)_\beta$ is increasing, if $\beta \ge \alpha$, then $P_\beta - P_\alpha \ge 0$, and so $(P_\beta - P_\alpha)^{\frac{1}{2}} \in \mathbb{M}$. Moreover, $0 \le P_\alpha \le P_\beta \le I$ implies $P_\beta - P_\alpha \le I - 0 = I$, and hence $(P_\beta - P_\alpha)^{\frac{1}{2}} \le I$. If $\beta \ge \alpha \ge \beta_0$, then

$$\begin{aligned} \|(P_\beta - P_\alpha)x\|^2 &\le \|(P_\beta - P_\alpha)^{\frac{1}{2}}\|^2 \; \|(P_\beta - P_\alpha)^{\frac{1}{2}} x\|^2 \\ &= \|P_\beta - P_\alpha\| \; \langle (P_\beta - P_\alpha)^{\frac{1}{2}} x, (P_\beta - P_\alpha)^{\frac{1}{2}} x\rangle \\ &\le \langle (P_\beta - P_\alpha)x, x\rangle < \varepsilon. \end{aligned}$$

Hence $(P_\beta x)_\beta$ is Cauchy. Since $\mathcal{H}$ is complete, $Px := \lim_\beta P_\beta x$ exists for all $x \in \mathcal{H}$. It is not hard to check that $P$ is linear, and $\langle Px, x\rangle = \lim_\beta \langle P_\beta x, x\rangle \ge 0$, so that $P \ge 0$. Since $P_\beta \to P$ in the SOT, we also have $P_\beta \to P$ in the WOT. Since the unit ball $\mathbb{M}_1$ of $\mathbb{M}$ is WOT-compact from above, and since $P_\beta \in \mathbb{M}_1$ for all $\beta$, we get $P \in \mathbb{M}_1$.

□

**1.27. Theorem.** ***Kaplansky's Density Theorem*** *Let $\mathcal{H}$ be a Hilbert space and let $I \in \mathbb{A} \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint algebra of operators. Then*

$$(\overline{\mathbb{A}}^{\mathrm{SOT}})_1^{sa} \subseteq \overline{(\mathbb{A}_1^{sa})}^{\mathrm{SOT}}.$$

**1.28. Lemma.** *Let $\mathbb{A}$ be a $C^*$-algebra and $\varphi \in \mathbb{A}^*$ be a self-adjoint linear functional. If $(\mathcal{H}, \rho, \nu)$ is the universal representation of $\mathbb{A}$, then there exist vectors $x, y \in \mathcal{H}$ so that $\varphi(a) = \langle \rho(a)x, y\rangle$ for all $a \in \mathbb{A}$. Furthermore, $x$ and $y$ can be chosen so that $\|x\|^2, \|y\|^2 \leq \|\varphi\|$.*

**1.29. Theorem.** *Let $\mathbb{A}$ be a $C^*$-algebra and $(\mathcal{H}, \rho, \nu)$ be the universal representation of $\mathbb{A}$. Then $\rho(\mathbb{A})''$ is isometrically isomorphic as a Banach space to $\mathbb{A}^{**}$ via an isomorphism that fixes $\mathbb{A}$.*

**1.30. Polar Decomposition.** Given a complex number $z$, we can write $z$ as a product of a positive number (its modulus) and a complex number of magnitude one. We wish to generalize this to operators on a Hilbert space. Our reason for waiting until this section to prove the result will be made clear from Propositon **??**.

**1.31. Definition.** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $V \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. We say that $V$ is a* ***partial isometry*** *if $\|Vx\| = \|x\|$ for all $x \in (\ker V)^\perp$. If $\ker V = \{0\}$, we say that $V$ is an* ***isometry****.*

*The space $(\ker V)^\perp$ is called the* initial space *of $V$, while* $\mathrm{ran}V$ *is called the* final space *of $V$. Observe that $\mathrm{ran}V$ is automatically closed in $\mathcal{H}_2$.*

**1.32. Example.** Fix $n \in \mathbb{N}$ and let $\mathcal{H} = \mathbb{C}^n$. Then $V$ is an isometry if and only if $V$ is unitary.

**1.33. Example.** Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^\infty$. Consider the unilateral forward shift $Se_n = e_{n+1}$, $n \geq 1$. Then $S$ is an isometry, and $S^*$ is a partial isometry with initial space $\{e_1\}^\perp$.

**1.34. Proposition.** *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}_1, \mathcal{H}_2$ be closed subspaces of $\mathcal{H}$ with $\dim \mathcal{H}_1 = \dim \mathcal{H}_2$. Then there exists a partial isometry $V$ with initial space $\mathcal{H}_1$ and final space $\mathcal{H}_2$.*

**Proof.**

□

**1.35. Proposition.** *Let $\mathcal{H}$ be a Hilbert space and $V \in \mathcal{B}(\mathcal{H})$. The following are equivalent:*

(a) *$V$ is a partial isometry.*
(b) *$V^*$ is a partial isometry.*
(c) *$VV^*$ is a projection - in which case it is the orthogonal projection onto the range of $V$.*
(d) *$V^*V$ is a projection, in which case it is the orthogonal projection onto the initial space of $V$.*

**Proof.**

□

**1.36. Theorem.** [***Polar Decomposition.***] *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. There there exists a positive operator $P$ and a partial isometry $V$ such that $T = VP$. Moreover, $P$ and $V$ are unique if we require that* $\ker P = \ker V = \ker T$.
**Proof.**

□

**1.37. Proposition.** *The partial isometry $V$ appearing in the polar decomposition of the operator $T = PV$ lies in the von Neumann algebra generated by $V$.*

**1.38. Example.** We include the following example which shows that $V$ need not belong to the $C^*$-algebra generated by $V$.

**1.39. Proposition.** *Let $\mathcal{H}$ be a Hilbert space, and $W \in \mathcal{B}(\mathcal{H})$ be an isometry. Then there exist a unitary $U$ and a cardinal number $\alpha$ so that $W \simeq U \oplus S^{(\alpha)}$, where $S$ is the forward unilateral shift operator.*
**Proof.**

□

*The love of honey is the root of all beehives.*

## 2. The spectral theorem for normal operators.

**2.1.** In this section we extend the functional calculus for normal operators on a separable Hilbert space beyond the continuous functional calculus we obtained in Chapter Four via the Gelfand transform. In the present setting, we show that if $\mathcal{H}$ is a separable Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ is normal, then the unital von Neumann algebra $W^*(N)$ generated by $N$ is isometrically $^*$-isomorphic to $L^\infty(\sigma(N), \mu)$, where $\mu$ is a finite, positive, regular Borel measure with support $\sigma(N)$. This identification leads us to an $L^\infty$-functional calculus for normal operators.

**2.2. Proposition.** *Let $(X, \mu)$ be a measure space, where $\mu$ is a finite, positive, regular Borel measure on $X$. Then a net $(f_\alpha)_{\alpha\in\Lambda}$ in $L^\infty(X, \mu)$ converges in the weak$^*$-topology to a function $f$ if and only if $(M_{f_\alpha})_\alpha$ converges in the WOT to $M_f$.*

**Proof.** Suppose $f_\alpha$ converges in the weak$^*$-topology to $f$. Then for all $g \in L^1(X, \mu)$,

$$\lim_\alpha \int_X f_\alpha g d\mu = \int_X f g d\mu.$$

If $h_1, h_2 \in L^2(X, \mu)$, then $h_1\overline{h_2} \in L^1(X, \mu)$ by Hölder's Inequality and so

$$\begin{aligned}
\lim_\alpha \langle M_{f_\alpha} h_1, h_2\rangle &= \lim_\alpha \langle f_\alpha h_1, h_2\rangle \\
&= \lim_\alpha \int_X f_\alpha h_1 \overline{h_2} d\mu \\
&= \int_X f h_1 \overline{h_2} d\mu \\
&= \langle f h_1, h_2\rangle \\
&= \langle M_f h_1, h_2\rangle.
\end{aligned}$$

That is, $(M_{f_\alpha})$ converges in the WOT to $M_f$.

Conversely, if $(M_{f_\alpha})_\alpha$ converges in the WOT to $M_f$, then given $g \in L^1(X, \mu)$, we can find $h_1, h_2 \in L^2(X, \mu)$ so that $g = h_1\overline{h_2}$. Then, as above,

$$\begin{aligned}
\lim_\alpha \int_X f_\alpha h_1 \overline{h_2} d\mu &= \lim_\alpha \langle M_{f_\alpha} h_1, h_2\rangle \\
&= \langle M_f h_1, h_2\rangle \\
&= \int_X f h_1 \overline{h_2} d\mu.
\end{aligned}$$

Thus $(f_\alpha)_\alpha$ converges in the weak$^*$-topology to $f$.

□

**2.3. Lemma.** *Suppose $X$ is a compact, Hausdorff space and that $\mu$ is a positive, regular Borel measure on $X$ with $\mu(X) = 1$. If $X$ can be written as the disjoint union of measurable sets $\{E_j\}_{j=1}^n$, $g \in L^1(X, \mu)$ and $\|g\|_1 = 1$, then for all $\varepsilon > 0$ there exist compact sets $K_1, K_2, ..., K_n$ such that $K_j \subseteq E_j$ and with $K = \cup_{j=1}^n K_j$,*

$$\int_{X \backslash K} |g| d\mu < \varepsilon.$$

**Proof.** For each $1 \leq j \leq n$, let $E_j(m) = \{x \in E_j : m - 1 \leq |g(x)| < m\}, m \geq 1$. Then $E_j(m)$ is measurable for all $m, j$ and

$$1 = \|g\|_1 = \sum_{j=1}^n \sum_{m=1}^\infty \int_{E_j(m)} |g| d\mu.$$

Let $\varepsilon > 0$. Then there exists $N > 0$ so that for each $1 \leq j \leq n$,

$$\sum_{m=N+1}^\infty \int_{E_j(m)} |g| d\mu < \varepsilon/2n.$$

For each $1 \leq j \leq n$, $1 \leq m \leq N$, the regularity of $\mu$ allows us to find a compact set $K_j(m) \subseteq E_j(m)$ so that $\mu(E_j(m) \backslash K_j(m)) < \varepsilon/2N^2n$.

Let $K_j = \cup_{m=1}^N K_j(m)$. Since each $K_m(j)$ is compact, so is $K_j$. It follows that if $K = \cup_{j=1}^n K_j$, then

$$\begin{aligned}
\int_{X \backslash K} |g| d\mu &= \sum_{j=1}^n \sum_{m=N+1}^\infty \int_{E_j(m)} |g| d\mu + \sum_{j=1}^n \sum_{m=1}^N \int_{E_j(m) \backslash K_j(m)} |g| d\mu \\
&\leq \sum_{j=1}^n \varepsilon/2n + \sum_{j=1}^n \sum_{m=1}^N (\varepsilon/2nN^2) \\
&< \varepsilon/2 + \sum_{j=1}^n \sum_{m=1}^N (\varepsilon/2nN^2) N \\
&= \varepsilon/2 + \sum_{j=1}^n \varepsilon/2n = \varepsilon.
\end{aligned}$$

□

**Remark:** If $L$ is compact and $K \subseteq L$, then $\int_{X \backslash L} |g| d\mu \leq \int_{X \backslash K} |g| d\mu < \varepsilon$.

**2.4. Proposition.** *Let $X$ be a compact, Hausdorff set and $\mu$ be a positive, regular Borel measure on $X$ with $\mu(X) = 1$. Then the unit ball $(\mathcal{C}(X))_1$ of $\mathcal{C}(X)$ is weak*-dense in $(L^\infty(X,\mu))_1$, and as such, $\mathcal{C}(X)$ is weak*-dense in $L^\infty(X,\mu)$.*

**Proof.** First observe that the simple functions in $(L^\infty(X,\mu))_1$ are norm dense in $(L^\infty(X,\mu))_1$, and hence they are weak*-dense. As such, it suffices to prove that each simple function can be approximated in the weak*-topology on $L^\infty(X,\mu)$ by continuous functions.

Consider $\varphi(x) = \sum_{j=1}^n a_j \chi_{E_j}$, where $E_j$ is measurable, $1 \leq j \leq n$ and $\cup_{j=1}^n E_j = X$. (We can suppose without loss of generality that the $E_j$'s are also disjoint. Suppose furthermore that $\|\varphi\|_\infty \leq 1$. Let $K_j \subseteq E_j$ be a compact set for all $1 \leq j \leq n$. Then $K = \cup_{j=1}^n K_j$ is compact, and so by Tietze's Extension Theorem we can find a function $f_K \in \mathcal{C}(X)$ so that $f_K(x) = a_j$ if $x \in K_j$ and $0 \leq f_K \leq 1$.

Let $\Lambda = \{K : K = \cup_{j=1}^n K_j, K_j \subseteq E_j \text{compact}\}$, and partially order $\Lambda$ by inclusion, so that $K_1 \leq K_2$ if $K_1 \subseteq K_2$. Then $\Lambda$ is a directed set and $(f_K)_{K\in\Lambda}$ is a net in $\mathcal{C}(X)$. Let $\varepsilon > 0$. For $g \in L^1(X,\mu)$, by Lemma 2.3 and the remark which follows it, we can find $K_0 \in \Lambda$ so that $K \geq K_0$ implies $\int_{X\setminus K} |g| d\mu < \varepsilon/2$. But then $K \geq K_0$ implies

$$\begin{aligned} \left|\int_X (f_K - \varphi) g d\mu\right| &\leq \sum_{j=1}^n \int_{E_j\setminus K_j} |f_K - \varphi|\,|g| d\mu \\ &\leq 2\sum_{j=1}^n \int_{E_j\setminus K_j} |g| d\mu \\ &= 2\int_{X\setminus K} |g| d\mu < \varepsilon, \end{aligned}$$

and so weak*-$\lim_K f_K = g$.

