The similarity problem for Fourier algebras and corepresentations of group von Neumann algebras

Michael Brannan [a], Ebrahim Samei [b],*

[a] *Department of Mathematics and Statistics, Queen's University at Kingston, 99 University Avenue, Kingston, Ontario, Canada, K7L 3N6*
[b] *Department of Mathematics and Statistics, University of Saskatchewan, 106 Wiggins Road, Saskatoon, Saskatchewan, Canada, S7N 5E6*

Communicated by N. Kalton

Abstract

Let G be a locally compact group, and let $A(G)$ and $VN(G)$ be its Fourier algebra and group von Neumann algebra, respectively. In this paper we consider the similarity problem for $A(G)$: Is every bounded representation of $A(G)$ on a Hilbert space H similar to a $*$-representation? We show that the similarity problem for $A(G)$ has a negative answer if and only if there is a bounded representation of $A(G)$ which is not completely bounded. For groups with small invariant neighborhoods (i.e. SIN groups) we show that a representation $\pi : A(G) \to B(H)$ is similar to a $*$-representation if and only if it is completely bounded. This, in particular, implies that corepresentations of $VN(G)$ associated to non-degenerate completely bounded representations of $A(G)$ are similar to unitary corepresentations. We also show that if G is a SIN, maximally almost periodic, or totally disconnected group, then a representation of $A(G)$ is a $*$-representation if and only if it is a complete contraction. These results partially answer questions posed in Effros and Ruan (2003) [7] and Spronk (2002) [25].
© 2010 Elsevier Inc. All rights reserved.

Keywords: Fourier algebras; Group von Neumann algebras; Completely bounded homomorphisms; Corepresentations; SIN groups

* Corresponding author.
  E-mail addresses: mbrannan@mast.queensu.ca (M. Brannan), samei@math.usask.ca (E. Samei).

1. Introduction

Let $\mathcal{A}$ be a Banach $*$-algebra. The *similarity problem for $\mathcal{A}$* is the following question: Is every bounded representation of $\mathcal{A}$ as operators on a Hilbert space similar to a $*$-representation? The study of similarity problems for various classes of Banach $*$-algebras has its origins in the study of group representations. Let G be a locally compact group and let H be a Hilbert space. A strongly continuous (unital) representation $\pi : G \to B(H)$ is said to be *uniformly bounded* if $\|\pi\|_\infty := \sup_{x \in G} \|\pi(x)\| < \infty$. The similarity problem for G asks whether every uniformly bounded representation $\pi : G \to B(H)$ is similar to a unitary representation? In other words, is there an invertible operator $S \in B(H)$ such that the representation $\sigma : G \to B(H)$ defined by $\sigma(x) = S\pi(x)S^{-1}$ is unitary for all $x \in G$? If this is the case, π is said to *unitarizable*. We say that G is *unitarizable* if every uniformly bounded representation of G is unitarizable. In 1950, Day [4] and Dixmier [5] independently showed that if G is an amenable locally compact group, then G is unitarizable. Later on, the existence of non-unitarizable uniformly bounded representations was shown for several non-amenable groups such as $SL(2, \mathbb{R})$ and the non-commutative free groups. See for example [2,8,11,17,18,24,27,28]. It is still an open problem whether or not every unitarizable locally compact group is necessarily amenable [23]. In recent decades, various authors have applied the theory of completely bounded maps to study this similarity problem. One major result, due to Pisier [22] (see also [26, Theorem 6.11]), states that G is amenable if and only if for every uniformly bounded representation $\pi : G \to B(H)$, there exists $S \in B(H)$ invertible such that $S\pi(\cdot)S^{-1}$ is a unitary representation and $\|S\|\|S^{-1}\| \leqslant \|\pi\|_\infty^2$. These results rely heavily on operator space techniques. For a detailed discussion see [23].

Let dx denote a fixed left-invariant Haar measure on G. It is well known that there is a one-to-one correspondence between the strongly continuous uniformly bounded unital representations of G and bounded non-degenerate representations of the Banach $*$-algebra $L^1(G) := L^1(G, dx)$. This correspondence is given by

$$\pi : G \to B(H) \quad \longleftrightarrow \quad \pi_1 : L^1(G) \to B(H)$$

$$\pi_1(f) = \int_G f(x)\pi(x)\,dx \quad \left(f \in L^1(G) \right).$$

Furthermore, it can be shown that $\|\pi_1\|_{L^1(G) \to B(H)} = \|\pi\|_\infty$, that π is unitary if and only if π_1 is a $*$-representation, and that this happens if and only if π_1 is a (complete) contraction [7]. In particular, this implies that π_1 is similar to a $*$-representation if and only if π is unitarizable, and so the similarity problem for $L^1(G)$ is equivalent to the question of G being unitarizable.

The similarity problem for C^*-algebras is more commonly known as the *Kadison Similarity Problem*: Is every bounded representation $\pi : \mathcal{A} \to B(H)$ of a C^*-algebra $\mathcal{A}$ similar to a $*$-representation? Many partial results concerning this problem have been obtained, most notably due to Christensen [3], Haagerup [14], and Pisier [23]. In particular, Haagerup showed that $\pi : \mathcal{A} \to B(H)$ is similar to a $*$-representation if and only if π is a *completely bounded* representation of $\mathcal{A}$. Hence an important consequence of Haagerup's result is that the similarity problem for a C*-algebra $\mathcal{A}$ has a negative solution if and only if there is a bounded representation of $\mathcal{A}$ which is not completely bounded.

Our goal in this paper is to study the dual version of the similarity problem for $L^1(G)$. That is, we consider the Fourier algebra $A(G)$, and the question of when a bounded representation $\pi : A(G) \to B(H)$ is similar to a $*$-representation. In the language of Kac algebras [9] (or

more generally locally compact quantum groups [16]), $A(G)$ is interpreted as the dual object of $L^1(G)$ in the sense of generalized Pontryagin duality. In particular, when G is abelian, with dual group $\widehat{G}$, then $A(G) \cong L^1(\widehat{G})$ via the Fourier transform. Thus for an abelian group G, the representation theory of $A(G)$ coincides with the representation theory of $L^1(\widehat{G})$. In the general non-abelian setting though, very few results have been obtained on the structure of the representations of $A(G)$. In [7], Effros and Ruan used operator space tensor products to define Hopf algebraic structures on the preduals of Hopf von Neumann algebras. In this context, they asked whether every completely contractive representation of $A(G)$ on a Hilbert space is in fact a $*$-representation. Independently, motivated by the work of Paulsen [21, Theorem 9.1] and Pisier [22], Spronk in [25] asked whether every completely bounded representation of $A(G)$ on a Hilbert space is similar to a completely contractive representation. In this paper, we give partial affirmative answers to both of these questions as follows.

In Section 2, we give a brief introduction on the Fourier algebra $A(G)$, the group von Neumann algebra $VN(G)$, and the correspondence between completely bounded representations of $A(G)$ and the corepresentations of $VN(G)$.

In Section 3, we show that a bounded representation $\pi : A(G) \to B(H)$ is similar to a $*$-representation if and only if one of the following equivalent conditions holds:

(i) π and $\check{\pi}$ are completely bounded representations,
(ii) π and π^* are completely bounded representations.

Here $\check{\pi}$ and π^* are the bounded representations of $A(G)$ given by

$$\check{\pi}(u) = \pi(\check{u}), \qquad \pi^*(u) = \pi(\bar{u})^* \quad \left(u \in A(G)\right),$$

where $\check{u}(x) = u(x^{-1})$ for all $x \in G$. Furthermore, if π is non-degenerate and either (and consequently both of) (i) or (ii) is satisfied, we show that there exists a similarity $S \in B(H)$ taking π to the $*$-representation $S\pi(\cdot)S^{-1}$ such that $\|S\|\|S^{-1}\| \leqslant \|\pi\|_{cb}^2 \|\pi^*\|_{cb}^2$. As a consequence of these results, we obtain an analogous characterization for Fourier algebras to that of Haagerup's for C*-algebras: the similarity problem for the Fourier algebra $A(G)$ has a negative answer if and only if there is a bounded representation of $A(G)$ which is not completely bounded.