Thus $(\mathcal{C}(X))_1$ is weak*-dense in $(L^\infty(X,\mu))_1$. The second statement is straightforward.

□

**2.5.** Recall that two positive measures $\mu_1$ and $\mu_2$ on a sigma algebra $(X, \mathcal{S})$ are *mutually absolutely continuous* if for $E \in \mathcal{S}$, $\mu_1(E) = 0$ is equivalent to $\mu_2(E) = 0$. We write $\mu_1 \sim \mu_2$ in this case.

**2.6. Theorem.** *Let $X$ be a compact, metric space and $\mu_1, \mu_2$ be finite, positive, regular Borel measures on $X$. Suppose that $\tau : L^\infty(X,\mu_1) \mapsto L^\infty(X,\mu_2)$ is an isometric *-isomorphism and $\tau(f) = f$ for all $f \in \mathcal{C}(X)$. Then $\mu_1 \sim \mu_2$, $L^\infty(X,\mu_1) = L^\infty(X,\mu_2)$, and $\tau(g) = g$ for all $g \in L^\infty(X,\mu_1)$.*

**Proof.** Suppose that $E \subseteq X$ is a Borel set. Then $\tau(\chi_E) \in L^\infty(X,\mu_2)$ is idempotent, and hence a characteristic function, say $\chi_F (= \chi_{F(E)})$. If we can show that $E = F$ $a.e. - \mu_2$, then

$$\begin{aligned} \mu_1(E) = 0 \quad &\text{iff} \quad \chi_E = 0 \text{ in } L^\infty(X, \mu_1) \\ &\text{iff} \quad \tau(\chi_E) = 0 \text{ in } L^\infty(X, \mu_2) \\ &\text{iff} \quad \chi_F = 0 \text{ in } L^\infty(X, \mu_2) \\ &\text{iff} \quad \mu_2(F) = 0 \\ &\text{iff} \quad \mu_2(E) = 0. \end{aligned}$$

From this it follows that $\mu_1 \sim \mu_2$ and therefore that $L^\infty(X, \mu_1) = L^\infty(X, \mu_2)$. Furthermore, since $\tau$ then fixes all characteristic functions, it fixes their spans, which are norm dense in $L^\infty(X, \mu_1)$. By continuity of $\tau$, we see that $\tau$ fixes the entire algebra, so $\tau$ is the identity map.

Note that $\chi_{X \backslash E} = 1 - \chi_E$, and hence $\tau(\chi_{X\backslash E}) = 1 - \tau(\chi_E) = 1 - \chi_F = \chi_{X\backslash F}$. As such, if we can prove that $E \subseteq X$ implies that $\mu_2(F\backslash E) = 0$, then $X\backslash E \subseteq X$ implies $\mu_2(E\backslash F) = \mu_2((X\backslash F)\backslash(X\backslash E)) = 0$. Letting $\Delta = (E\backslash F) \cup (F\backslash E)$, we have $\mu_2(\Delta) = 0$, and hence $E = F$ $a.e. - \mu_2$.

Case One: $E$ is compact: For each $n \geq 1$, define $f_n \in \mathcal{C}(X)$ as follows:

$$f_n(x) = \begin{cases} 1 - n \operatorname{dist}(x, E) & \text{if } \operatorname{dist}(x, E) \leq 1/n \\ 0 & \text{otherwise} \end{cases}$$

Then $f_n \geq \chi_E$ for all $n \geq 1$, and $f_n(x) \to \chi_E(x)$ as $n \to \infty$ for all $x \in X$. Since $\tau$ is a $^*$-homomorphism, it is positive, and as such, it preserves order. Thus $\tau(\chi_E) \leq \tau(f_n)$ for all $n \geq 1$. But $f_n \in \mathcal{C}(X)$ implies $\tau(f_n) = f_n$ so that $\chi_F = \tau(\chi_E) \leq f_n$ for all $n \geq 1$. Hence $\chi_F \leq \chi_E$ in $L^\infty(X, \mu_2)$. Thus $\mu_2(F\backslash E) = 0$, as required.

Case Two: $E \subseteq X$ is Borel: Since $\mu_1, \mu_2$ are regular, we can find an increasing sequence $(K_n)_n$ of compact subseteq of $E$ so that $\mu_i(E\backslash K_n) \to 0$ as $n \to \infty$, $i = 1, 2$. (Indeed, choose $K_1$ so that $\mu_1(E\backslash K_1) < 1$, $K_2 \geq K_1$ so that $\mu_2(E\backslash K_2) < 1/2, etc.$).

Now $\tau$ preserves order, and therefore it also preserves suprema. That is, if $\sup g_n = g$ in $L^\infty(X, \mu_1)$, then $\sup \tau(g_n) = \tau(g)$ in $L^\infty(X, \mu_2)$. In our case,

$$\sup \chi_{K_n} = \chi_E \text{ in } L^\infty(X, \mu_1).$$

Thus $\sup \tau(\chi_{K_n}) = \tau(\chi_E) = \chi_F$ in $L^\infty(X, \mu_2)$. Since $\tau(\chi_{K_n}) \leq \chi_{K_n}$ by Case One, we have

$$\chi_E = \sup \chi_{K_n} \geq \sup \tau(\chi_{K_n}) = \chi_F$$

in $L^\infty(X, \mu_2)$, and so again, $\mu_2(F\backslash E) = 0$, completing the proof.

$\square$

**2.7. Definition.** *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be an algebra. A vector $x \in \mathcal{H}$ is said to be **cyclic** for $\mathcal{A}$ if $[\mathcal{A}x] = \mathcal{H}$. Also, $x$ is said to be **separating** for $\mathcal{A}$ if $A \in \mathcal{A}$ and $Ax = 0$ imply that $A = 0$.*

**2.8. Example.** Let $X \subseteq \mathbb{C}$ be a compact set and $\mu$ be a positive regular Borel measure with $\text{supp}\mu = X$.

Let $q(z) = z,\ z \in X$, and consider $M_q \in \mathcal{B}(L^2(X, \mu))$. Then $e(z) = 1, z \in X$ is cyclic for $C^*(M_q)$. Indeed, since $C^*(M_q) \simeq \mathcal{C}(X)$, we get $[C^*(M_1)e] = [\mathcal{C}(X)] = L^2(X, \mu)$.

Note that $e$ is also separating for $C^*(M_q)$, since $T \in C^*(M_q)$ implies $T = M_f$ for some $f \in \mathcal{C}(X)$, and hence $Te = f = 0$ if and only if $T = 0$. This is not a coincidence.

**2.9. Lemma.** *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be an abelian algebra. If $x$ is cyclic for $\mathcal{A}$, then $x$ is separating for $\mathcal{A}$.*

**Proof.** Suppose $A \in \mathcal{A}$ and $Ax = 0$. Then for all $B \in \mathcal{A}$, $ABx = BAx = 0$. By continuity of $A$, $Ay = 0$ for all $y \in [\mathcal{A}x] = \mathcal{H}$. Thus $A = 0$ and $x$ is separating for $\mathcal{A}$.

□

**2.10. Theorem.** [***The Spectral Theorem. Cyclic Case***] *Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ be normal. Suppose that $x \in \mathcal{H}$ is a cyclic vector for $C^*(N)$. Then there exists a finite, positive, regular Borel measure $\mu$ with* $\text{supp}\mu = \sigma(N)$ *and a unitary $U : \mathcal{H} \to L^2(\sigma(N), \mu)$ so that*

$$\begin{array}{rccc} \Gamma^* : & W^*(N) & \mapsto & \mathcal{B}(L^2(\sigma(N), \mu)) \\ & T & \mapsto & UTU^* \end{array}$$

*is an isometric *-isomorphism onto $\mathcal{M}^\infty(\sigma(N), \mu)$. Furthermore, up to the isomorphism between $\mathcal{M}^\infty(\sigma(N), \mu)$ and $L^\infty(\sigma(N), \mu)$, $\Gamma^*|_{C^*(N)} = \Gamma$, the Gelfand transform.*

**Proof.** First we observe that since $C^*(N)$ is separable and $x \in \mathcal{H}$ is cyclic for $C^*(N)$, it follows that $\mathcal{H}$ is separable as well. Without loss of generality, we may assume that $\|x\| = 1$.

Consider

$$\begin{array}{rccc} \varphi : & C^*(N) & \to & \mathbb{C} \\ & T & \mapsto & \langle Tx, x \rangle. \end{array}$$

Then $\varphi$ is a positive linear functional. Also, $\Gamma : C^*(N) \to \mathcal{C}(\sigma(N))$ is an isometric *-isomorphism, so

$$\varphi \circ \Gamma^{-1} : \mathcal{C}(\sigma(N)) \to \mathbb{C}$$

is a positive linear functional on $\mathcal{C}(\sigma(N))$. By the Riesz-Markov Theorem, there exists a finite, positive, regular Borel measure $\mu$ on $\sigma(N)$ such that

$$\varphi(f(N)) = \varphi \circ \Gamma^{-1}(f) = \int_{\sigma(N)} f d\mu.$$

We claim that $\text{supp}\mu = \sigma(N)$. For otherwise, there exists $G \subseteq \sigma(N)$ open so that $\mu(G) = 0$. Choose a non-zero positive continuous function $f$

with $f \le \chi_G$. Then $0 \ne f(N)$ and hence

$$\begin{aligned}\varphi(f(N)) &= \varphi((f(N)^{1/2})^2) \\ &= \|f(N)^{1/2}x\|^2 \\ &\ne 0\end{aligned}$$

since $f(N)^{1/2} \in C^*(N)$ and $x$ is cyclic, hence separating for $C^*(N)$. But then $0 \ne \varphi(f(N)) = \int_{\sigma(N)} f d\mu \le \int_G 1 d\mu = \mu(G) = 0$, a contradiction. Thus $\mathrm{supp}\mu = \sigma(N)$.

Consider

$$\begin{aligned}U_0 : \quad C^*(N) &\rightarrow \mathcal{C}(\sigma(N)) \\ g(N)x &\mapsto g.\end{aligned}$$

Then

$$\begin{aligned}\|f\|_2^2 &= \int_{\sigma(N)} |g|^2 d\mu \\ &= \varphi \circ (|g|^2(N)) \\ &= \langle |g|^2(N)x, x\rangle \\ &= \langle g(N)^* g(N)x, x\rangle \\ &= \|g(N)x\|^2,\end{aligned}$$

so $U_0$ is isometric. We can and do extend $U_0$ to an isometry $U : \mathcal{H} = [C^*(N)x] \rightarrow [\mathcal{C}(\sigma(N)] = L^2(\sigma(N), \mu)$.

Now set

$$\begin{aligned}\Gamma^* : \quad W^*(N) &\rightarrow \mathcal{B}(L^2(\sigma(N), \mu)) \\ T &\mapsto UTU^*.\end{aligned}$$

Then $\Gamma^*$ is an isometric $^*$-preserving map. For $f, g \in \mathcal{C}(\sigma(N))$, $\Gamma^*(f(N))g = U(f(N))U^*g = Uf(N)g(N) = fg$, so that $\Gamma^*(f(N)) = M_f$.

Now $\Gamma^*$ is WOT-WOT continuous. Indeed, suppose $f_\alpha(N) \rightarrow f(N)$ in the WOT. Then for all $g, h \in \text{Ł}^2(\sigma(N), \mu)$, $\langle Uf_\alpha U^*(Ug), (Uh)\rangle = \langle f_\alpha g, h\rangle \rightarrow \langle fg, h\rangle = \langle UfU^*(Ug), (Uh)\rangle$. Since the WOT on $\mathcal{M}^\infty(\sigma(N), \mu)$ is just the weak$^*$-topology on $L^\infty(\sigma(N), \mu)$, and since $\mathcal{C}(\sigma(N))$ is weak$^*$-dense in $L^\infty(\sigma(N), \mu)$ by Proposition 2.2, it follows that $\operatorname{ran} \gamma^* \supseteq \overline{\mathcal{M}_{\mathcal{C}(\sigma(N))}}^{\text{WOT}} = \mathcal{M}^\infty(\sigma(N), \mu)$.

□

**2.11.** We remark that the measure $\mu$ above is unique in the sense that if $\nu$ is a second finite, positive, regular Borel measure with support equal to $\sigma(N)$ and $\Gamma^*_\nu : W^*(N) \rightarrow \mathcal{B}(L^2(\sigma(N), \nu))$ extends the Gelfand map as $\Gamma^*$ does, then $\mu \sim \nu$, $L^\infty(\sigma(N), \mu) = L^\infty(\sigma(N), \nu)$, and $\Gamma^*_\nu = \Gamma^*$.

Indeed, $\Gamma^*_\nu \circ (\Gamma^*)^{-1} : \mathcal{M}^\infty(\sigma(N), \mu) \rightarrow \mathcal{M}^\infty(\sigma, \nu)$ is an isometric $^*$-isomorphism which, through $\Gamma$, induces an isometric $^*$-isomorphism $\tau$ from $L^\infty(\sigma(N)), \mu$ to $L^\infty(\sigma(N), \nu)$ which fixes the continuous functions. By Theorem 2.6, $\tau$ is the identity map, so that $\Gamma^*_\nu = \Gamma^*$.