In Section 4, we show that there is a close connection between the similarity problem for $A(G)$ and the invertibility of corepresentations of $VN(G)$. One major result we obtain is that a non-degenerate completely bounded representation $\pi : A(G) \to B(H)$ is similar to a $*$-representation if and only if its associated corepresentation $V_\pi \in VN(G) \overline{\otimes} B(H)$ is an invertible operator. This, in particular, implies that V_π is similar to a unitary corepresentation.

When G is a SIN group, we improve our results in Section 3 and show that π is similar to a $*$-representation if and only if π is completely bounded, and that π is a $*$-representation if and only if it is completely contractive (Section 5). Furthermore, if π is non-degenerate, we show that there exists a similarity $S \in B(H)$ taking π to the $*$-representation $S\pi(\cdot)S^{-1}$ such that $\|S\|\|S^{-1}\| \leqslant \|\pi\|_{cb}^4$.

Finally, in Section 6 we use structure theory for locally compact groups to extend some of these results, and conclude that every completely contractive representation $\pi : A(G) \to B(H)$ is a $*$-representation whenever G is a totally disconnected, maximally almost periodic, or SIN group.

2. Preliminaries

2.1. The Fourier algebra

Let G be a locally compact group with a fixed left-invariant Haar measure dx. We denote by $\lambda : G \to B(L^2(G))$ the left regular representation of G on $L^2(G)$ and let $VN(G) = \lambda(G)'' \subseteq B(L^2(G))$ be the *group von Neumann algebra* of G. $VN(G)$ is a co-involutive Hopf von Neumann algebra with weak-$*$ continuous coproduct $\Gamma : VN(G) \to VN(G) \, \overline{\otimes} \, VN(G)$ defined by the equation

$$\Gamma\big(\lambda(x)\big) = \lambda(x) \otimes \lambda(x) \quad (x \in G),$$

and weak-$*$ continuous co-involution $\kappa : VN(G) \to VN(G)$ defined by

$$\kappa\big(\lambda(x)\big) = \lambda\big(x^{-1}\big) \quad (x \in G).$$

We refer to [9] for details regarding this.

The *Fourier algebra*, $A(G)$, is defined as the predual of $VN(G)$. By considering the pre-adjoint of the coproduct Γ on $VN(G)$, we obtain an associative product $\Gamma_* : A(G) \otimes A(G) \to A(G)$, making $A(G)$ a commutative completely contractive Banach algebra. The co-involution $\kappa : VN(G) \to VN(G)$ also induces an anti-linear completely isometric involution, $u \mapsto \bar{u}$ on $A(G)$, defined by

$$\langle \bar{u}, T \rangle = \overline{\langle u, \kappa(T)^* \rangle} \quad \big(u \in A(G), \ T \in VN(G)\big).$$

We can identify $A(G)$ with a dense $*$-subalgebra of $C_0(G)$ via the injective $*$-homomorphism $\hat{\lambda} : A(G) \to C_0(G)$ given by

$$\hat{\lambda}(u)(x) := \langle u, \lambda(x) \rangle \quad \big(u \in A(G), \ x \in G\big).$$

From now on, we will identify $A(G)$ with the $*$-subalgebra $\hat{\lambda}(A(G)) \subseteq C_0(G)$. Note that $A(G)$ consists precisely of those functions in $C_0(G)$ which are coefficients of the left regular representation. That is,

$$A(G) = \big\{ x \mapsto \langle \lambda(x) f | g \rangle = (\bar{g} * \check{f})(x) : \ f, g \in L^2(G) \big\},$$

where $*$ denotes the convolution of functions on G and $\check{f}(x) := f(x^{-1})$. Furthermore, the norm on $A(G)$ is given by

$$\|u\|_{A(G)} = \inf\big\{ \|f\|_2 \|g\|_2 : \ u = \langle \lambda(\cdot) f | g \rangle = \bar{g} * \check{f} \big\}.$$

We refer to [6] and the fundamental paper of Eymard [10] for details on these and other properties of the Fourier algebra.

For any left and right translation invariant space E of functions on G, we denote by L and R the natural left and right actions of G on E:

$$L_x f(y) = f\big(x^{-1}y\big), \qquad R_x f(y) = f(yx) \quad (x, y \in G, \ f \in E).$$

Given a complex function on G, we will also make frequent use of the so-called "check map" $f \mapsto \check{f}$ and "tilde map" $f \mapsto \tilde{f}$ where

$$\check{f}(x) = f(x^{-1}), \qquad \tilde{f}(x) = \overline{f(x^{-1})} \quad (x \in G).$$

Note in particular that the check map takes $A(G)$ onto itself and is isometric, since it can be readily seen as the pre-adjoint of the (isometric) co-involution $\kappa : VN(G) \to VN(G)$. This map however defines a completely bounded map on $A(G)$ if and only if G contains an open abelian subgroup of finite index. See [12, Proposition 1.5].

In this paper we will make extensive use of the natural structure of $A(G)$ as a left Banach $VN(G)$-module. We will quickly outline this structure here. See [10, Section 3.16] for a more detailed discussion. Given $u \in A(G)$, and $T \in VN(G)$, define $T \cdot u \in A(G)$ by

$$\langle T \cdot u, S \rangle := \langle u, \kappa(T)S \rangle \quad (S \in VN(G)).$$

It is readily checked that the operation $(T, u) \mapsto T \cdot u$ is indeed a contractive left action of $VN(G)$ on $A(G)$, and that pointwise, we have

$$(T \cdot u)(x) = \langle L_x \check{u}, T \rangle \quad (u \in A(G), \ T \in VN(G), \ x \in G). \tag{1}$$

Now suppose that $f : G \to \mathbb{C}$ is a Haar measurable function such that $f * L^2(G) \subseteq L^2(G)$. We will denote by $\lambda(f) \in VN(G) \subseteq B(L^2(G))$ the left-convolution operator associated to f. If $f \in L^2(G)$ and $T \in VN(G)$, we will always denote by $Tf \in L^2(G)$ the image of f under the operator T. We note here the very important fact that whenever $f \in A(G) \cap L^2(G)$, then $T \cdot f \in A(G) \cap L^2(G)$ and $(T \cdot f)(x) = (Tf)(x)$ for almost every $x \in G$ (see [10, Proposition 3.17]). Finally, we remark that if $f \in A(G) \cap L^1(G) \subseteq A(G) \cap L^2(G)$, then again $T \cdot f = Tf$ (almost everywhere) and consequently we have the equality of convolution operators

$$T\lambda(f) = \lambda(Tf) = \lambda(T \cdot f) \in VN(G). \tag{2}$$

2.2. Representations of $A(G)$ and corepresentations of $VN(G)$

Our standard reference for operator spaces and completely bounded maps will be [6]. In particular, we recall that $A(G)$, being the predual of a von Neumann algebra, comes equipped with a canonical operator space structure.

Let G be a locally compact group and let H be a Hilbert space. Given a bounded representation $\pi : A(G) \to B(H)$, we say that π is *completely bounded* if

$$\|\pi\|_{cb} := \sup_{n \in \mathbb{N}} \|\pi^{(n)}\| < \infty,$$

where $\pi^{(n)} : M_n(A(G)) \to M_n(B(H))$ is the nth amplification of π,

$$\pi^{(n)}[u_{ij}] = [\pi(u_{ij})] \quad ([u_{ij}] \in M_n(A(G))).$$

We say that π is a *completely contractive* representation if $\|\pi\|_{cb} \leqslant 1$.

If M and N are two von Neumann algebras with preduals M_* and N_*, recall that there is a completely isometric identification

$$CB(M_*, N) \cong (M_* \widehat{\otimes} N_*)^* \cong M \overline{\otimes} N, \tag{3}$$

where $CB(M_*, N)$ is the operator space of completely bounded linear maps from M_* into N, $\widehat{\otimes}$ is the operator space projective tensor product, and $\overline{\otimes}$ denotes the von Neumann spatial tensor product [6, Theorem 7.2.4]. The identification between the first two spaces is given by the dual pairing

$$\langle \omega_2, T\omega_1 \rangle = \langle \omega_1 \otimes \omega_2, \Phi_T \rangle,$$

where $T \in CB(M_*, N)$, $\omega_1 \in M_*$, $\omega_2 \in N_*$, and $\Phi_T \in (M_* \widehat{\otimes} N_*)^*$. The identification between the last two spaces is a non-commutative Fubini theorem, which relies on showing that $M_* \widehat{\otimes} N_* \cong (M \overline{\otimes} N)_*$ completely isometrically.