**2.12. Proposition.** *Suppose $\mathcal{H}$ is a Hilbert space, $\mathbb{A} \subseteq \mathcal{B}(\mathcal{H})$ is an abelian $C^*$-algebra. Then there exists a masa $\mathbb{M}$ of $\mathcal{B}(\mathcal{H})$ so that $\mathbb{A} \subseteq \mathbb{M}$.*
**Proof.** This is a straightforward application of Zorn's Lemma and the proof is left to the reader.

$\square$

**2.13. Theorem.** *Let $\mathcal{H}$ be a separable Hilbert space and $\mathbb{M} \subseteq \mathcal{B}(\mathcal{H})$ be a masa. Then $\mathbb{M}$ admits a cyclic vector $x$.*
**Proof.** The key to the first half of the proof is that if $y$ and $z$ are two non-zero vectors and $z$ is orthogonal to $[\mathbb{M}y]$, then $[\mathbb{M}z]$ is orthogonal to $[\mathbb{M}y]$. This follows from the fact that $[\mathbb{M}y]$ is reducing for $\mathbb{M}$.

Now consider the family $\mathfrak{F} = \{\{x_\alpha\}_\alpha \in \Lambda \subseteq \mathcal{H} : \|x_\alpha\| = 1 \text{ for all } \alpha, [\mathbb{M}x_{\alpha_1}] \perp [\mathbb{M}x_{\alpha_2}] \text{ if } \alpha_1 \neq \alpha_2\}$, partially ordered with respect to inclusion. If $\mathfrak{J} = \{(J_\beta)_\beta\}$ is a chain in $\mathfrak{F}$, it is routine to verify that $\cup_\beta J_\beta$ lies in $\mathfrak{F}$ and is an upper bound for $\mathfrak{J}$. By Zorn's Lemma, $\mathfrak{F}$ has a maximal element, say $\{x_\gamma\}_{\gamma\in\Xi}$. If $\mathcal{H}_0 = \vee[\mathbb{M}x_\gamma] \neq \mathcal{H}$, then we can choose a unit vector $y \in \mathcal{H}_0$. From the comment in the first paragraph, we deduce that $\{x_\gamma\}_\gamma \cup \{y\} \in \mathfrak{F}$ and is greater than $\{x_\gamma\}_\gamma$, contradicting the maximality of $\{x_\gamma\}_\gamma$. Thus $\vee[\mathbb{M}x_\gamma] = \mathcal{H}$.

Since $\mathbb{M}$ is a masa, $I \in \mathbb{M}$ and so $x_\gamma \in [\mathbb{M}x_\gamma]$ for each $\gamma$ and thus $\dim[\mathbb{M}x_\gamma] \geq 1$. Since $\dim \mathcal{H} = \aleph_0 \geq \sum_\gamma \dim[\mathbb{M}x_\gamma]$, it follows that the cardinality of $\Xi$ is at most $\aleph_0$. Write $\Xi = \{n\}_{n=1}^m$, $m \leq \aleph_0$. Let $x = \sum_{n<m+1} x_n/n$. (The index set of the sum is merely a device to allow us to handle the cases where $\Xi$ is infinite and where $\Xi$ is finite simultaneously.) For each $n$, the orthogonal projection $P_n$ onto $[\mathbb{M}x_n]$ lies in $\mathbb{M}' = \mathbb{M}$, so that $[\mathbb{M}x_n] = [\mathbb{M}P_n x] \subseteq [\mathbb{M}x]$ for all $n < m+1$. Thus $\mathcal{H} = \vee_{n<m+1}[\mathbb{M}x_n] \subseteq [\mathbb{M}x] \subseteq \mathcal{H}$, and $x$ is a cyclic vector for $\mathbb{M}$.

$\square$

**2.14. Corollary.** *Let $\mathcal{H}$ be a separable Hilbert space and $\mathbb{A} \subseteq \mathcal{B}(\mathcal{H})$ be an abelian $C^*$-algebra. Then $\mathbb{A}$ has a separating vector.*
**Proof.** By Proposition 2.12, $\mathbb{A} \subseteq \mathbb{M}$ for some masa $\mathbb{M}$ of $\mathcal{B}(\mathcal{H})$. By Theorem 2.13, $\mathbb{M}$ has a cyclic vector $x$, and $x$ is separating for $\mathbb{M}$ by Lemma 2.9. Finally, if $x$ is separating for $\mathbb{M}$, then trivially $x$ is also separating for $\mathbb{A}$.

$\square$

Let $\mathcal{H}$ be a separable Hilbert space, $\mathbb{A} \subseteq \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra, and $x \in \mathcal{H}$. Denote by $\mathcal{H}_x$ the space $[\mathbb{A}x]$, and for $Z \in \mathcal{B}(\mathcal{H})$, denote by $Z_x$ the compression of $Z$ to $\mathcal{H}_x$.

**2.15. Proposition.** *Let $\mathcal{H}$ be a separable Hilbert space, $\mathbb{A} \subseteq \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra, and $x \in \mathcal{H}$ be a separating vector for $\mathbb{A}$. The map*

$$\begin{array}{rccc} \Phi : & \mathbb{A} & \to & \mathcal{B}(\mathcal{H}_x) \\ & T & \mapsto & T_x \end{array}$$

*is an isometric* $^*$*-isomorphism of* $\mathbb{A}$ *onto* $\operatorname{ran}\Phi$. *Moreover,* $\sigma(T) = \sigma(T_x)$ *for all* $T \in \mathbb{A}$.

**Proof.** Recall from Lemma 1.17 that the orthogonal projection $P_x$ onto $\mathcal{H}_x$ lies in $\mathbb{A}'$, and $x \in \operatorname{ran} P_x$. From this the fact that $\Phi$ is a $^*$-homomorphism easily follows.

Suppose $0 \neq T \in \mathbb{A}$. Then $T_x(x) = TP_x(x) = Tx \neq 0$, as $x$ is separating for $\mathbb{A}$. Thus $\ker \Phi = 0$, and so $\Phi$ is an isometric map as well. Now $\Phi(\mathbb{A})$ is a $C^*$-algebra by ??, and so in particular,

$$\sigma(T) = \sigma_{\mathbb{A}}(T) = \sigma_{\Phi(\mathbb{A})}(\Phi(T)) = \sigma_{\mathcal{B}(\mathcal{H}_x)}(T_x) = \sigma(T_x),$$

completing the proof.

□

**2.16. Theorem.** [***The Spectral Theorem for normal operators***] *Let* $\mathcal{H}$ *be a separable Hilbert space and* $N \in \mathcal{B}(\mathcal{H})$ *be normal. Then there exists a finite, positive, regular Borel measure* $\mu$ *with support equal to* $\sigma(N)$ *and an isometric* $^*$*-isomorphism*

$$\Gamma^* : W^*(N) \to \mathcal{M}^\infty(\sigma(N), \mu)$$

*which extends the Gelfand map* $\Gamma_m : C^*(N) \to \mathcal{M}(\mathcal{C}(\sigma(N), \mu))$, $\Gamma_m(f(N)) = M_f$.

*Moreover,* $\mu$ *is unique up to mutual absolute continuity, while* $\Gamma_m^*$ *and* $M^\infty(\sigma(N), \mu)$ *are unique.*

**Proof.** By Corollay 2.14, $W^*(N)$ an abelian $C^*$-algebra implies that $W^*(N)$ has a separating vector $x$, which we may assume has norm one. Let $\mathcal{H}_x = [W^*(N)x]$, and consider (using the same notation as before)

$$\begin{array}{cccc} \Phi : & W^*(N) & \to & \mathcal{B}(\mathcal{H}_x) \\ & T & \mapsto & T_x. \end{array}$$

By Proposition 2.15, $\Phi$ is an isometric $^*$-isomorphism, and $\sigma(T_x) = \sigma(T)$ for all $T \in W^*(N)$ - in particular, $\sigma(N_x) = \sigma(N)$. By identifying $W^*(N)$ with its range $\Phi(W^*(N))$, we may assume that $W^*(N)$ already has a cyclic vector. But $\overline{C^*(N)}^{\text{WOT}} = W^*(N)$, and so if $T \in W^*(N)$, then there exists a net $(T_\alpha)_\alpha \in C^*(N)$ so that $Tx = \lim_\alpha T_\alpha x \in [C^*(N)x]$. It follows that $\mathcal{H}_x = [C^*(N)x]$, so that $x$ is also a cyclic vector for $C^*(N)$.

By the Cyclic Version of the Spectral Theorem for normal operators, Theorem 2.10, we obtain a finite, positive, regular Borel measure $\mu$ with support $\sigma(N)$ so that $\Gamma_m^* : W^*(N) \to \mathcal{B}(\mathcal{H}_x)$ is an isometric $^*$-isomorphism. Also, $\operatorname{ran}\Gamma_m^* = \mathcal{M}^\infty(\sigma(N), \mu)$. From the proof of that Theorem, we saw that $\Gamma_m^*$ is WOT-WOT continuous, and so $\Gamma^* = \Gamma_m^* \circ \Phi$ is WOT-WOT continuous as well. Also, $\Gamma^*$ extends the Gelfand map because $\Gamma_m^*$ does.

Finally, $\Phi$ surjective implies that $\operatorname{ran}\Gamma^* = \mathcal{M}^\infty(\sigma(N), \mu)$. Uniqueness follows as before.

□

**2.17. Remark.**

(i) Let $\mathcal{H}$ be a separable Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ be normal. We can now define an $L^\infty$-functional calculus for $N$.
(ii) $N$ can be approximated by linear combinations of projections.
(iii) We can define a spectral measure on $\sigma(N)$.

*Where there's a will, there's a wake. [Old Irish Proverb]*

****************************************

# Appendix A: The essential spectrum

**A.1. Definition.** *Given an operator $T \in \mathcal{B}(\mathcal{H})$, we define the* ***essential spectrum*** *of $T$ to be the spectrum of the image $\pi(T)$ in the Calkin algebra $\mathcal{A}(\mathcal{H})$.*

In this note, we wish to prove a result due to Putnam and Schechter , namely:

**A.2. Theorem.** *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Suppose that $\lambda \in \partial\sigma(T)$. Then either $\lambda$ is isolated in $\sigma(T)$, or $\lambda \in \sigma_e(T)$.*

The proof below uses a description of the *singular points* of the semi-Fredholm domain of $T$, due to C. Apostol [**?**].

**A.3. Definition.** *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then the* ***semi-Fredholm domain*** *$\rho_{\mathrm{sF}}(T)$ of $T$ is the set of all complex numbers $\lambda$ such that $\lambda 1 - \pi(T)$ is either left or right invertible in the Calkin algebra.*

*If $\mu \in \mathbb{C}$, then $\mu$ is called a $(T)-$**singular point** if the function*

$$\lambda \mapsto P_{\ker(T-\lambda)}$$

*is discontinuous at $\mu$. Otherwise, $\mu$ is said to be $(T)-$**regular** .*

*If $\mu \in \rho_{\mathrm{sF}}(T)$ and $\mu$ is singular (resp. $\mu$ is regular), then we write $\mu \in \rho^{\mathrm{s}}_{\mathrm{sF}}(T)$ (resp. $\rho^{\mathrm{r}}_{\mathrm{sF}}(T)$).*

**A.4. Lemma.** *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that $\mu$ is a regular point of the semi-Fredholm domain of $T$. Then*

$$\ker(T-\mu)^* \subseteq (\text{ span } \{\ker(T-\lambda) : \lambda \in \mathbb{C}\})^{\perp}.$$

**Proof.** First note that $\operatorname{ran}(T-\mu) \supseteq \ker(T-\lambda)$ for all $\lambda \neq \mu$. For if $x \in \ker(T-\lambda)$ and $\lambda \neq \mu$, then $(T-\mu)x = (\lambda-\mu)x$ and so $x \in \operatorname{ran}(T-\mu)$.

Also $\overline{\operatorname{span}}\{\ker(T-\lambda) : \lambda \in \mathbb{C}\} = \overline{\operatorname{span}}\{\ker(T-\lambda) : \lambda \neq \mu\}$. This follows from the regularity of $\mu$. Basically, we must show that $\ker(T-\mu) \subseteq \overline{\operatorname{span}}\{\ker(T-\lambda) : \lambda \neq \mu\}$. But if $x \in (T-\mu)$ and $\|x\| = 1$, then again by the regularity of $\mu$, for any $\epsilon > 0$ we can find $\lambda_n \to \mu$ such that $\|P_{\ker(T-\lambda_n)} - P_{\ker(T-\mu)}\| < \epsilon$.

But then

$$\begin{aligned} \epsilon &> \|P_{\ker(T-\lambda_n)}x - P_{\ker(T-\mu)}x\| \\ &= \|P_{\ker(T-\lambda_n)}x - x\|, \end{aligned}$$

and so $x \in \ker(T-\mu) \subseteq \overline{\operatorname{span}}\{\ker(T-\lambda) : \lambda \neq \mu\}$.

Combining these two arguments,

$$\begin{aligned} \ker(T-\mu)^* &= (\operatorname{ran}(T-\mu))^\perp \\ &\subseteq \overline{\operatorname{span}}\{\ker(T-\lambda) : \lambda \in \mathbb{C}\}. \end{aligned}$$

□

**A.5. Theorem.** *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

(i) $\rho_{sF}^{r}(T)$ *is open;*
(ii) $\rho_{sF}^{r}(T) = \rho_{sF}^{r}(T^*)^* := \{\overline{\lambda} : \lambda \in \rho_{sF}^{r}(T)\}$*;*
(iii) $\rho_{sF}^{r}(T)$ *has no accumulation points in* $\rho_{sF}^{s}(T)$*.*

**Proof.**

(i) Let $\mu \in \rho_{sF}^{r}(T)$ and put $Y = \overline{\operatorname{span}}\{\ker(T-\lambda) : \lambda \in \mathbb{C}\}$. We claim that $TY \subseteq Y$.