Given a Hilbert space H, an operator $V \in VN(G) \overline{\otimes} B(H)$ is called a *corepresentation of VN(G) on H* [20, Lemma 2.6] if

$$(\Gamma \otimes id)V = V_{1,3}V_{2,3}. \tag{4}$$

Here, we are using the standard *leg notation* for $V_{1,3}$ and $V_{2,3}$ [1, p. 428]: $V_{1,3}$ is the linear operator on the Hilbert space tensor product $L^2(G) \otimes^2 L^2(G) \otimes^2 H$ that acts as V on the first and the third tensor factor and as the identity on the second one. $V_{2,3}$ is defined similarly. If we let $\pi \in CB(A(G), B(H))$ be the completely bounded map corresponding to the operator $V \in VN(G) \overline{\otimes} B(H)$, it is readily checked that condition (4) on V is equivalent to π being multiplicative. Therefore the completely bounded representations of $A(G)$ on H are in one-to-one correspondence with the corepresentations of $VN(G)$ on H. Concretely, this correspondence is given by

$$\pi \longleftrightarrow V_\pi$$

where

$$\langle V_\pi(f \otimes \xi) | g \otimes \eta \rangle = \langle \pi(\bar{g} * \check{f})\xi | \eta \rangle \tag{5}$$

for all elementary tensors $f \otimes \xi, g \otimes \eta \in L^2(G) \otimes^2 H$. Furthermore, since the identification (3) is (completely) isometric, we always have

$$\|\pi\|_{cb} = \|V_\pi\|_{B(L^2(G) \otimes^2 H)}.$$

We note that it is shown in [19, Theorem A.1] that V_π is a unitary operator (i.e. a unitary corepresentation of $VN(G)$ on H) if and only if π is a non-degenerate $*$-representation of $A(G)$ on H.

Finally, given a bounded representation $\pi : A(G) \to B(H)$, observe that we can construct three additional bounded representations on H from π. These are

$$\check{\pi}, \pi^*, \tilde{\pi} : A(G) \to B(H)$$

and are defined by the formulae

$$\check{\pi}(u) = \pi(\check{u}), \qquad \pi^*(u) = \pi(\bar{u})^*, \qquad \tilde{\pi} = (\check{\pi})^* \quad (u \in A(G)). \tag{6}$$

Since the maps $u \mapsto \check{u}$, $u \mapsto \bar{u}$ on $A(G)$, and the adjoint map on $B(H)$ are all norm-preserving, it follows that

$$\|\pi\| = \|\check{\pi}\| = \|\pi^*\| = \|\tilde{\pi}\|.$$

Since the adjoint on $B(H)$ (for H infinite-dimensional) is never completely bounded and the check map on $A(G)$ is not completely bounded unless G has an abelian subgroup of finite index [12], we cannot infer complete boundedness for $\check{\pi}$ and π^* from the complete boundedness of π (and visa versa). However, we will show in Lemma 2 that for any completely bounded representation π, $\tilde{\pi}$ is completely bounded and

$$\|\pi\|_{cb} = \|\tilde{\pi}\|_{cb}.$$

Also if either of $\check{\pi}$ or π^* is completely bounded, then the other one is also completely bounded and we have

$$\|\check{\pi}\|_{cb} = \|\pi^*\|_{cb}.$$

3. Completely bounded representations of $A(G)$

Recall that if (X, μ) is a measure space and H is a Hilbert space, then $L^2(X) \otimes^2 H$ can be canonically identified with $L^2(X, H)$, the Hilbert space of strongly measurable functions $\varphi : X \to H$ such that $\int_X \|\varphi(x)\|^2 \, d\mu(x) < \infty$. This identification is given μ-almost everywhere by

$$(f \otimes \xi)(x) = f(x)\xi \quad \left(f \in L^2(X), \ \xi \in H\right).$$

Proposition 1. *Let H be a Hilbert space, let $\Phi \in CB(A(G), B(H))$ be a completely bounded map, and let $V_\Phi \in VN(G) \overline{\otimes} B(H)$ be the unique operator corresponding to Φ. Then for any $f \in A(G) \cap L^2(G)$ and $\xi \in H$, we have*

$$V_\Phi(f \otimes \xi)(x) = \Phi(L_x \check{f})\xi, \tag{7}$$

almost everywhere. In particular

$$\left\| V_\Phi(f \otimes \xi) \right\|_{L^2(G) \otimes^2 H} = \left(\int\limits_G \left\| \Phi(L_x \check{f})\xi \right\|^2 dx \right)^{1/2}.$$

Proof. Fix $f \in A(G) \cap L^2(G)$ and $\xi \in H$. It is well known that for $u \in A(G)$ the map $x \mapsto L_x u$ is continuous from G into $A(G)$. Consequently the function $x \mapsto \Phi(L_x \check{f})\xi$ belongs to $C_b(G, H)$, the Banach space of bounded continuous functions from G into H. For each $\eta \in H$ let $T_{\eta,\xi}^\Phi \in VN(G)$ be the coefficient operator defined by the dual pairing

$$\langle u, T_{\eta,\xi}^\Phi \rangle = \langle \Phi(u)\xi \,|\, \eta \rangle \quad \left(u \in A(G)\right).$$

Then for any $g \in L^2(G) \cap L^1(G)$,

$$\int_G \langle \Phi(L_x \check{f})\xi | \eta \rangle \overline{g(x)} \, dx = \int_G (T_{\eta,\xi}^{\Phi} \cdot f)(x) \overline{g(x)} \, dx \quad \text{(by (1))}$$

$$= \int_G (T_{\eta,\xi}^{\Phi} f)(x) \overline{g(x)} \, dx \quad \text{(since } f \in A(G) \cap L^2(G))$$

$$= \langle T_{\eta,\xi}^{\Phi} f | g \rangle$$

$$= \langle \Phi(\bar{g} * \check{f})\xi | \eta \rangle$$

$$= \langle V_{\Phi}(f \otimes \xi) | g \otimes \eta \rangle.$$

This shows that the conjugate-linear functional

$$\psi : (L^2(G) \cap L^1(G)) \otimes H \to \mathbb{C}$$

defined by

$$\psi(g \otimes \eta) = \int_G \langle \Phi(L_x \check{f})\xi | \eta \rangle \overline{g(x)} \, dx$$

coincides with the conjugate-linear functional

$$g \otimes \eta \mapsto \langle V_{\Phi}(f \otimes \xi) | g \otimes \eta \rangle.$$

From the density of $(L^2(G) \cap L^1(G)) \otimes H$ in $L^2(G) \otimes^2 H$, this implies that the function $x \mapsto \Phi(L_x \check{f})\xi$ belongs to $L^2(G, H) \cong L^2(G) \otimes^2 H$ and coincides with $V_{\Phi}(\xi \otimes f) \in L^2(G, H) \cong L^2(G) \otimes^2 H$ almost everywhere. $\square$

Lemma 2. *Let $\pi : A(G) \to B(H)$ be a bounded representation and consider the representations $\tilde{\pi}$, $\check{\pi}$, and π^* defined in (6). Then*:

(i) *π is completely bounded if and only if $\tilde{\pi}$ is completely bounded. In either case, $V_{\tilde{\pi}} = V_{\pi}^* \in VN(G) \overline{\otimes} B(H)$ and $\|\pi\|_{cb} = \|\tilde{\pi}\|_{cb}$.*

(ii) *$\check{\pi}$ is completely bounded if and only if π^* is completely bounded. In either case, $V_{\pi^*} = V_{\check{\pi}}^* \in VN(G) \overline{\otimes} B(H)$ and $\|\check{\pi}\|_{cb} = \|\pi^*\|_{cb}$.*