Consider $y \in \operatorname{span}\{\ker(T-\lambda) : y \in \mathbb{C}\}$, say $y = \sum_{n=1}^{m} y_n$ with each $y_n \in \ker(T-\lambda_n)$. Then $Ty = \sum_{n=1}^{m} Ty_n = \sum_{n=1}^{m} \lambda_n y_n$ which lies in $\operatorname{span}\{\ker(T-\lambda_n) : 1 \leq n \leq m\}$. By the continuity of $T$, we have $TY \subseteq Y$. Let $T_Y = T|_Y$.

Since $\ker(T-\mu) \subseteq Y$, $\operatorname{ran}(T_Y - \mu)$ is closed. To see this, suppose that $\{x_n\}$ is a sequence in $\operatorname{ran}(T_Y-\mu)$ such that $\{x_n\}$ converges to $x \in Y$. Then there exists a sequence $\{y_n\} \subseteq Y$ such that $(T_Y - \mu)y_n = x_n$.

In fact, since $\ker(T-\mu) \subseteq Y$, we can let

$$z_n = P_{Y \ominus \ker(T-\mu)} y_n$$

and then

$$(T-\mu)z_n = (T_Y-\mu)z_n = (T_Y-\mu)y_n = x_n$$

for all $n \geq 1$.

Since $\operatorname{ran}(T-\mu)$ is closed, (i.e. $\mu \in \rho_{sF}$), there exists $z \in \mathcal{H}$ such that $(T-\mu)z = x$. But $(T-\mu)$ is bounded below on $(\ker(T-\mu))^\perp$, and therefore $(T_Y - \mu)$ is bounded below on $Y \ominus \ker(T-\lambda)$. From this we get a $\delta > 0$ such that

$$\begin{aligned} \|x_n - x\| &= \|(T-\mu)z_n - (T-\mu)z\| \\ &\geq \delta\|z_n - z\| \end{aligned}$$

for all $n \geq 1$.

But then $z = \lim_{n\to\infty} z_n$, and so $z \in Y$. This gives us $(T_Y - \mu)z = (T - \mu)z = x$, and so $x \in \operatorname{ran}(T_Y - \mu)$, i.e. $\operatorname{ran}(T_Y - \mu)$ is closed.

We next claim that

$$\begin{aligned}(T_Y - \mu)Y &\supseteq \overline{\operatorname{span}}\{\ker(T-\lambda) : \lambda \neq \mu\} \\ &= \overline{\operatorname{span}}\{\ker(T-\lambda) : \lambda \in \mathbb{C}\} \\ &= Y.\end{aligned}$$

The first equality we saw in the previous Lemma, while the second *is* the definition of $Y$. As for the containment, let $y \in \overline{\operatorname{span}}\{\ker(T-\lambda) : \lambda \neq \mu\}$, say $y = \sum_{n=1}^m y_n$. Then

$$\begin{aligned}(T_Y - \mu)y &= \sum_{n=1}^m (T_Y - \mu)y_n \\ &= \sum_{n=1}^m (\lambda_n - \mu)y_n\end{aligned}$$

where $y_n \in \ker(T - \lambda_n)$. Thus if $z = \sum_{n=1}^m (\lambda_n - \mu)^{-1} y_n$, we have $z \in Y$ and $(T_Y - \mu)z = y$. Since $\operatorname{ran}(T_Y - \mu)$ is closed, the desired conclusion follows.

Since $(T_Y - \mu)$ is onto, we have $\mu \in \rho_{\mathrm{r}}(T_Y)$, the right resolvent set of $T_Y$. For $A \in \mathcal{B}(\mathcal{H})$, define the *right resolvent* as

$$R_{\mathrm{r}}(\lambda; A) = (\lambda - A)^*[(\lambda - A)(\lambda - A)^*]^{-1}$$

so that

$$P_{\ker(A-\lambda)} = I - R_{\mathrm{r}}(\lambda; A)(\lambda - A)$$

for all $\lambda \in \rho_{\mathrm{r}}(A)$.

Since $\ker(T_Y - \mu) = \ker(T - \mu)$ for all $\lambda \in \mathbb{C}$, we infer that the map $\lambda \mapsto P_{\ker(T-\mu)}$ is continuous in an open neighbourhood $G_\mu$ of $\mu$, as $\rho_{\mathrm{r}}(T)$ is open. Thus $\mu$ is an interior point of $\rho_{\mathrm{sF}}^{\mathrm{r}}(T)$, and so $\rho_{\mathrm{sF}}^{\mathrm{r}}(T)$ is open.

(ii) $\rho_{\mathrm{sF}}^{\mathrm{r}}(T) = \rho_{\mathrm{sF}}^{\mathrm{r}}(T^*)^*$

Let $Z = (\overline{\operatorname{span}}\{\ker(T-\lambda)^* : \lambda \in \mathbb{C}\})^\perp$. Then the proof of (i) shows that $\rho_{\mathrm{sF}}^{\mathrm{r}}(T) \subseteq \rho_{\mathrm{r}}(T_Y)$ and $\rho_{\mathrm{sF}}^{\mathrm{r}}(T^*) \subseteq \rho_{\mathrm{r}}(T^*_{Z^\perp})$.

We now claim that $\rho_{\mathrm{sF}}^{\mathrm{r}}(T) \subseteq \rho_{\mathrm{l}}(T_{Y^\perp}) = \rho_{\mathrm{r}}(T^*_{Y^\perp})^*$. For suppose that $\lambda \in \rho_{\mathrm{r}}(T_Y)$. If $w \in \ker(T_{Y^\perp} - \lambda)$, then

$$(T-\lambda)\begin{bmatrix} 0 \\ w \end{bmatrix} = \begin{bmatrix} T_y - \lambda & T_Z \\ 0 & T_{Y^\perp} - \lambda \end{bmatrix} \begin{bmatrix} 0 \\ w \end{bmatrix} = \begin{bmatrix} T_Z w \\ 0 \end{bmatrix}.$$

Since $T_Y - \lambda$ is right invertible, $(T_Y - \lambda)R = I$ for some $R \in \mathcal{B}(Y)$ and so $(T_Y - \lambda)R(-T_Z w) = (-T_Z w)$. Letting $v = -RT_Z w$, we have

$$(T-\lambda)\begin{bmatrix} v \\ w \end{bmatrix} \text{ and so } \begin{bmatrix} v \\ w \end{bmatrix} \in \ker(T-\lambda) \subseteq Y.$$

Thus $w = 0$. But then $T_{Y^\perp}$ is injective.

If $\lambda \in \rho^{\mathrm{r}}_{\mathrm{sF}}(T)$, then $\operatorname{ran}(T_{Y^\perp} - \lambda) = P_{Y^\perp}(\operatorname{ran}(T - \lambda))$ and since $\operatorname{ran}(T - \lambda)$ is closed, so is $\operatorname{ran}(T_{Y^\perp} - \lambda)$. Thus $\lambda \in \rho^{\mathrm{r}}_{\mathrm{sF}}(T)$ implies that $\ker(T_{Y^\perp} - \lambda) = \{0\}$ and $\operatorname{ran}(T_{Y^\perp} - \lambda)$ is closed, so that $\lambda \in \rho_{\mathrm{l}}(T_{Y^\perp}) = \rho_{\mathrm{r}}(T^*_{Y^\perp})^*$.

Similarly, $\rho^{\mathrm{r}}_{\mathrm{sF}}(T^*) \subseteq \rho_{\mathrm{r}}(T_Z)^*$.

Since the maps

$$\begin{aligned} \lambda &\mapsto P_{\ker(T^*_{Y^\perp} - \overline{\lambda})} \lambda \in \rho^{\mathrm{r}}_{\mathrm{sF}}(T) \\ \lambda &\mapsto P_{\ker(T^*_Z - \overline{\lambda})} \lambda \in \rho^{\mathrm{r}}_{\mathrm{sF}}(T^*) \end{aligned}$$

are continuous, and by the first Lemma we have

$$\begin{aligned} \ker(T - \lambda)^* &= \ker(T^*_{Y^\perp} - \overline{\lambda}) \text{ for all } \lambda \in \rho^{\mathrm{r}}_{\mathrm{sF}}(T) \\ \ker(T - \overline{\lambda}) &= \ker(T_Z - \overline{\lambda}) \text{ for all } \lambda \in \rho^{\mathrm{r}}_{\mathrm{sF}}(T^*), \end{aligned}$$

we infer that

$$\begin{aligned} \rho^{\mathrm{r}}_{\mathrm{sF}}(T)^* &\subseteq \rho^{\mathrm{r}}_{\mathrm{sF}}(T^*) \\ \rho^{\mathrm{r}}_{\mathrm{sF}}(T^*) &\subseteq \rho^{\mathrm{r}}_{\mathrm{sF}}(T)^* \end{aligned}$$

so that $\rho^{\mathrm{r}}_{\mathrm{sF}}(T) = \rho^{\mathrm{r}}_{\mathrm{sF}}(T^*)^*$.

(iii) $\rho^{\mathrm{s}}_{\mathrm{sF}}(T)$ has no accumulation points in $\rho_{\mathrm{sF}}(T)$.

Let $z \in \rho^{\mathrm{s}}_{\mathrm{sF}}(T)$. Then we may assume that $z \in \rho_{\mathrm{le}}(T)$, for otherwise, by (2), we may consider $\overline{z}$ and $T^*$. Put $Y_0 = \overline{\operatorname{span}}\{\ker(T - \lambda) : \lambda \neq z\}$. As $T_{Y_0} - z$ has dense range (the proof follows as from (1)), and since $\operatorname{ran}(T_{Y_0})$ is closed (i.e. $z \in \rho_{\mathrm{le}}(T)$), we get $z \in \rho_{\mathrm{r}}(T_{Y_0})$.

Now for $\lambda \neq z$, we have $\ker(T - \lambda) = \ker(T_{Y_0} - \lambda)$. Since the map

$$\lambda \mapsto R_{\mathrm{r}}(\lambda; T_{Y_0}) \qquad \lambda \in \rho_{\mathrm{r}}(T_{Y_0})$$

is continuous, we have that

$$\lambda \mapsto P_{\ker(T_{Y_0} - \lambda)} = I - R_{\mathrm{r}}(\lambda; T_{Y_0})(\lambda - T_{Y_0}) \qquad \lambda \in \rho_{\mathrm{r}}(T_{Y_0})$$

is continous, and so

$$\lambda \mapsto P_{\ker(T - \lambda)}$$

is continous in some punctured neighbourhood of $z$. Since $\rho_{\mathrm{sF}}(T)$ is open, we have that $z$ is an isolated point in $\rho^{\mathrm{s}}_{\mathrm{sF}}(T)$.

Finally, suppose $\rho^{\mathrm{s}}_{\mathrm{sF}}(T)$ has an accumulation point $\mu \in \rho_{\mathrm{sF}}(T)$. Then by (1), $\mu \in \rho^{\mathrm{s}}_{\mathrm{sF}}(T)$ and $\mu$ is isolated, a contradiction. This concludes the proof.

$\square$

**A.6. Proposition.** *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then*

$$\sigma(T) = \sigma_{\mathrm{e}}(T) \cup \sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{p}}(T^*)^*,$$

*where $\sigma_{\mathrm{p}}(T^*)^* = \{\overline{\lambda} : \lambda \in \sigma_{\mathrm{p}}(T^*)\}$.*

**Proof.** Suppose $\lambda \notin \cup \sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{p}}(T^*)^*$. Then $\mathrm{nul}(T-\lambda) = \mathrm{nul}(T-\lambda)^* = 0$.

Thus $(T-\lambda)$ is injective and has dense range. If $\lambda \notin \sigma_{\mathrm{e}}$, then $(T-\lambda)$ is Fredholm and thus $\mathrm{ran}\,(T-\lambda)$ is closed. But then $(T-\lambda)$ is bijective and hence $\lambda \notin \sigma(T)$. Thus $\sigma(T) \subseteq \sigma_{\mathrm{e}}(T) \cup \sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{p}}(T^*)^*$. The other inclusion is obvious.

□

**A.7. Theorem.** *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose $\lambda \in \partial\sigma(T)$. Then either $\lambda$ is isolated or $\lambda \in \sigma_{\mathrm{e}}(T)$.*

**Proof.** Suppose $\lambda \notin \sigma_{\mathrm{e}}(T)$. Then by the above Proposition, we may assume that $\lambda \in \sigma_{\mathrm{p}}(T)$ (otherwise consider $\overline{\lambda}$ and $T^*$. Since $\lambda \in \partial\sigma(T)$, we can find a sequence $\{\lambda\}_n \subseteq \rho(T)$ such that $\lambda = \lim_{n\to\infty} \lambda_n$.

Since $\ker(T-\lambda_n) = \{0\}$ for all $n \geq 1$ while $\ker(T-\lambda) \neq \{0\}$, we conclude that $\lambda \in \rho^{\mathrm{s}}_{\mathrm{sF}}(T)$. Since $\rho^{\mathrm{s}}_{\mathrm{sF}}(T)$ has no accumulation points in $\rho_{\mathrm{sF}}(T)$, and since $\lambda \notin \sigma_{\mathrm{e}}(T)$, we conclude that $\lambda$ is isolated in $\sigma(T)$.