Proof. Note that it suffices to prove (i) because $\pi^* = \tilde{\check{\pi}}$ and therefore (ii) follows from (i) by applying (i) to the representation $\sigma = \check{\pi}$. We now prove (i). Suppose that π is completely bounded with associated corepresentation $V_{\pi} \in VN(G) \overline{\otimes} B(H)$. Then for any $\xi, \eta \in H$, $f, g \in L^2(G)$, we have

$$\langle \tilde{\pi}(\bar{g} * \check{f})\xi | \eta \rangle = \langle (\check{\pi})^*(\bar{g} * \check{f})\xi | \eta \rangle$$

$$= \langle \pi(\bar{f} * \check{g})^* \xi | \eta \rangle$$

$$= \langle \xi | \pi(\bar{f} * \check{g})\eta \rangle$$

$$= \langle f \otimes \xi | V_{\pi}(g \otimes \eta) \rangle$$

$$= \langle V_{\pi}^*(f \otimes \xi) | g \otimes \eta \rangle.$$

Thus the canonical identification (5) implies that $\tilde{\pi} \in CB(A(G), B(H))$, $V_{\tilde{\pi}} = V_{\pi}^*$, and so $\|\tilde{\pi}\|_{cb} = \|V_{\tilde{\pi}}\| = \|V_{\pi}^*\| = \|\tilde{\pi}\|_{cb}$. Since $\pi = \tilde{\tilde{\pi}}$, the converse is also true, completing the proof. $\square$

Now let $\pi : A(G) \to B(H)$ be a bounded representation. For each $\xi, \eta \in H$ let $T_{\eta,\xi}^{\pi} \in VN(G)$ be the coefficient operator defined by

$$\langle u, T_{\eta,\xi}^{\pi} \rangle = \langle \pi(u)\xi \,|\, \eta \rangle \quad \left(u \in A(G) \right). \tag{8}$$

Theorem 3. *Let $\pi : A(G) \to B(H)$ be a representation such that both π and π^* are completely bounded. (Or equivalently, by Lemma 2, π and $\check{\pi}$ are completely bounded.) Then for any $\xi, \eta \in H$ there exists a unique complex regular Borel measure $\mu_{\eta,\xi}^{\pi} \in M(G)$ such that $T_{\eta,\xi}^{\pi} = \lambda(\mu_{\eta,\xi}^{\pi})$. Furthermore,*

$$\left\| \mu_{\eta,\xi}^{\pi} \right\|_{M(G)} \leqslant \|\pi\|_{cb} \|\pi^*\|_{cb} \|\xi\| \|\eta\|. \tag{9}$$

Proof. Fix $\xi, \eta \in H$. To show that $T_{\eta,\xi}^{\pi} = \lambda(\mu_{\eta,\xi}^{\pi}) \in \lambda(M(G))$ with the claimed norm estimate, it suffices by Wendel's theorem [19, Theorem 1] to show that $T_{\eta,\xi}^{\pi}$ defines a right centralizer of $L^1(G)$ with the same norm estimate as the right-hand side of (9). Since $VN(G) = \lambda(G)'' = \rho(G)'$, where $\rho : G \to B(L^2(G))$ is the right regular representation of G, the operator $T_{\eta,\xi}^{\pi}$ automatically commutes with right translations by elements from G. We therefore only need to show that for any $f \in L^1(G)$,

$$T_{\eta,\xi}^{\pi} \lambda(f) \in \lambda\left(L^1(G) \right)$$

with

$$\left\| T_{\eta,\xi}^{\pi} \lambda(f) \right\|_{L^1(G)} \leqslant \|\pi\|_{cb} \|\pi^*\|_{cb} \|\xi\| \|\eta\| \|f\|_1.$$

To begin, let $f \in L^1(G)$ be of the form $f = gh$ with $g, h \in A(G) \cap L^2(G)$. Note that in this case $f = gh \in A(G) \cap L^1(G) \subseteq A(G) \cap L^2(G)$, and therefore by (2), we have $T_{\eta,\xi}^{\pi} \lambda(f) = \lambda(T_{\eta,\xi}^{\pi} \cdot f)$. Thus,

$$\left\| T_{\eta,\xi}^{\pi} \lambda(f) \right\|_{L^1(G)} = \left\| \lambda\left(T_{\eta,\xi}^{\pi} \cdot f \right) \right\|_{L^1(G)}$$

$$= \left\| T_{\eta,\xi}^{\pi} \cdot f \right\|_{L^1(G)}$$

$$= \int_G \left| (T_{\eta,\xi}^{\pi} \cdot f)(x) \right| dx$$

$$= \int_G \left| \langle L_x(\check{g}\check{h}), T_{\eta,\xi}^{\pi} \rangle \right| dx$$

$$= \int_G \left| \langle \pi\left((L_x \check{g})(L_x \check{h}) \right) \xi \,|\, \eta \rangle \right| dx$$

$$= \int_G \left| \langle \pi(L_x \check{g}) \pi(L_x \check{h}) \xi \,|\, \eta \rangle \right| dx$$

$$= \int_G \left| \langle \pi(L_x \check{h}) \xi \,|\, \pi(L_x \check{g})^* \eta \rangle \right| dx$$

$$\leqslant \int_G \left\| \pi(L_x \check{h}) \xi \right\| \left\| \pi(L_x \check{g})^* \eta \right\| dx$$

$$\leqslant \left(\int_G \left\| \pi(L_x \check{h}) \xi \right\|^2 dx \right)^{1/2} \left(\int_G \left\| \pi(L_x \check{g})^* \eta \right\|^2 dx \right)^{1/2}.$$

Now let $V_\pi, V_{\pi^*} \in VN(G) \,\overline{\otimes}\, B(H)$ denote the corepresentations associated to π and π^*, respectively. By Proposition 1 we have

$$\left(\int_G \left\| \pi(L_x \check{h}) \xi \right\|^2 dx \right)^{1/2} = \left\| V_\pi (h \otimes \xi) \right\|_{L^2(G) \otimes^2 H},$$

and

$$\left(\int_G \left\| \pi(L_x \check{g})^* \eta \right\|^2 dx \right)^{1/2} = \left\| V_{\pi^*} (\overline{g} \otimes \eta) \right\|_{L^2(G) \otimes^2 H},$$

giving

$$\left\| T_{\eta,\xi}^\pi \lambda(f) \right\|_{L^1(G)} \leqslant \left\| V_\pi (h \otimes \xi) \right\|_{L^2(G) \otimes^2 H} \left\| V_{\pi^*} (\overline{g} \otimes \eta) \right\|_{L^2(G) \otimes^2 H}$$

$$\leqslant \left\| V_\pi \right\| \|h\|_2 \|\xi\| \left\| V_{\pi^*} \right\| \|g\|_2 \|\eta\|$$

$$= \left\| \pi \right\|_{cb} \left\| \pi^* \right\|_{cb} \|g\|_2 \|h\|_2 \|\xi\| \|\eta\|.$$

Now suppose $f \in L^1(G)$ is arbitrary. Let $g, h \in L^2(G)$ be chosen so that $f = gh$ and $\|f\|_1 = \|g\|_2 \|h\|_2$. Since $A(G) \cap L^2(G)$ is norm dense in $L^2(G)$, one can easily show (by approximating g and h by sequences in $A(G) \cap L^2(G)$) that the preceding inequality extends by continuity to this situation. That is,

$$\left\| T_{\eta,\xi}^\pi \lambda(f) \right\|_{L^1(G)} \leqslant \left\| \pi \right\|_{cb} \left\| \pi^* \right\|_{cb} \|g\|_2 \|h\|_2 \|\xi\| \|\eta\| = \left\| \pi \right\|_{cb} \left\| \pi^* \right\|_{cb} \|f\|_1 \|\xi\| \|\eta\|.$$

Therefore $T_{\eta,\xi}^\pi$ is a right centralizer of $L^1(G)$ with norm no larger than

$$\left\| \pi \right\|_{cb} \left\| \pi^* \right\|_{cb} \|\xi\| \|\eta\|,$$

completing the proof. $\square$

Interestingly, Theorem 3 provides an elementary "operator space" proof of Eymard's theorem [10, Theorem 3.34] characterizing the Gelfand spectrum of the Fourier algebra.