□

**A.8. Corollary.** *Let $T \in \mathcal{B}(\mathcal{H})$. Then $\sigma(T) = \sigma_{\mathrm{e}}(T) \cup \Omega$, where $\Omega$ consists of some bounded components of the Fredholm domain of $T$ and a sequence of isolated points in the Fredholm domain which converge to $\sigma_{\mathrm{e}}(T)$.*

*"I regret that I have only sixteen lives to give to my country."*
***Sybill***

*************************************

# Appendix B. von Neumann algebras as dual spaces

Let $\mathcal{H}$ be a separable Hilbert space. In this note we show the von Neumann algebras are precisely the class of C*-algebras of $\mathcal{B}(\mathcal{H})$ which can be identified with the dual space of some Banach space $\mathfrak{X}$. Much of the material in the second half of this note is borrowed from the book of Pedersen [**?**] .

Let us first recall how $\mathcal{B}(\mathcal{H})$ is itself a dual space. By $\mathcal{K}(\mathcal{H})$ we denote the set of compact operators on $\mathcal{H}$.

Given an operator $K \in \mathcal{K}(\mathcal{H})$, we may consider $|K| = (K^*K)^{\frac{1}{2}} \in \mathcal{K}(\mathcal{H})$. Then $|K| \geq 0$, and so by the Spectral Theorem for Compact Normal Operators, we know that $\sigma(|K|) = \{s_n(K)\}_{n=1}^{\infty}$, where $s_n(K) \geq 0$ for all $n \geq 1$ and $\lim_{n\to\infty} s_n(K) = 0$.

**B.1. Definition.** *We write $K \in \mathcal{C}_1(\mathcal{H})$ and say that $K$ is a* ***trace class operator on*** *$\mathcal{H}$ if $K$ is compact and $\sum_{n=1}^{\infty} s_n(K) < \infty$. The numbers $s_n = s_n(K)$ are called the* ***singular numbers*** *for $K$.*

*More generally, we write $K \in \mathcal{C}_p(\mathcal{H})$ if $\sum_{n=1}^{\infty} {s_n}^p < \infty$.*

We shall require the following two facts. Their proofs may be found in [**?**].

**Facts:**

- For each $p$, $1 \leq p < \infty$, $\mathcal{C}_p(\mathcal{H})$ is an ideal of $\mathcal{B}(\mathcal{H})$ called the *Schatten p-ideal.* Moreover, $\mathcal{C}_p(\mathcal{H})$ is closed in the $\mathcal{C}_p-$ *norm topology* which is the topology determined by the norm

$$\|K\|_p = (\sum_{n=1}^{\infty} {s_n}^p)^{1/p}.$$

- If $T \in \mathcal{B}(\mathcal{H})$, $K \in \mathcal{C}_1(\mathcal{H})$ and $\{e_n\}_{n=1}^\infty$ is an orthonormal basis of $\mathcal{H}$, then we can define $\operatorname{tr}(TK) = \sum_{n=1}^\infty a_{nn}$, where $TK = [a_{ij}]_{i,j\geq 1}$ with respect to the orthonormal basis $\{e_n\}_{n=1}^\infty$. One can then show that $\operatorname{tr}(TK)$ is well-defined; that is, it is independent of the orthonormal basis chosen.

From the above two facts, we see that given $T \in \mathcal{B}(\mathcal{H})$, we can define

$$\begin{array}{rccc} \phi_T : & \mathcal{C}_1(\mathcal{H}) & \to & \mathbb{C} \\ & K & \mapsto & \operatorname{tr}(TK) \end{array}.$$

The map that sends a trace class operator $T$ to the functional $\phi_T$ proves to be an isometric isomorphism between $\mathcal{C}_1(\mathcal{H})^*$ and $\mathcal{B}(\mathcal{H})$, so that $\mathcal{B}(\mathcal{H})$ is a dual space and as such is endowed with the weak*-topology induced by its predual, $\mathcal{C}_1(\mathcal{H})$. This turns out to be precisely the ultraweak or $\sigma$-weak topology on $\mathcal{B}(\mathcal{H})$.

An alternate approach to this result is to realize $\mathcal{C}_1(\mathcal{H})$ as the closure of $\mathcal{H} \otimes \mathcal{H}$ in $\mathcal{B}(\mathcal{H})^*$.

In order to prove that every von Neumann algebra $\mathbb{A}$ is a dual space, we require some basic results form Linear Analysis.

**B.2. Definition.** *Let $\mathfrak{X}$ be a Banach space and $M \subset \mathfrak{X}$, $N \subseteq \mathfrak{X}^*$ be linear manifolds. Then*

$$\begin{aligned} M^\perp &= \{f \in \mathfrak{X}^* : f(m) = 0 \text{ for all } m \in M\} \\ {}^\perp N &= \{x \in \mathfrak{X} : g(x) = 0 \text{ for all } g \in N\}. \end{aligned}$$

**B.3. Proposition.** *Let $\mathfrak{X}$ be a Banach space and $M \subseteq \mathfrak{X}$, $N \subseteq \mathfrak{X}^*$ be linear manifolds. Then*

1. *$M^\perp$ is a weak*-closed subspace of $\mathfrak{X}^*$.*
2. *${}^\perp N$ is a norm closed subspace of $\mathfrak{X}$.*

**Proof.**

1. Suppose $\{f_\alpha\}_{\alpha\in\Lambda}$ is a net in $M^\perp$ and $f_\alpha$ converges to $f$ in the weak*-topology. Then for all $x$ in $\mathfrak{X}$, $\lim_{\alpha\in\Lambda} f_\alpha(x) = f(x)$, and so in particular, $f(m) = \lim_\alpha f_\alpha(m) = 0$ for all $m \in M$, implying that $f \in M^\perp$. Thus $M$ is weak*-closed.
2. If $\{x_n\}_{n=1}^\infty \subseteq^\perp N$ and $x = \lim_{n\to\infty} x_n$, then $g(x) = \lim_{n\to\infty} g(x_n) = 0$ for all $g \in N$. Thus $x \in^\perp N$ and the latter is norm closed.

$\square$

**B.4. Theorem.** *Let $\mathfrak{X}$ be a Banach space and let $M \subseteq \mathfrak{X}$ and $N \subseteq \mathfrak{X}^*$ be linear manifolds. Then $({}^\perp N)^\perp$ is the weak*-closure of $N$ in $\mathfrak{X}^*$.*

**Proof.** Clearly, if $g \in N$, then $g(x) = 0$ for all $x \in {}^{\perp}N$, and so $g \in ({}^{\perp}N)^{\perp}$. But $({}^{\perp}N)^{\perp}$ is now a weak*-closed subspace of $\mathfrak{X}^*$ which contains the weak*-closure of $N$.

If $f$ does not lie in the weak*-closure of $N$, then by the Hahn-Banach Theorem applied to $\mathfrak{X}^*$ with its weak*-topology (which separates points from convex sets), there exists $x \in {}^{\perp}N$ such that $f(x) \neq 0$. But then $f \notin ({}^{\perp}N)^{\perp}$, completing the proof.

□

**B.5. Theorem.** *Let $\mathfrak{X}$ be a Banach space and $M \subseteq \mathfrak{X}$ be a subspace of $\mathfrak{X}$. Let $\pi : \mathfrak{X} \to \mathfrak{X}/M$ denote the canonical quotient map. Then the map*

$$\begin{array}{cccc} \tau : & (\mathfrak{X}/M)^* & \to & M^{\perp} \\ & f & \mapsto & f \circ \pi \end{array}$$

*is an isometric isomorphism.*

**Proof.** First we shall show that $\tau$ is injective.

If $\tau(f) = f \circ \pi = g \circ \pi = \tau(g)$, then

$$f(\pi(x)) = (f \circ \pi)(x) = (g \circ \pi)(x) = g(\pi(x)) \text{ for all } x \in \mathfrak{X},$$

and so $f = g$ as elements of $(\mathfrak{X}/M)^*$.

Next we show that $\tau$ is surjective.

Let $\phi \in M^{\perp}$ and define $g \in (\mathfrak{X}/M)^*$ by $g(\pi(x)) = \phi(x)$. To see that $g$ is well-defined, note that if $\pi(x) = \pi(y)$, then

$$g(\pi(x)) - g(\pi(y)) = \phi(x) - \phi(y) = \phi(x - y).$$

But $\pi(x - y) = 0$ implies that $x - y \in M$, and so $\phi(x - y) = 0$. Thus $g$ is well-defined, and since $\tau(g) = g \circ \pi = \phi$, $\tau$ is surjective.

Finally we show that $\tau$ is isometric. Let $\tau \in (\mathfrak{X}/M)^*$. Then

$$\begin{aligned} \|\tau(g)\| &= \|g \circ \pi\| \\ &= \sup_{\|x\|=1} \|g \circ \pi(x)\| \\ &= \sup_{\|\pi(x)\|=1} \|g(\pi(x))\| \\ &= \|g\|. \end{aligned}$$

□

**B.6. Theorem.** *Let $\mathbb{A} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$. Then $\mathbb{A}$ is isometrically isomorphic to the dual space of some Banach space.*

**Proof.** Let $\mathfrak{X} = \mathcal{C}_1(\mathcal{H}) = (\mathcal{B}(\mathcal{H}))_*$, and let $M = {}^\perp\mathbb{A}$, so that $M$ is closed in $\mathfrak{X}$. By Theorem B.5 above, we have

$$(\mathfrak{X}/M)^* \simeq M^\perp,$$

and this isomorphism is isometric. But then

$$\left(\mathcal{C}_1(\mathcal{H})/({}^\perp\mathbb{A})\right)^* \simeq \left({}^\perp\mathbb{A}\right)^\perp = \mathbb{A}^{-weak^*}.$$

But $\mathbb{A}$ is a von Neumann algebra and hence $\mathbb{A}$ is closed in the weak-operator topology, which is weaker than the weak*-topology on $\mathcal{B}(\mathcal{H})$. Thus $\mathbb{A}$ is weak*-closed as well, and so

$$\left(\mathcal{C}_1(\mathcal{H})/({}^\perp\mathbb{A})\right)^* \simeq \mathbb{A},$$

where the isomorphism is once again isometric.

□

To complete the analysis, one needs to show that if a C*-algebra $\mathbb{A}$ is isometrically isomorphic to the dual space of some Banach space $\mathfrak{X}$, then $\mathbb{A}$ is a von Neumann algebra. This is by far the more difficult implication.

We begin with the following Proposition, which may be found in [**?**].

**B.7. Proposition.** *Let $\mathbb{A}$ be a C*-algebra and let $S$ denote its unit sphere. Then $S$ has an extreme point if and only if $\mathbb{A}$ has an identity.*

**Proof.** Suppose first that $\mathbb{A}$ has an identity, say 1. We shall show that 1 is an extreme point in $S$. If $1 = (a+b)/2$ with $a, b \in S$, then put $c = (a+a^*)/2$ and $d = (b+b^*)/2$. Then $1 = (c+d)/2$ with $c, d \in S$. Since $d = 2-c$, $d$ commutes with $c$ and both $c$ and $d$ are self-adjoint.

Representing the C*-algebra generated by $1, c$, and $d$ as continuous functions on some compact Hausdorff space, we can easily see that $c = d = 1$. Hence $a^* = 2-a$, so that $a$ is normal. But then $a = a^* = 1$, again by norm considerations, so that $b = 1$ and thus 1 is an extreme point.

Conversely, suppose $x$ is an extreme point in $S$. Let $C_0(\Omega)$ be the C*-subalgebra of $\mathbb{A}$ generated by $x^*x$. Then, since every C*-algebra has an approximate identity, we can take a sequence $\{y_n\}$ of positive elements in $C_0(\Omega)$ such that $\|y_n\| \leq 1$ for all $n$, $\lim_{n\to\infty} \|(x^*x)y_n - (x^*x)\| = 0$, and $\lim_{n\to\infty} \|(x^*x)y_n^2 - (x^*x)\| = 0$. (This last step follows from the fact that if $\{y_n\}$ is a bounded approximate identity for $C_0(\Omega)$, then so is $\{y_n^2\}$.)

Suppose that at some point $t$ of $\Omega$, $x^*x$ takes a non-zero value less than one. Then we can take a positive element $c$ of $C_0(\Omega)$, non-zero at $t$, such that $\gamma_n = y_n + c$, $s_n = y_n - c$, $\|(x^*x)\gamma_n^2\| \leq 1$, and $\|(x^*x)s_n^2\| \leq 1$. Hence $x\gamma_n$ and $xs_n$ are in $S$.

On the other hand,

$$\|(xy_n - x)^*(xy_n - x)\| = \|x^*xy_n^2 - x^*xy_n - x^*xy_n + x^*x\|,$$

and this tends to 0 as $n$ tends to $\infty$. Hence $\lim_{n\to\infty} xy_n = x$, so that $x\gamma_n \to x + xc$ and $xs_n \to x - xc$.

Since $x+xc,\ x-xc \in S$ and $x = \dfrac{(x+xc)+(x-xc)}{2}$, $x = x+xc = x-xc$. Hence $xc = 0$ and so $\|cx^*xc\| = \|x^*xc^2\| = 0$. This is a contradiction, because $x^*x(t)c^2(t) \neq 0$.

Therefore, $x^*x$ has no non-zero value less than one in $\Omega$. In other words, $x^*x$ is a projection.

Put $x^*x + xx^* = h$, and let $\mathbb{B}$ be a maximal commutative C*-algebra of $\mathbb{A}$ containing $h$. Suppose $h$ is not invertible in $\mathbb{B}$. Then there exists a sequence $\{z_n\}$ of positive elements belonging to $\mathbb{B}$ which satisfies $\|z_n^2\| = 1$ for all $n$ and $\lim_{n\to\infty} \|hz_n^2\| = 0$. Hence,

$$\|xz_n\| = \|z_n x^*\| = \|z_n x^* x z_n\|^{\frac{1}{2}} \leq \|z_n h z_n\|^{\frac{1}{2}} \to 0 \quad (n \to \infty),$$

and analogously, $\|z_n x\| = \|x^* z_n\| \to 0$ $(n \to \infty)$. Therefore

$$\lim_{n\to\infty} \|z_n - xx^*z_n - z_n x^* x + xx^* z_n x^* x\| = 1.$$

Now we use the symbolic notation: $y(1-x) = y - yx,\ (1-x)y = y - xy$.

We shall show that $(1-xx^*)\mathbb{A}(1-x^*x) = 0$. Suppose

$$a \in (1-xx^*)\mathbb{A}(1-x^*x),$$

and $\|a\| \leq 1$. Then

$$\|x \pm a\| = \|(x^* \pm a^*)(x \pm a)\|^{\frac{1}{2}} = \|x^*x \pm (x^*a + a^*x) + a^*a\|^{\frac{1}{2}}.$$

Since $a^*xx^*a = 0,\ x^*a = a^*x = 0$ and $x^*xa^*a = x^*x(1-x^*x)a^*a = 0$. Hence$\|x \pm a\| = \max(\|x^*x\|^{\frac{1}{2}}, \|a^*a\|^{\frac{1}{2}}) \leq 1$, so that by the extremity of $x$, $a = 0$.

On the other hand,

$$z_n - xx^*z_n - z_n x^* x + xx^* z_n x^* x \in (1-xx^*)\mathbb{A}(1-x^*x);$$

hence it is zero, a contradiction.

Therefore $h$ is invertible in $\mathbb{B}$, $h^{-1}h$ is the identity of $\mathbb{B}$, and so it is a projection in $\mathbb{A}$ and the identity of $h^{-1}h\mathbb{A}h^{-1}h$.

Suppose $\mathbb{A}(1-h^{-1}h) \neq 0$. Then there exists an element $a \neq 0$ in $\mathbb{A}(1-h^{-1}h)$. Since $a^*ah^{-1}h = 0$, $a^*a$ commutes with $h^{-1}h\mathbb{A}h^{-1}h \supseteq \mathbb{B}$. But $a \notin \mathbb{B}$, since $a \neq 0,\ h^{-1}h = 1_{\mathbb{B}}$, and $ah^{-1}h = 0$. This contradicts the maximality of $\mathbb{B}$. Hence $h^{-1}h$ is the identity of $\mathbb{A}$, completing the proof.

□

Recall the following:

**B.8. Theorem. [The Krein-Smulian Theorem]** *A convex set in the dual space $\mathfrak{X}^*$ of a Banach space $\mathfrak{X}$ is weak*-closed if and only if its intersection with every positive multiple of the closed unit ball in $\mathfrak{X}^*$ is weak*-closed.*

We shall use the Krein-Smulian Theorem to prove the following Lemma.

**B.9. Lemma.** *Let $\mathbb{A}$ be a $C^*$-algebra and $\mathfrak{X}$ be a Banach space such that $\mathbb{A}$ is isomorphic as a Banach space to $\mathfrak{X}^*$. Then $\mathbb{A}_h := \{a \in \mathbb{A} : a = a^*\}$is weak*-closed.*

*Furthermore, the positive cone $\mathbb{A}_+$ of $\mathbb{A}$ is also weak*-closed.*

**Proof.** By the Krein-Smulian Theorem above, it is sufficient to show that the unit ball $B_1(\mathbb{A}_h)$ is weak*-closed, for $\mathbb{A}_h$ is clearly convex. To that end, let $\{x_\alpha\}$ be a weak*-convergent net in $B_1(\mathbb{A}_h)$ and write the limit as $x + iy$, with $x, y \in \mathbb{A}_h$. Here, $x + iy \in B_1(\mathbb{A})$, which is weak*-closed by Alaoglu's Theorem. Then $\{x_\alpha + in\}$ is weak*-convergent to $x + i(y + n)$ for every $n$. Since $\|x_\alpha + in\| \leq (1 + n^2)^{\frac{1}{2}}$ and the norm is weak*-lower semicontinuous, we have

$$(1 + n^2)^{\frac{1}{2}} \geq \|x + i(n + y)\| \geq \|n + y\|.$$

If $y \neq 0$, we may assume that $\sigma(y)$ contains a number $\lambda > 0$ (passing, if necessary, to $\{-x_\alpha\}$). But then

$$\lambda + n \leq \|n + y\| \leq (1 + n^2)^{\frac{1}{2}}$$

for all $n$, a contradiction. Thus $y = 0$. Again, since the norm is weak*-lower semicontinuous, we also have $\|x\| \leq 1$, that is, $x \in B_1(\mathbb{A}_h)$.

As for the positive cone, it again suffices to show that the unit ball $B_1(\mathbb{A}_+)$ of $\mathbb{A}$ is weak*-closed. But then simply note that $B_1(\mathbb{A}_+) = \frac{1}{2}(B_1(\mathbb{A}_h) + 1)$, and translation and contraction do not affect weak*-closures.

□

**B.10. Definition.** *A $C^*$-algebra $\mathbb{A}$ is said to be* ***monotone complete*** *if each bounded increasing net in $\mathbb{A}_h$ has a least upper bound in $\mathbb{A}_h$.*

**B.11. Example.** *The most important example of a monotone complete $C^*$-algebra for our purposes is the space $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$. To see that this is indeed monotone complete, it suffices (by translation) to show that increasing bounded nets of positive operators have a least upper bound. We do this by showing that such nets converge strongly.*

*Let $\mathcal{H}$ be a Hilbert space and let $\{P_\alpha\}_{\alpha \in \Lambda}$ be a net of positive operators on $\mathcal{H}$ such that $0 \leq P_\alpha \leq P_\beta \leq I$ for $\alpha,\ \beta \in \Lambda$ with $\alpha \leq \beta$. Then there exits $P \in \mathcal{B}(\mathcal{H})$ such that $0 \leq P_\alpha \leq P \leq I$ for all $\alpha$ and the net $\{P_\alpha\}_{\alpha \in \Lambda}$ converges to $P$ strongly.*

***Proof.*** *Indeed, if $Q \in \mathcal{B}(\mathcal{H})$ with $0 \leq Q \leq 1$, then $0 \leq Q \leq Q^2 \leq I$, since $Q$ commutes with $(I - Q)^{\frac{1}{2}}$ by the functional calculus and*

$$\begin{aligned} < (Q - Q^2)x, x > &= < Q(I - Q)^{\frac{1}{2}}x, (I - Q)^{\frac{1}{2}}x > \\ &\geq 0 \end{aligned}$$

*for all $x \in \mathcal{H}$.*

*Moreover, for all $x$, the net $\{< P_\alpha x, x >\}$ is nondecreasing and is bounded above by $\|x\|^2$, and thus is a Cauchy net. Now for $\alpha \leq \beta$, we have*

$$\begin{aligned} \|(P_\beta - P_\alpha)x\|^2 &= < (P_\beta - P_\alpha)^2 x, x > \\ &\leq < (P_\beta - P_\alpha)x, x > \\ &= < P_\beta x, x > - < P_\alpha x, x > \end{aligned}$$

*and so $\{P_\alpha x\}$ is a Cauchy net with respect to the Hilbert space norm.*

*For $x \in \mathcal{H}$, let $Px = \lim_\alpha P_\alpha x$. Then $P$ is linear and $\|Px\| = \lim_\alpha \|P_\alpha x\| \leq \|x\|$, so that $\|P\| \leq 1$. Also,*

$$0 \leq \lim_\alpha < P_\alpha x, x > = < Px, x >,$$

*so that $P \geq 0$. This completes the proof.*

□

**B.12. Lemma.** *Let $\mathbb{A}$ be a $C^*$-algebra and $\mathfrak{X}$ be a Banach space such that $\mathbb{A}$ is isomorphic as a Banach space to $\mathfrak{X}^*$. Then $\mathbb{A}$ is monotone complete.*

**Proof.** Let $\{x_i\}$ be a bounded increasing monotone net of self-adjoint elements of $\mathbb{A}$. Since $B_1(\mathbb{A}_h)$ is weak*-compact (being convex, norm bounded and weak*-closed), there is a subnet $\{x_j\}$ of $\{x_i\}$ which is weak*-convergent to an element $x \in \mathbb{A}_h$.

For each $x_i$ we eventually have $x_j \geq x_i$ for $j \geq i$, and thus $x \geq x_i$ since $\mathbb{A}_+$ is weak*-closed. That is, consider the subnet $\{x_j - x_i\}_{j \geq i}$ which eventually lie in $\mathbb{A}_+$ and converges in the weak*-topology to $x - x_i$. In particular, $x$ is an upper bound for $\{x_i\}$ in $\mathbb{A}_h$.

If $y \in \mathbb{A}_h$ and $y \geq x_i$ for all $i$ then $y \geq x_j$ for all $j$, so that

$$y \geq \text{weak}^* - \lim x_j = x$$

as above. As such, $x$ is the least upper bound for $x_i$ and so $\mathbb{A}$ is monotone complete.

□

**B.13. Definition.** *Given a subset $\mathbb{M}$ of self-adjoint operators on some Hilbert space $\mathcal{H}$, we denote by $\mathbb{M}^m$ (resp. $\mathbb{M}_m$) the set of operators obtained by taking strong limits of increasing (resp. decreasing) nets in $\mathbb{M}$.*

Note that if $\mathbb{A}$ is a C*-algebra and $\mathbb{M} = \mathbb{A}_{sa}$, then $\mathbb{M}^m = \mathbb{M}_m$.

**B.14. Lemma.** *Let $\mathbb{A}$ be a $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and let $\mathbb{M}$ denote the strong operator closure of $\mathbb{A}$. If $P$ is a projection in $\mathbb{M}$ then given $x \in ranP$ and $y \in \operatorname{ran}P^{\perp}$ there is an element $B \in (\mathbb{M}_{sa})^m$ such that $Bx = x$ and $By = 0$.*

**Proof.** By Kaplansky's Density Theorem, we find find operators $A_n \in \mathbb{M}_+^1$ such that $\|A_n x - x\| < \frac{1}{2}$ and $\|A_n y\| < \frac{1}{n}2^{-n}$.

For $n < m$ define $B_{nm} = (1 + \sum_{k=n}^m kA_k)^{-1} \sum_{k=n}^m kA_k$. By spectral theory, $\|B_{nm}\| \le 1$, $B_{nm} \in \mathbb{M}_+$, and $B_{nm} \le \sum_{k=n}^m kA_k$.

Thus $< B_{nm}y, y > \le < \sum_{k=n}^m kA_k y, y > \le \sum_{k=n}^m 2_{-k} < 2^{-n+1}$.

Since $\sum_{k=n}^m kA_k \ge mA_m$, we have $B_{nm} \ge (1 + mA_m)^{-1} mA_m$ and so $1 - B_{nm} \le (1 + mA_m)^{-1}$. But $A_m \in \mathbb{M}_+^1$ implies that $(1 + mA_m) \le (1 + m)$, and hence $(1+m)^{-1} \le (1 + mA_m)^{-1}$. Then $(mA_m)^{\frac{1}{2}}(1+m)^{-1}(mA_m)^{\frac{1}{2}} \le (mA_m)^{\frac{1}{2}}(1 + mA_m)^{-1}(mA_m)^{\frac{1}{2}}$, and hence $(mA_m)(1+m)^{-1} \le (mA_m)(1 + mA_m)^{-1}$. It follows that $1 - (mA_m)(1 + mA_m)^{-1} \le 1 - (mA_m)(1+m)^{-1}$, i.e. $(1 + mA_m)^{-1} \le (1+m)^{-1}((1+m) - mA_m)$, so that

$$1 - B_{nm} \le ((1+m) - mA_m).$$

Thus

$$\begin{aligned} < (1 - B_{nm})x, x > \quad &\le < (1+m)^{-1}(1 + m(1 - A_m))x, x > \\ &= (1+m)^{-1}(< x, x > + m < (1 - A_m)x, x >) \\ &\le (1+m)^{-1}(1 + m(\tfrac{1}{m})) \\ &= 2(1+m)^{-1}. \end{aligned}$$

For fixed $n$, the sequence $\{B_{nm}\}$ is monotone increasing, and since it is norm bounded, it is strongly convergent to an element $0 \le B_n \in (\mathbb{M}_{sa})^m$. Moreover, $\|B_n\| \le 1$.

Since $B_{n+1\,m} \le B_{nm}$ for each $m > (n+1)$, we see that $B_{n+1} \le B_n$, so that the sequence $\{B_n\}$ is monotone decreasing and bounded. Again, it is strongly convergent to an element $B \ge 0$, which lies in $(\mathbb{M}_{sa})^m$, again, as $(\mathbb{M}_{sa})^m = (\mathbb{M}_{sa})_m$.

Note that $\| < B_n y, y > \| = \| \lim_m < B_{nm} y, y > \ \| \le 2^{-n+1}$, and $\| < (1 - B_n)x, x > \ \| = \| \lim_m (1 - B_{nm})x, x > \| \le 0$. Since $0 \le B_n \le 1$, we deduce that $< (1 - B_n)x, x >= 0$, and hence that $B_n x = x$.