Corollary 4 *(Eymard's theorem). For any locally compact group G, the Gelfand spectrum $\Sigma_{A(G)}$ of $A(G)$ is precisely the group G itself.*

Proof. Let $\chi \in \Sigma_{A(G)}$ be any character of $A(G)$. Then $\check{\chi}$ is also a character of $A(G)$. Since all bounded linear functionals on an operator space are automatically completely bounded, we may apply Theorem 3 to the one-dimensional representation $\chi : A(G) \to B(\mathbb{C}) = \mathbb{C}$ to get that $\chi = T_{1,1}^{\check{\chi}} = \lambda(\mu^\chi)$ for some measure $\mu^\chi \in M(G)$. Since $A(G)$ is a dense subalgebra of $C_0(G)$, $\chi = \lambda(\mu^\chi)$ extends uniquely to a character of $C_0(G)$. By Gelfand theory for the commutative C*-algebra $C_0(G)$, μ^χ must correspond to point evaluation at some $x \in G$. Conversely, any $x \in G$ gives rise to a character of $A(G)$ by evaluation at x, completing the proof. $\quad\square$

Before stating the main result of this section, we would first like to make the following remark concerning the possible degeneracy of the representations of $A(G)$ that we consider.

Remark 5. Let $\mathcal{A}$ be a Banach algebra, H a Hilbert space, and let $\pi : \mathcal{A} \to B(H)$ be a bounded representation. Recall that the *essential space of π* is the closed subspace $H_e := \overline{\text{span}}\{\pi(\mathcal{A})H\} \subseteq H$, that π is *non-degenerate* if $H_e = H$, and that π is *degenerate* if $H_e \neq H$. It is clear that for any representation $\pi : \mathcal{A} \to B(H)$, the subrepresentation

$$\pi_e := \pi(\cdot)|_{H_e} : \mathcal{A} \to B(H_e)$$

is always non-degenerate. We call π_e the *essential part of π*.

In the literature (see [3,14,21,23] for example), authors generally only consider the similarity problem for non-degenerate representations of Banach $*$-algebras. However, this assumption of non-degeneracy is not really needed as long one assumes the Banach $*$-algebra $\mathcal{A}$ under consideration has a bounded two-sided approximate identity. This useful fact is probably well known, but we present a proof this here for completeness.

Proposition 6. *Let $\mathcal{A}$ be a Banach $*$-algebra with a bounded two-sided approximate identity $\{e_\alpha\}_\alpha$, and let $\pi : \mathcal{A} \to B(H)$ be a bounded representation with essential part π_e. If π_e is similar to a $*$-representation, then so is π.*

Proof. Let $Q \in B(H)$ be a weak operator topology cluster point of the bounded net $\{\pi(e_\alpha)\}_\alpha \subseteq B(H)$. A routine calculation shows that Q is an idempotent with range equal to H_e, the essential space of π. Furthermore, if $M = \sup_\alpha \|e_\alpha\|$, then $\|Q\| \leqslant M \|\pi\|$. Write H as the orthogonal direct sum $H = H_e \oplus H_e^\perp$, and relative to this decomposition define $S \in B(H) = B(H_e \oplus H_e^\perp)$ to be the invertible operator given by

$$S(\xi_1, \xi_2) = (\xi_1 + Q\xi_2, \xi_2) \quad (\xi_1 \in H_e, \ \xi_2 \in H_e^\perp),$$

$$S^{-1}(\xi_1, \xi_2) = (\xi_1 - Q\xi_2, \xi_2).$$

Since $\pi(a)Q = \pi(a)$ for all $a \in \mathcal{A}$, we have

$$S\pi(a)S^{-1}(\xi_1, \xi_2) = S\pi(a)(\xi_1 - Q\xi_2, \xi_2)$$

$$= S\big(\pi(a)\xi_1 - \pi(a)Q\xi_2 + \pi(a)\xi_2, 0\big)$$

$$= S\big(\pi(a)\xi_1, 0\big)$$
$$= \big(\pi(a)\xi_1, 0\big) = \big(\pi_e(a)\xi_1, 0\big).$$

That is,

$$S\pi(\cdot)S^{-1} = \pi_e \oplus 0_{H_e^\perp}.$$

Now suppose that π_e is similar to a $*$-representation. Then there exists a $*$-representation $\sigma : \mathcal{A} \to B(H_e)$ and an invertible operator $T \in B(H_e)$ such that $\pi_e = T\sigma T^{-1}$. This implies that

$$\pi = S^{-1}(\pi_e \oplus 0_{H_e^\perp})S = S^{-1}(T \oplus I_{H_e^\perp})(\sigma \oplus 0_{H_e^\perp})(T \oplus I_{H_e^\perp})^{-1}S,$$

so π is similar to the $*$-representation $\sigma \oplus 0_{H_e^\perp}$. $\square$

Remark 7. Note that if we assume in Proposition 6 that $\mathcal{A}$ has a contractive approximate identity and $\|\pi\| \leqslant 1$, then the idempotent Q constructed above is a contraction. Therefore Q is actually the orthogonal projection from H onto the essential space H_e, and it follows from this that $\pi = \pi_e \oplus 0_{H_e^\perp}$.

We are now ready to state the main result of this section.

Theorem 8. *Let $\pi : A(G) \to B(H)$ be a bounded representation. Then the following are equivalent*:

(i) π *is similar to a $*$-representation.*
(ii) π *and $\check{\pi}$ are completely bounded representations.*
(iii) π *and π^* are completely bounded representations.*

Furthermore, if π is non-degenerate and (i)–(iii) are true, then there exists an invertible operator $S \in B(H)$ such that $S\pi(\cdot)S^{-1}$ is a $*$-representation of $A(G)$ and

$$\|S\|\big\|S^{-1}\big\| \leqslant \|\pi\|_{cb}^2\big\|\pi^*\big\|_{cb}^2 = \|\pi\|_{cb}^2\|\check{\pi}\|_{cb}^2.$$

Proof. The proof of (ii) $\Leftrightarrow$ (iii) follows from Lemma 2.

We now prove that (iii) $\Rightarrow$ (i): If π and π^* are both completely bounded representations, then for each $\xi, \eta \in H$, Theorem 3 implies the existence of a unique measure $\mu_{\eta,\xi}^\pi \in M(G)$ such that the coefficient operator $T_{\eta,\xi}^\pi \in VN(G)$ defined in (8) is given by $T_{\eta,\xi}^\pi = \lambda(\mu_{\eta,\xi}^\pi)$, and

$$\big\|\mu_{\eta,\xi}^\pi\big\|_{M(G)} \leqslant \|\pi\|_{cb}\big\|\pi^*\big\|_{cb}\|\xi\|\|\eta\|.$$

Thus for any $u \in A(G)$ we have

$$\big\|\pi(u)\big\| = \sup_{\{\xi,\eta \in H:\ \|\xi\|=\|\eta\|=1\}} \big|\langle \pi(u)\xi \,|\, \eta\rangle\big|$$

$$= \sup_{\{\xi,\eta \in H:\ \|\xi\|=\|\eta\|=1\}} \big|\langle u, T_{\eta,\xi}^\pi\rangle\big|$$

$$
= \sup_{\{\xi, \eta \in H: \ \|\xi\| = \|\eta\| = 1\}} \left| \int_G u(x) \, d\mu_{\eta, \xi}^{\pi}(x) \right|
$$

$$
\leqslant \|u\|_{\infty} \sup_{\{\xi, \eta \in H: \ \|\xi\| = \|\eta\| = 1\}} \left\| \mu_{\eta, \xi}^{\pi} \right\|_{M(G)}
$$

$$
\leqslant \|u\|_{\infty} \|\pi\|_{cb} \left\| \pi^* \right\|_{cb}.
$$

Consequently π is continuous with respect to the $\| \cdot \|_{\infty}$-norm on $A(G)$. Since $A(G)$ is dense in $C_0(G)$, π extends uniquely to a bounded representation $\pi_0 : C_0(G) \to B(H)$ with $\|\pi_0\| \leqslant \|\pi\|_{cb} \|\pi^*\|_{cb}$. Since $C_0(G)$ is a commutative (in particular nuclear) C^*-algebra, [3, Theorem 4.1] together with Proposition 6 imply that π_0 is similar to a $*$-representation $\sigma : C_0(G) \to B(H)$. From this, we get that $\pi = \pi_0|_{A(G)}$ is similar to the $*$-representation $\sigma|_{A(G)}$. In particular, if π (and therefore π_0) is non-degenerate, then by [3, Theorem 4.1], there exists an invertible operator $S \in B(H)$ such that $S^{-1}\pi_0(\cdot)S$ (and therefore $S^{-1}\pi(\cdot)S$) is a $*$-representation, and

$$
\|S\| \|S^{-1}\| \leqslant \|\pi_0\|^2 \leqslant \|\pi\|_{cb}^2 \|\pi^*\|_{cb}^2.
$$