Finally, as $0 \le B \le 1$, $\| < By, y > \| = \| \lim_n < B_n y, y > \| = 0$, implying that $By = 0$. Similarly, $Bx = \lim_n B_n x = x$, completing the proof. □

**B.15. Theorem.** *Let $\mathcal{H}$ be a Hilbert space. A unital $C^*$-algebra $\mathbb{M}$ of $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if $(\mathbb{M}_{sa})^m = \mathbb{M}_{sa}$.*

**Proof.** Suppose that $\mathbb{M}$ is a von Neumann algebra. Let $T \in (\mathbb{M}_{sa})^m$. Then $T \in \mathbb{M}$ as the latter is closed in the strong operator topology. Since $T \in \mathcal{B}(\mathcal{H})_{sa}$ by definition, $T \in \mathbb{M}_{sa}$.

Conversely, to prove that $\mathbb{M}$ is a von Neumann algebra, it suffices to show that each projection $P$ in the strong closure of $\mathbb{M}$ actually belongs to $\mathbb{M}$.

Suppose that $x \in P\mathcal{H}$ and $y \in (I-P)\mathcal{H}$. Then Lemma B.14 shows that there exists $R \in \mathbb{M}_+$ such that $Rx = x$ and $Ry = 0$. The range projection $P_{(x,y)}$ of $R$ belongs to $\mathbb{M}$. Indeed, the sequence $(\frac{1}{n} + R)^{-1}$ is monotone increasing, and converges strongly to $P_{(x,y)}$. Thus $P_{(x,y)}x = x$, and $P_{(x,y)}y = 0$. The projections $P_{(x,y_1)} \wedge P_{(x,y_2)} \wedge \ldots \wedge P_{(x,y_n)}$ form a decreasing net in $\mathbb{M}_+$, when $\{y_1, y_2, \ldots, y_n\}$ runs through the finite subsets of $(I-P)\mathcal{H}$. Thus the limit projection $P_x \leq P$, and lies in $\mathbb{M}_{sa}$. Clearly, $P$ is the limit of the increasing net of projections $P_{x_1} \vee P_{x_2} \vee \ldots \vee P_{x_k}$ where $\{x_1, x_2, \ldots, x_k\}$ runs over the finite subsets of $P\mathcal{H}$. Thus $P \in (\mathbb{M}_{sa})^m \subseteq \mathbb{M}$, completing the proof.

□

**B.16. Definition.** *Let $\mathbb{A}$ be a von Neumann algebra. Then $\phi \in \mathbb{A}^*$ is said to be* ***normal*** *if for each bounded monotone increasing net $\{x_i\}$ in $\mathbb{A}_h$ with $\lim x_i = x$ we have $\{\phi(x_i)\}$ converging to $\phi(x)$.*

*More generally, if $\mathbb{A}$ and $\mathbb{B}$ are von Neumann algebras, then a positive linear map $\rho$ of $\mathbb{A}$ into $\mathbb{B}$ is said to be* ***normal*** *if for each bounded monotone increasing net $\{x_i\}$ in $\mathbb{A}_h$, the net $\{\rho(x_i)\}$ increases to $\rho(x)$ in $\mathbb{B}_h$.*

**B.17. Lemma.** *If $\mathbb{A}$ is a unital monotone complete C*-algebra with a separating family of normal states, then there is a normal isomorphism of $\mathbb{A}$ onto a von Neumann algebra.*

**Proof.** Let $\mathcal{F}$ denote the separating family of normal states of $\mathbb{A}$ and consider the representation $\pi_{\mathcal{F}} = \oplus_{\phi\in\mathcal{F}}\pi_\phi$, acting on $\mathcal{H}_{\mathcal{F}} = \oplus_{\phi\in\mathcal{F}}\mathcal{H}_\phi$. Then $(\pi_{\mathcal{F}}, \mathcal{H}_{\mathcal{F}})$ is faithful. Indeed, if $x \geq 0$ lies in the kernel of $\pi_{\mathcal{F}}$, then

$$\begin{aligned}\phi(x) &= < \pi_\phi(x)\xi_\phi, \xi_\phi > \\ &= 0\end{aligned}$$

for each $\phi \in \mathcal{F}$, so that $x = 0$. Since $\ker\pi_{\mathcal{F}}$ is a C*-algebra, it is spanned by its positive elements, and therefore $\ker\pi_{\mathcal{F}} = \{0\}$.

Now if $\{x_\alpha\}$ is a bounded montone increasing net in $\mathbb{A}_{sa}$, then $\{x_\alpha\}$ has a least upper bound $x \in \mathbb{A}_{sa}$, as $\mathbb{A}$ is monotone complete. Also $\{\pi_{\mathcal{F}}(x_\alpha)\}_\alpha$ is a bounded monotone decreasing net in $\mathcal{B}(\mathcal{H}_{\mathcal{F}})$ as $\pi_{\mathcal{F}} \geq 0$, and thus $\{\pi_{\mathcal{F}}(x_\alpha)\}_\alpha$ has a least upper bound $y$ in $\mathcal{B}(\mathcal{H}_{\mathcal{F}})$, as $\mathcal{B}(\mathcal{H}_{\mathcal{F}})$ is monotone complete. Since $x \geq x_\alpha$ for all $\alpha$, $\pi_{\mathcal{F}}(x) \geq \pi_{\mathcal{F}}(x_\alpha)$ for all $\alpha$, and hence $\pi_{\mathcal{F}}(x) \geq y$.

However, if $\phi \in \mathcal{F}$, and $(\pi_\phi, \mathcal{H}_\phi, z_\phi)$ is the cyclic representation associated with $\phi$ via the GNS construction, then for all unitaries $u$ in $\mathbb{A}$,

$$\begin{aligned} < \pi_\phi(x)\pi_\phi(u)z_\phi, \pi_\phi(u)z_\phi > &= \phi(u^*xu) \\ &= \lim \phi(u^*x_\alpha u) \quad \text{as } \phi \text{ is normal} \\ &= \lim < \pi_\phi(x_\alpha)\pi_\phi(u)z_\phi, \pi_\phi(u)z_\phi > . \end{aligned}$$

Thus $(\pi_\phi(x) - y)\pi_\phi(u)z_\phi = 0$. But $\mathbb{A}$ is spanned by its unitaries, and hence

$$(\pi_\phi(x) - y)\,[\pi_\phi(\mathbb{A})z_\phi] = 0.$$

As $\mathcal{H}_\mathcal{F} = \oplus_{\phi\in\mathcal{F}}\mathcal{H}_\phi$, we conclude that $\pi_\mathcal{F}(x) = y$. Thus $\pi_\mathcal{F}(\mathbb{A})$ is monotone complete. By Theorem B.15, $\pi_\mathcal{F}(\mathbb{A})$ is a von Neumann algebra.

□

**B.18. Theorem.** *Let $\mathbb{A}$ be a $C^*$-algebra and $\mathfrak{X}$ be a Banach space such that $\mathbb{A}$ is isomorphic as a Banach space to $\mathfrak{X}^*$. Then $\mathbb{A}$ has a faithful representation as a von Neumann algebra with $\mathbb{A}_* = \mathfrak{X}$.*

**Proof.** Consider the weak*-topology on $\mathbb{A}$ arising from $\mathfrak{X}$, and identify $\mathfrak{X}$ with the weak*-continuous elements of $\mathfrak{X}^*$. Since the unit ball $B_1(\mathbb{A})$ is weak*-compact, it has an extremal point, by the Krein-Milman Theorem. Hence $\mathbb{A}$ is unital, by Proposition B.7.

By Lemma B.9, $\mathbb{A}_h$ is weak*-closed, as well as the positive cone $\mathbb{A}_+$ of $\mathbb{A}$.

It now follows that the positive cone of $\mathfrak{X}$, namely $\mathfrak{X}_+$, is separating for $\mathbb{A}$. For if $a \in \mathbb{A}_h$ and $-a \notin \mathbb{A}_+$, then since $\mathbb{A}_+$ is a weak*-closed cone in $\mathbb{A}_h$, by the Hahn-Banach Theorem there exists an element $\phi \in \mathfrak{X}_h$ such that $\phi(\mathbb{A}_+) \geq 0$ and $\phi(a) > 0$. Namely, we can think of $\mathbb{A}_h$ as a real vector space and obtain a real linear functional on $\mathbb{A}_h$ satisfying these conditions. Then we complexify $\phi$ to $\mathbb{A}$.

By Lemma B.12, $\mathbb{A}$ is monotone complete.

Suppose $\phi \in \mathfrak{X}_+$. Then $\phi$ is normal, since if $\{x_i\}$ is a monotone increasing net in $\mathbb{A}$ with least upper bound $x$, then

$$\lim \phi(x_i) \leq \phi(x) = weak^* - \lim \phi(x_j) \leq \lim \phi(x_i).$$

Note that $\lim \phi(x_i)$ exists since it is a bounded monotone increasing net in $\mathbb{R}$. Thus $\mathbb{A}$ is a monotone complete C*-algebra with a separating family (namely $B_1(\mathfrak{X})_+$) of normal states.

By Lemma B.17, $\mathbb{A}$ has a faithful representation as a von Neumann algebra. Moreover, from the GNS construction, we have that $\mathfrak{X}_+ \subseteq \mathbb{A}_*$.

Also, if $x \in \mathbb{A}_h$ and $x \neq 0$, then $\phi(x) \neq 0$ for some $\phi \in \mathfrak{X}^+$. Thus the linear span of $\mathfrak{X}_+$ is norm dense in $\mathfrak{X}$, from which we conclude that $\mathfrak{X} \subseteq \mathbb{A}_*$.

Since the compact topology in $B_1(\mathbb{A})$ is unique, the weak*- and the $\sigma$-weak topologies coincide. Hence $\mathbb{A}_* = \mathfrak{X}$.

□

# Lab Questions

**Question 1.** Let $\mathcal{A}$ be a Banach algebra. Find $\mathcal{A}^{-1}$ and $\Lambda_{\mathcal{A}} = \mathcal{A}^{-1}/\mathcal{A}_0^{-1}$ when:

(i) $\mathcal{A} = \mathbb{M}_n$;
(ii) $\mathcal{A} = \mathcal{T}_n$;
(iii) $\mathcal{A} = \mathcal{C}([0,1])$.

**Question 2.** Let $X$ denote a compact, Hausdorff space, and let $f \in \mathcal{C}(X)$.

(i) Determine $\sigma(f)$.
(ii) Is the answer the same when $f \in \mathcal{A}(\mathbb{D})$, the disk algebra?

**Question 3.** Let $f \in \mathcal{C}([0,1])$. Consider

$$\begin{array}{rccc} M_f : & \mathcal{C}([0,1]) & \to & \mathcal{C}([0,1]) \\ & g & \mapsto & fg. \end{array}$$

(i) Determine $\sigma(M_f)$;
(ii) Determine $\|M_f\|$.
(iii) Consider the eigenvalues of $M_f$. In particular, what happens if $f(x) = x$ for all $x \in [0,1]$?

**Question 4.** Let $\{e_n\}_{n=1}^{\infty}$ be the standard orthonormal basis for $\mathcal{H} = \ell^2(\mathbb{N})$. Consider the *diagonal operator*

$$\begin{array}{rccc} D : & \mathcal{H} & \to & \mathcal{H} \\ & e_n & \mapsto & d_n\, e_n, \end{array}$$

where $\{d_n\}_n$ is a bounded sequence of complex numbers.

(i) Determine $\sigma(D)$ and $(D-\lambda)^{-1}$ when $\lambda \notin \sigma(D)$.
(ii) Determine $\|D\|$.
(iii) Determine the set of eigenvalues of $D$. How does this set compare to $\sigma(D)$?

**Question 5.** Consider the following more general version of Question 4. Let $A_n \in \mathbb{M}_{k_n}$, $n \geq 1$. Suppose that there exists $M > 0$ such that $\|A_n\| \leq M$ for all $n \geq 1$.

(i) Is it true that $\sigma(\oplus_{n=1}^{\infty} A_n) = \overline{\cup_{n=1}^{\infty} \sigma(A_n)}$?
(ii) Now suppose that the sequence $\{k_n\}$ is bounded above. Does this make any difference to the solution of $(i)$?

**Question 6.** Find an example of an operator $A$ acting on an infinite dimensional space such that $\|A\| = 1$ but $\operatorname{spr}(A) = 1/2$.

**Question 7.** Let $F \in \mathcal{C}([0,1], \mathbb{M}_n)$. Describe $\int_0^1 F(t)\,dt$.

**Question 8.** Let $A \in \mathbb{M}_n$ and suppose that $f$ is analytic on an open neighbourhood of $\sigma(A)$. Describe $f(A)$. **Hint:** Consider Jordan canonical forms.

**Question 9.** Find a Banach algebra $\mathcal{A}$ such that for each $\varepsilon > 0$, there exist elements $a, b \in \mathcal{A}$ such that $\|a - b\| < \varepsilon$ and

(a) $\sigma(a)$ has only one component;
(b) $\sigma(b)$ has infinitely many components.

Now find a Banach algebra $\mathcal{B}$ for which given *any* $\varepsilon > 0$, there *do not* exist elements $a$ and $b$ of $\mathcal{B}$ satisfying the above two conditions.

**Question 10.** If $a \in \mathbb{M}_2$, does there exist $b \in \mathbb{M}_2$ such that $b^2 = a$? More generally, under what circumstances does $a$ have a square root?

**Question 11.** Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_n\}_{n=1}^\infty$. Let $S$ be the operator satisfying $Se_n = e_{n+1}$ for all $n \geq 1$. (Extend $S$ by linearity and continuity to all of $\mathcal{H}$.) Does there exist $B \in \mathcal{B}(\mathcal{H})$ such that $B^2 = S$?