Finally, we show that (i) $\Rightarrow$ (ii): First note that if $\sigma : A(G) \to B(H)$ is any $*$-representation, then σ is a complete contraction. Indeed, since any $*$-representation of $A(G)$ extends uniquely to a $*$-representation of the universal enveloping C^*-algebra $C^*(A(G)) \cong C_0(G)$, it follows that for any $[u_{ij}] \in M_n(A(G))$,

$$
\left\| \sigma^{(n)}[u_{ij}] \right\|_{M_n(B(H))} \leqslant \left\| [u_{ij}] \right\|_{M_n(C_0(G))} \leqslant \left\| [u_{ij}] \right\|_{M_n(A(G))}.
$$

Now, if we suppose that the representation $\pi : A(G) \to B(H)$ is similar to the $*$-representation $\sigma : A(G) \to B(H)$, then π is similar to a complete contraction. In particular, π must be completely bounded. Furthermore, since $\check{\pi}$ will also be similar to the $*$-representation $\check{\sigma}$, which is again completely contractive, we get that $\check{\pi}$ is completely bounded as well. $\quad\square$

Corollary 9. *Let $\pi : A(G) \to B(H)$ be a completely bounded representation. Then π is similar to a $*$-representation if and only if there is an invertible operator $S \in B(H)$ such that the representation $S^{-1}\pi(\cdot)S$ maps $A(G)$ into a subhomogeneous von Neumann algebra.*

Proof. Suppose that there is an invertible operator $S \in B(H)$ and a subhomogeneous von Neumann algebra $M \subset B(H)$ such that

$$
\rho(u) = S^{-1}\pi(u)S \in M \quad \left(u \in A(G) \right).
$$

Then ρ is a completely bounded representation of $A(G)$ on H. Moreover, since the adjoint map is completely bounded on M, it follows that ρ^* is also completely bounded. Therefore by the preceding theorem ρ is similar to a $*$-representation, and so, the same holds for π as well.

Conversely, suppose that there is an invertible operator $T \in B(H)$ and a $*$-representation $\sigma : A(G) \to B(H)$ such that $\sigma(u) = T^{-1}\pi(u)T$ for every $u \in A(G)$. Then $T\sigma(A(G))T^{-1}$ is commutative $*$-subalgebra of $B(H)$ so that the von Neumann algebra generated by $T\sigma(A(G))T^{-1}$ is commutative, and in particular, subhomogeneous. $\quad\square$

Note that if H is a finite-dimensional Hilbert space and $\pi : A(G) \to B(H)$ is any bounded representation, then $B(H) \cong M_n(\mathbb{C})$ is subhomogeneous and π is automatically completely bounded (see [6, Proposition 2.2.2]), therefore we obtain the following corollary.

Corollary 10. *Every bounded representation $\pi : A(G) \to B(H)$ with H a finite-dimensional Hilbert space is similar to a $\ast$-representation.*

We finish this section with the following corollary which is analogous to the main result of Haagerup in [14].

Corollary 11. *Let G be a locally compact group. Then the similarity problem for $A(G)$ has a negative solution if and only if there is a bounded representation of $A(G)$ which is not completely bounded.*

Proof. If every bounded representation of $A(G)$ is similar to a $\ast$-representation, then every such representation is automatically completely bounded by Theorem 8. Conversely, suppose that there is a bounded representation $\pi : A(G) \to B(H)$ that is not similar to a $\ast$-representation. Then, by Theorem 8, either π or π^* is not completely bounded. $\square$

4. Invertible corepresentations

Theorem 8 says that a completely bounded representation $\pi : A(G) \to B(H)$ is similar to a $\ast$-representation if and only if the bounded representation π^* (or equivalently $\check{\pi}$) is also completely bounded. In this section we show that if the corepresentation $V_\pi \in VN(G) \, \overline{\otimes} \, B(H)$ associated to π is assumed to be an invertible operator, then π^* and $\check{\pi}$ are automatically completely bounded, and therefore π is similar to a $\ast$-representation. To obtain this result, we need a few preparatory lemmas.

Lemma 12. *Let $\pi : A(G) \to B(H)$ be a bounded representation. Fix $u \in A(G) \cap C_c(G)$, $\xi, \eta \in H$, and consider the coefficient operator $T^\pi_{\eta, \pi(u)\xi} \in VN(G)$ defined in (8). Then for any $f \in A(G) \cap C_c(G)$ we have*

$$T^\pi_{\eta, \pi(u)\xi} \cdot f \in C_c(G) \cap A(G)$$

and

$$\int_G \left(T^\pi_{\eta, \pi(u)\xi} \cdot f \right)(x) \, dx = \left(\int_G f(x) \, dx \right) \langle \pi(u)\xi \, | \, \eta \rangle.$$

Proof. Consider the function $T^\pi_{\eta, \pi(u)\xi} \cdot f \in A(G)$. We have from (1) and (8) that

$$\left(T^\pi_{\eta, \pi(u)\xi} \cdot f \right)(x) = \left\langle \pi \big((L_x \check{f}) u \big) \xi \, | \, \eta \right\rangle$$

for all $x \in G$. Since f and u are compactly supported, the continuous map $x \mapsto (L_x \check{f})u$ from G into $A(G)$ is compactly supported. Indeed, $(L_x \check{f})u \neq 0$ only if $x \in \operatorname{supp}(u) \operatorname{supp}(f)$. In particular, $T^\pi_{\eta, \pi(u)\xi} \cdot f$ is only non-zero on the compact set $\operatorname{supp}(u) \operatorname{supp}(f) \subseteq G$.

Now choose $\varphi \in C_c(G)$ so that $\varphi = 1$ on $\operatorname{supp}(u) \operatorname{supp}(f)$. Then by the above considerations we have

$$
\int_G \left(T^\pi_{\eta, \pi(u)\xi} \cdot f \right)(x)\, dx = \int_G \varphi(x) \left(T^\pi_{\eta, \pi(u)\xi} \cdot f \right)(x)\, dx
$$
$$
= \left\langle T^\pi_{\eta, \pi(u)\xi} f \,|\, \bar\varphi \right\rangle
$$
$$
= \left\langle \varphi * \check f, T^\pi_{\eta, \pi(u)\xi} \right\rangle
$$
$$
= \left\langle \pi\left((\varphi * \check f) u \right)\xi \,|\, \eta \right\rangle.
$$

But for all $z \in \operatorname{supp}((\varphi * \check f)u) \subseteq \operatorname{supp}(u)$, we have

$$
\left((\varphi * \check f) u \right)(z) = u(z) \int_G \varphi(zx) f(x)\, dx
$$
$$
= u(z) \int_G f(x)\, dx.
$$

That is $(\varphi * \check f) u = (\int_G f(x)\, dx) u$, giving

$$
\int_G \left(T^\pi_{\eta, \pi(u)\xi} \cdot f \right)(x)\, dx = \left\langle \pi\left((\varphi * \check f) u \right)\xi \,|\, \eta \right\rangle = \left(\int_G f(x)\, dx \right) \left\langle \pi(u)\xi \,|\, \eta \right\rangle. \qquad \square
$$

Lemma 13. *Let $\pi : A(G) \to B(H)$ be a completely bounded representation, and let $V_\pi \in VN(G) \,\bar\otimes\, B(H)$ be the associated corepresentation of π. If V_π has dense range, then π is non-degenerate.*

Proof. Suppose V_π has dense range. We need to show that $H_0 = \operatorname{span}\{\pi(A(G))H\}$ is dense in H, or equivalently, that $H_0^\perp = \{0\}$. Let $\eta \in H_0^\perp$. Then for all $\xi \in H$ and $f, g \in L^2(G)$, we have

$$
0 = \left\langle \pi(\bar g * \check f)\xi \,|\, \eta \right\rangle = \left\langle V_\pi(f \otimes \xi) \,|\, g \otimes \eta \right\rangle.
$$

By linearity and the density of $V_\pi(L^2(G) \otimes H)$ in $L^2(G) \otimes^2 H$, this implies that $g \otimes \eta = 0$ for all $g \in L^2(G)$. Therefore $\eta = 0$. $\quad\square$

Remark 14. The converse of Lemma 13 is in fact also true: If $\pi : A(G) \to B(H)$ is a non-degenerate completely bounded representation, then V_π has dense range. Since we will not directly use this fact, we shall omit the proof.

Theorem 15. *Let $\pi : A(G) \to B(H)$ be a completely bounded representation such that the associated corepresentation $V_\pi \in VN(G) \,\bar\otimes\, B(H)$ is invertible. Then $\check\pi$ and π^* are completely bounded representations, and $V_\pi^{-1} = V_{\check\pi}$.*

Proof. From Lemma 2 we know that $V_{\tilde{\pi}} = V_{\pi}^*$ where $\tilde{\pi}$ is the representation defined in (6). Since V_{π} and $V_{\tilde{\pi}} = V_{\pi}^*$ are both surjective operators, Lemma 13 implies that both representations π and $\tilde{\pi}$ are non-degenerate. Since $\tilde{\pi} = (\pi^*)^{\vee}$, and $u \mapsto \check{u}$ is an automorphism of $A(G)$, we see that π^* is also non-degenerate.

Let $H_0 = \text{span}\{\pi^*(A(G) \cap C_c(G))H\}$. Since $A(G) \cap C_c(G)$ is dense in $A(G)$ and π^* is non-degenerate, H_0 is a dense subspace of H. We now define a linear map

$$\Lambda_{\pi^*} : (A(G) \cap C_c(G)) \otimes H_0 \to C_c(G, H) \subset L^2(G) \otimes^2 H$$

by the equation

$$\Lambda_{\pi^*}(g \otimes \eta)(x) = \pi^*(L_x \check{g})\eta \quad (g \in A(G) \cap C_c(G), \ \eta \in H_0). \tag{10}$$

To see that Λ_{π^*} is well defined, we need to verify that $\Lambda_{\pi^*}(g \otimes \eta) \in C_c(G, H)$ for any $g \in A(G) \cap C_c(G)$ and $\eta \in H_0$. To see this, it suffices by linearity to assume $\eta = \pi^*(u)\eta_0$ for some $u \in A(G) \cap C_c(G)$ and $\eta_0 \in H$. But then the function

$$x \mapsto \Lambda_{\pi^*}(g \otimes \eta)(x) = \pi^*((L_x \check{g})u)\eta_0$$

belongs to $C_c(G, H)$ since π^* is bounded and the function $x \mapsto (L_x \check{g})u$ belongs to $C_c(G, A(G))$.

Now let $f, g \in A(G) \cap C_c(G)$ and $\xi \in H$ and $\eta \in H_0$. Write $\eta = \sum_{i=1}^n \pi^*(u_i)\eta_i$ with $u_i \in A(G) \cap C_c(G)$, $\eta_i \in H$. Then, by (10) and Proposition 1, we have

$$\langle V_{\pi}(f \otimes \xi) | \Lambda_{\pi^*}(g \otimes \eta) \rangle = \int_G \langle \pi(L_x \check{f})\xi | \pi^*(L_x \check{g})\eta \rangle \, dx$$

$$= \sum_{i=1}^n \int_G \langle \pi(L_x \check{f})\xi | \pi(L_x \check{g})^* \pi(\bar{u}_i)^* \eta_i \rangle \, dx$$

$$= \sum_{i=1}^n \int_G \langle \pi(L_x(f\bar{g})^{\vee})\pi(\bar{u}_i)\xi | \eta_i \rangle \, dx$$

$$= \sum_{i=1}^n \int_G (T_{\eta_i, \pi(\bar{u}_i)\xi}^{\pi} \cdot f\bar{g})(x) \, dx$$

$$= \left(\int_G f(x)\overline{g(x)} \, dx \right) \sum_{i=1}^n \langle \pi(\bar{u}_i)\xi | \eta_i \rangle$$

$$= \langle f | g \rangle \langle \xi | \eta \rangle$$

$$= \langle f \otimes \xi | g \otimes \eta \rangle$$

$$= \langle V_{\pi}(f \otimes \xi) | (V_{\pi}^{-1})^*(g \otimes \eta) \rangle.$$

As $(A(G) \cap C_c(G)) \otimes H$ is dense in $L^2(G) \otimes^2 H$ and V_{π} is continuous and surjective, the above calculation shows that

$$\Lambda_{\pi^*}(g \otimes \eta) = (V_{\pi}^{-1})^*(g \otimes \eta) \quad (g \in A(G) \cap C_c(G), \ \eta \in H_0).$$

Therefore Λ_{π^*} and $(V_\pi^{-1})^*$ agree on the dense subspace $(A(G) \cap C_c(G)) \otimes H_0 \subseteq L^2(G) \otimes^2 H$. Since $(V_\pi^{-1})^*$ is a bounded operator, this implies that Λ_{π^*} is bounded, and

$$\Lambda_{\pi^*} = (V_\pi^{-1})^* \in VN(G) \,\overline{\otimes}\, B(H).$$

From relation (5) we see that the completely bounded map from $A(G)$ to $B(H)$ corresponding to Λ_{π^*} is π^*. Therefore π^* is a completely bounded representation and $V_{\pi^*} = \Lambda_{\pi^*}$. By Lemma 2 (ii), $\check{\pi}$ is also a completely bounded representation, and $V_{\check{\pi}} = V_{\pi^*}^* = (\Lambda_{\pi^*})^* = V_\pi^{-1}$. $\quad\square$

We now state the main theorem of this section.

Theorem 16. *Let $\pi : A(G) \to B(H)$ be a non-degenerate completely bounded representation. Then π is similar to a $*$-representation if and only if its associated corepresentation $V_\pi \in VN(G) \,\overline{\otimes}\, B(H)$ is an invertible operator. In either case, V_π is similar to a unitary corepresentation.*

Proof. If V_π is invertible, then Theorem 15 implies that π^* is a completely bounded representation. Therefore π is similar to a $*$-representation by Theorem 8.

Now suppose that $\pi = S\sigma(\cdot)S^{-1}$ where $S \in B(H)$ is an invertible operator and $\sigma : A(G) \to B(H)$ is a $*$-representation. Since non-degeneracy is preserved under similarities, σ is non-degenerate. By [19, Theorem A.1], the corepresentation $V_\sigma \in VN(G) \,\overline{\otimes}\, B(H)$ is unitary, and

$$V_\pi = (id \otimes S) V_\sigma (id \otimes S^{-1})$$

is similar to a unitary corepresentation (and therefore invertible). $\quad\square$

Remark 17. If $\pi : L^1(G) \to B(H)$ is a bounded non-degenerate representation, then the associated corepresentation $V_\pi \in L^\infty(G) \,\overline{\otimes}\, B(H)$ is always an invertible operator. This suggests to us that the same should be true for the Fourier algebra: Given a non-degenerate completely bounded representation $\pi : A(G) \to B(H)$, we expect that the corepresentation $V_\pi \in VN(G) \,\overline{\otimes}\, B(H)$ should always be invertible (and therefore similar to a unitary corepresentation by Theorem 16). We are unable to prove this conjecture for arbitrary locally compact groups G. However, in the following section we show that this conjecture is true for the class of SIN groups (see Section 5).