**Question 12.** Let $f \in L^\infty([0,1], dx)$ and consider

$$\begin{array}{cccc} M_f: & L^2([0,1], dx) & \to & L^2([0,1], dx) \\ & g & \mapsto & fg. \end{array}$$

Find $\sigma(M_f)$.

**Question 13.** Which diagonal operators on $\ell^2(\mathbb{N})$ are compact? Which have dense range? Which are unitary? Which are positive? Which are self-adjoint?

**Question 14.** Suppose $T = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ as an operator on $\mathcal{H} \oplus \mathcal{H}$.

(i) Is $\sigma(A) \subseteq \sigma(T)$?
(ii) Is $\sigma(D) \subseteq \sigma(T)$?
(iii) What can be said about the sets of eigenvalues of $A$ and $D$ with respect to those of $T$?
(iv) Is $\sigma(T) \subseteq \sigma(A) \cup \sigma(D)$?

**Question 15.** Find an operator $T \in \mathcal{B}(\mathcal{H})$ such that $T$ is injective, the range of $T$ is dense, but $T$ is not invertible.

**Question 16.** Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Let $\{w_n\}_{n=1}^{\infty} \in \ell^{\infty}(\mathbb{N})$ and define

$$\begin{array}{rccc} W: & \mathcal{H} & \to & \mathcal{H} \\ & e_n & \mapsto & w_n e_{n+1} \end{array}$$

for each $n \geq 1$. Extend $W$ by linearity and continuity to all of $\mathcal{H}$. Such an operator is called a *unilateral forward weighted shift with weight sequence* $\{w_n\}_{n=1}^{\infty}$.

(i) Calculate $\|W\|$.
(ii) Calculate $\sigma(W)$ in the case where $w_n = 1$ for all $n$. This particular operator is called *the* unilateral shift.
(iii) Calculate $\mathrm{spr}(W)$ in general.
(iv) When is $W$ nilpotent?
(v) When is $W$ compact?
(vi) If $W$ is compact, compute $\sigma(W)$.
(vii) When is $W$ quasinilpotent? (Recall that $T \in \mathcal{B}(\mathcal{H})$ is *quasinilpotent* if $\sigma(W) = \{0\}$.)
(viii) Find a unilateral weighted shift $W$ of norm 1 such that $W$ is quasinilpotent but not nilpotent. Is it possible to find one that is nilpotent but not quasinilpotent?

**Question 17.** Let $\mathrm{Nil}(\mathbb{C}^n) = \{T \in \mathbb{M}_n : T^k = 0 \text{ for some } k \geq 1\}$. Find the operator norm closure of $\mathrm{Nil}(\mathbb{C}^n)$.

**Question 18.** Find the norm closure of the invertibles in the following Banach algebras:

(i) $\mathcal{A} = \mathbb{M}_n$.
(ii) $\mathcal{A} = \mathcal{T}_n$.
(iii) $\mathcal{A} = \mathcal{C}([0,1])$.
(iv) $\ell^{\infty}(\mathbb{N})$.

**Question 19.** Let $S$ be the unilateral shift opeator acting on a Hilbert space $\mathcal{H}$. Show that the distance from $S$ to the set of invertible operators on $\mathcal{H}$ is exactly 1.

**Question 20.** Find elements $R, T \in \mathcal{B}(\mathcal{H})$ such that

(a) $T$ is right invertible but not left invertible.
(b) $R$ is left invertible but not right invertible.

**Question 21.** Show that the set of left invertible elements of a Banach algebra $\mathcal{A}$ is open. (Alternatively, show that the set of right invertible elements of a Banach algebra is open.)

**Question 22.** Find a left topological divisor of 0 in $\mathcal{B}(\mathcal{H})$ which is not a right topological divisor of 0.

**Question 23.** Are the invertibles in $\mathcal{B}(\mathcal{H})$ dense?

**Question 24.** Let $T \in \mathcal{B}(\mathcal{H})$ and $p$ be a polynomial. Suppose that $p(T) \in \mathcal{K}(\mathcal{H})$. Must $T$ be compact?

**Question 25.** Suppose $\mathcal{H}$ is a Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that the matrix of $T$ with respect to this basis is $[t_{ij}]$ Finally, suppose that

$$\|T\|_2 := \left( \sum_{i,j} |t_{ij}|^2 \right)^{\frac{1}{2}} < \infty.$$

(i) Show that $\|T\| < \|T\|_2$.
(ii) Show that $T \in \mathcal{K}(\mathcal{H})$.
(iii) The set of all operators $T$ for which $\|T\|_2$ is finite is called the *Hilbert-Schmidt class* and is sometimes denoted by $\mathcal{C}_2(\mathcal{H})$. Show that $\mathcal{C}_2(\mathcal{H})$ is a *proper* subset of $\mathcal{K}(\mathcal{H})$.

**Question 26.** Find an example of a Banach algebra $\mathcal{A}$ with no multiplicative linear functionals.

**Question 27.** Determine all of the multiplicative linear functionals on $\ell_n^{\infty}$.

**Question 28.** Consider $\mathcal{T}_n$, the set of upper triangular $n \times n$ matrices.
(i) Find all of the maximal ideals of $\mathcal{T}_n$.
(ii) Find the Jacobson radical $\mathcal{J}_n$ of $\mathcal{T}_n$.
(iii) Let $a \in \mathcal{T}_n$. Let $\pi : \mathcal{T}_n \to \mathcal{T}_n/\mathcal{J}_n$ be the canonical map. Find $\|\pi(a)\|$.

**Question 29.** Consider the disk algebra $\mathcal{A}(\mathbb{D})$.
(i) Show that $f^*(z) = \overline{f(\overline{z})}$ defines an isometric involution on $\mathcal{A}(\mathbb{D})$.
(ii) Show that not every multiplicative linear functional on $\mathcal{A}(\mathbb{D})$ is self-adjoint.

**Question 30.** Let $\mathcal{F}$ be a self-adjoint family of operators on a Hilbert space $\mathcal{H}$. Let

$$\mathcal{F}' = \{T \in \mathcal{B}(\mathcal{H}) : TF = FT \text{ for all } F \in \mathcal{F}\}.$$

Show that if $\{T_\alpha\}$ is a net in $\mathcal{F}'$, and if $T \in \mathcal{B}(\mathcal{H})$ satisfies

$$(T_\alpha x, y) \to (Tx, y)$$

for all $x, y \in \mathcal{H}$, then $T \in \mathcal{F}'$. (This is the statement that $\mathcal{F}'$ is closed in the *weak operator toplogy* (WOT).)

**Question 31.** Let $\mathcal{A}$ be a $C^*$-algebra, and let $a, b$ lie in the positive cone of $\mathcal{A}$.

(i) Show that $a \leq b$ implies $a^{1/2} \leq b^{1/2}$.
(ii) Show that $a \leq b$ does NOT imply that $a^2 \leq b^2$.

**Question 32.** Give an example of a non-closed ideal in the $C^*$-algebra $\mathcal{C}(\mathbb{D})$ that is not self-adjoint.

**Question 33.** Show that the strong operator topology, the weak operator topology, and the norm topology coincide when the underlying Hilbert space is finite dimensional.

**Question 34.** Show that for infinite dimensional Hilbert spaces, the weak operator topology is stricly weaker than the strong operator topology, which in turn is strictly weaker than the norm topology.

**Question 35.** Show that the map $^* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ which takes $T \mapsto T^*$ is continuous in the weak operator topology, but not in the strong operator topology. Show that this is not the case for the map $L_B(A) = BA$ for all $A \in \mathcal{B}(\mathcal{H})$, where $B \in \mathcal{B}(\mathcal{H})$ is fixed.

**Question 36.** If $M$ and $N$ are compact normal operators with the same spectrum, then $C^*(M)$ is isometrically isomorphic to $C^*(N)$. Do $M$ and $N$ have to be unitarily equivalent?

**Question 37.** Find all of the multiplicative linear functionals on $c_0$.

**Question 38.** Let $W$ be a unilateral weighted shift as defined in Question 16. Suppose that the weight sequence for $W$ is $\{w_n\}_{n=1}^{\infty}$. Show that $W$ is unitarily equivalent to a weighted shift $V$ whose weight sequence is $\{|w_n|\}_{n=1}^{\infty}$.

Show that the conclusion does not change if we assume that $W$ is a *bilateral weighted shift*; that is, $W$ is defined as in Question 16, but the basis is indexed by $\mathbb{Z}$ rather than $\mathbb{N}$.

For this reason, in many applications it suffices to consider weighted shifts with non-negative weight sequence.

**Question 39.** Which weighted shifts (bilateral or unilateral) are:

(i) normal?
(ii) self-adjoint?
(iii) unitary?
(iv) essentially unitary?
(v) essentially normal?
(vi) essentially self-adjoint?

**Question 40.** Find the topological divisors of 0 in:

(i) $\mathcal{C}([0,1])$.
(ii) $\mathbb{M}_n$.
(iii) $\mathcal{C}_0(\mathbb{R})$.

**Question 41.** Show that two injective unilateral weighted shifts $W$ with weight sequence $\{w_n\}_{n=1}^{\infty}$ and $V$ with weight sequence $\{v_n\}_{n=1}^{\infty}$ are unitarily equivalent if and only if $|w_n| = |v_n|$ for all $n \geq 1$.

Is the same true for injective bilateral shifts?

**Question 42.** Give necessary and sufficient conditions for two diagonal operators to be unitarily equivalent.

**Question 43.** Give necessary and sufficient conditions for two diagonal operators to be similar. Compare this with Question 42. What can you conclude from this?

**Question 44.** What is the distance from the unilateral shift $S$ to the set $\mathcal{K}(\mathcal{H})$ of compact operators?

**Question 45.** Let $\ell_n^{\infty} = (\mathbb{C}^n, \|\cdot\|_{\infty})$ as a Banach algebra under pointwise multiplication. Find $\sum_{\ell_n^{\infty}}$.

# Bibliography


[AFHV84] C. Apostol, L.A. Fialkow, D.A. Herrero, and D. Voiculescu. *Approximation of Hilbert space operators II*, volume 102 of *Research Notes in Math.* Pitman Advanced Publishing Program, Boston, London, Melbourne, 1984.

[Ber71] I.D. Berg. An extension of the Weyl-von Neumann theorem to normal operators. *Trans. Amer. Math. Soc.*, 160:365–371, 1971.

[Bol90] B. Bollabás. *Linear Analysis, an introductory course*. Cambridge University Press, Cambridge, 1990.

[Bro78] S.W. Brown. Some invariant subspaces for subnormal operators. *Integr. Equat. Oper. Th.*, 1:310–333, 1978.

[CHM89] J.B. Conway, D.A. Herrero, and B.B. Morrel. *Completing the Riesz-Dunford Functional Calculus*, volume 417. Mem. Amer. Math. Soc., 1989.

[CM87] J.B. Conway and B.B. Morrel. Roots and logarithms of bounded operators on a hilbert space. *J. Funct. Anal.*, 70:171–193, 1987.

[Con78] J.B. Conway. *Functions of one complex variable*. Springer Verlag, New York, 1978.

[Con85] J.B. Conway. *A course in functional analysis*, volume 96. Springer Verlag, New York, 1985.

[CPY74] S.R. Caradus, W.E. Pfaffenberger, and B. Yood. *Calkin algebras and algebras of operators on Banach spaces*, volume 9 of *Lecture Notes Pure Appl. Math.* Marcel Dekker, Inc., New York, 1974.

[DS57] N. Dunford and J.T. Schwartz. *Linear Operators, Part I: General Theory.* Interscience, New York, 1957.

[Enf73] P. Enflo. A counterexample to the approximation problem in Banach spaces. *Acta Math.*, 130:309–317, 1973.

[Fol95] G.B. Folland. *A Course in Abstract Harmonic Analysis.* CRC Press, 1995.

[Hal68] P.R. Halmos. Quasitriangular operators. *Acta Sci. Math. (Szeged)*, 29:283–293, 1968.

[HNRR80] D.H. Hadwin, E.A. Nordgren, H. Radjavi, and P. Rosenthal. An operator not satisfying Lomonosov's hypothesis. *J. Funct. Anal.*, 38:410–415, 1980.

[Lom73] V.. Lomonosov. On invariant subspaces of families of operators commuting with a completely continuous operator. *Funkcional. Anal. i Priložen*, 7:55–56, 1973.

[Mur81] G.J. Murphy. Continuity of the spectrum and spectral radius. *Proc. Amer. Math. Soc.*, 82:619–621, 1981.

[Pry73] J.D. Pryce. *Basic methods of linear functional analysis.* Hutchinson University Press, 1973.

[Rea84] C.J. Read. A solution to the invariant subspace problem. *Bull. London Math. Soc.*, 16:337–401, 1984.

[Rea85] C.J. Read. A solution to the invariant subspace problem on the space $\ell_1$. *Bull. London Math. Soc.*, 17:305–317, 1985.

[Rie11] F. Riesz. Sur certains systèms singuliers d'équations intégrales. *Ann. Ecole Norm. Sup.*, 28:33–62, 1911.

[RR69] H. Radjavi and P. Rosenthal. The set of irreducible operators is dense. *Proc. Amer. Math. Soc.*, 21:256, 1969.

[Voi74] D. Voiculescu. Norm-limits of algebraic operators. *Rev. Roumaine Math. Pures Appl.*, 19:371–378, 1974.

[Voi76] D. Voiculescu. A non-commutative Weyl-von Neumann theorem. *Rev. Roumaine Math. Pures Appl.*, 21:97–113, 1976.

# Index