5. Groups with small invariant neighborhoods

In this section, we will restrict our attention to the class of locally compact groups with small invariant neighborhoods (called SIN groups). Recall that a locally compact group G is a *SIN group* if it has a neighborhood base $\mathcal{U}$ at the identity e consisting of open neighborhoods which are invariant under the inner automorphisms of G. That is, for all $U \in \mathcal{U}$ and $g \in G$, we have $gUg^{-1} = U$. Typical examples of SIN groups are discrete, abelian, and compact groups. We will show that for any SIN group G, every completely bounded representation of $A(G)$ on a Hilbert space H is similar to a $*$-representation of $A(G)$. In other words, for SIN groups, the completely bounded representation theory of $A(G)$ on Hilbert spaces is very simple – every completely bounded representation of $A(G)$ on a Hilbert space arises as the restriction of a bounded representation of $C_0(G)$ on H. The basic idea in our approach is that when G is a SIN group and

$\pi : A(G) \to B(H)$ is a completely bounded representation, we can show that π is similar to a
$*$-representation without having to a priori assume anything about the complete boundedness of
the associated representations $\check{\pi}$ and π^* defined in (6).

We begin with the following lemma which will be needed for our considerations of SIN
groups. Recall that a locally compact group G is said to be *unimodular* if $\Delta = 1$, where
$\Delta : G \to \mathbb{R}_+$ is the Haar modular function for G. Below, we will use the notation $ZL^1(G)$ to
denote the center of the group algebra $L^1(G)$.

Lemma 18. *Let G be a unimodular locally compact group, and let $\pi : A(G) \to B(H)$ be a completely bounded representation. Fix $\xi, \eta \in H$ and let $T^\pi_{\eta,\xi} \in VN(G)$ be the coefficient operator
introduced in* (8). *Then, for any $\psi \in A(G) \cap L^2(G)$ and $\varphi \in A(G) \cap ZL^1(G)$ such that $\check{\varphi} = \varphi$,
we have*

$$T^\pi_{\eta,\xi} \lambda(\varphi\psi) \in \lambda\big(L^1(G)\big),$$

with

$$\big\| T^\pi_{\eta,\xi} \lambda(\varphi\psi) \big\|_{L^1(G)} \leqslant \|\pi\|^2_{cb} \|\xi\| \|\eta\| \|\psi\|_2 \|\varphi\|_2.$$

Proof. Let φ and ψ be as above. Since $\varphi\psi \in A(G) \cap L^1(G) \subseteq L^2(G)$ we have $T^\pi_{\eta,\xi}(\varphi\psi) \in L^2(G)$ and $T^\pi_{\eta,\xi}(\varphi\psi) = T^\pi_{\eta,\xi} \cdot (\varphi\psi) \in A(G) \cap L^2(G)$ almost everywhere. Therefore, by (2) we
have

$$T^\pi_{\eta,\xi} \lambda(\varphi\psi) = \lambda\big(T^\pi_{\eta,\xi}(\varphi\psi)\big) = \lambda\big(T^\pi_{\eta,\xi} \cdot (\varphi\psi)\big) \in VN(G),$$

and so

$$
\begin{aligned}
\big\| T^\pi_{\eta,\xi} \lambda(\varphi\psi) \big\|_{L^1(G)} &= \big\| T^\pi_{\eta,\xi} \cdot (\varphi\psi) \big\|_{L^1(G)} \\
&= \int\limits_G \big| T^\pi_{\eta,\xi} \cdot (\varphi\psi)(x) \big| \, dx \\
&= \int\limits_G \big| \big\langle \pi\big(L_x(\varphi\psi)^{\check{}}\big)\xi \,|\, \eta \big\rangle \big| \, dx \\
&= \int\limits_G \big| \big\langle \pi(L_x\check{\psi})\xi \,|\, \pi(L_x\check{\varphi})^*\eta \big\rangle \big| \, dx \\
&\leqslant \int\limits_G \big\| \pi(L_x\check{\psi})\xi \big\| \big\| \pi(L_x\check{\varphi})^*\eta \big\| \, dx \\
&\leqslant \left(\int\limits_G \big\| \pi(L_x\check{\psi})\xi \big\|^2 \, dx \right)^{1/2} \left(\int\limits_G \big\| \pi(L_x\check{\varphi})^*\eta \big\|^2 \, dx \right)^{1/2} \\
&= \big\| V_\pi(\psi \otimes \xi) \big\|_{L^2(G)\otimes^2 H} \cdot \left(\int\limits_G \big\| \pi(L_x\check{\varphi})^*\eta \big\|^2 \, dx \right)^{1/2},
\end{aligned}
$$

where the last equality is obtained from Proposition 1. We now consider the term $(\int_G \|\pi(L_x\check{\varphi})^*\eta\|^2\,dx)^{1/2}$ above. Note that since $\varphi \in A(G) \cap ZL^1(G)$ and $\tilde{\varphi} = \varphi$, we have $L_x\check{\varphi} = (L_{x^{-1}}\check{\varphi})^{\sim}$ for all $x \in G$. Indeed, for any $x, y \in G$

$$
\begin{aligned}
(L_{x^{-1}}\check{\varphi})^{\sim}(y) &= \overline{(L_{x^{-1}}\check{\varphi})(y^{-1})} \\
&= \overline{\check{\varphi}(xy^{-1})} \\
&= \tilde{\varphi}(xy^{-1}) \\
&= \varphi(xy^{-1}) \quad (\text{since } \tilde{\varphi} = \varphi) \\
&= \varphi(y^{-1}x) \quad (\text{since } \varphi \in ZL^1(G) \text{ and } G \text{ is unimodular}) \\
&= \check{\varphi}(x^{-1}y) \\
&= (L_x\check{\varphi})(y).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\left(\int_G \|\pi(L_x\check{\varphi})^*\eta\|^2\,dx\right)^{1/2} &= \left(\int_G \|\pi((L_{x^{-1}}\check{\varphi})^{\sim})^*\eta\|^2\,dx\right)^{1/2} \\
&= \left(\int_G \|\pi((L_x\check{\varphi})^{\sim})^*\eta\|^2\,dx\right)^{1/2} \quad (\text{since } G \text{ is unimodular}) \\
&= \left(\int_G \|\tilde{\pi}((L_x\check{\varphi}))\eta\|^2\,dx\right)^{1/2} \\
&= \|V_{\tilde{\pi}}(\varphi \otimes \eta)\| \quad (\text{applying Lemma 2}) \\
&= \|V_\pi^*(\varphi \otimes \eta)\|.
\end{aligned}
$$

This finally gives,

$$
\|T_{\eta,\xi}^\pi \lambda(\varphi\psi)\|_{L^1(G)} \leqslant \|V_\pi(\psi \otimes \xi)\| \cdot \|V_\pi^*(\varphi \otimes \eta)\| \leqslant \|\pi\|_{cb}^2 \|\xi\|\|\eta\|\|\varphi\|_2\|\psi\|_2. \qquad \square
$$

Remark 19. In Lemma 18, we only considered functions $\psi \in A(G) \cap L^2(G)$ and $\varphi \in A(G) \cap ZL^1(G)$ with $\varphi = \tilde{\varphi}$. It is however obvious from the above proof that we can use the density of $A(G) \cap L^2(G)$ in $L^2(G)$ to extend the conclusion of Lemma 18 to arbitrary $\psi \in L^2(G)$. More precisely, we have for any $\psi \in L^2(G)$ and $\varphi \in A(G) \cap Z(L^1(G))$ with $\varphi = \tilde{\varphi}$, that the image of the vector $\varphi\psi \in L^2(G) \cap L^1(G)$ under the operator $T_{\eta,\xi}^\pi \in VN(G)$ satisfies $T_{\eta,\xi}^\pi(\varphi\psi) \in L^2(G) \cap L^1(G)$ with

$$
\|T_{\eta,\xi}^\pi(\varphi\psi)\|_{L^1(G)} \leqslant \|\pi\|_{cb}^2 \|\xi\|\|\eta\|\|\varphi\|_2\|\psi\|_2.
$$

We are now in a position to prove our main result for SIN groups.

Theorem 20. *Let G be a SIN group and let $\pi : A(G) \to B(H)$ be a completely bounded representation. Then π extends continuously to a bounded (hence completely bounded) representation $\pi_0 : C_0(G) \to B(H)$ with norm no larger than $\|\pi\|_{cb}^2$. In particular, every completely bounded representation $\pi : A(G) \to B(H)$ is similar to a $*$-representation, and when π is non-degenerate, the similarity $S \in B(H)$ taking π to a $*$-representation can be chosen so that*

$$\|S\|\|S^{-1}\| \leqslant \|\pi\|_{cb}^4.$$

Proof. To show that π extends continuously to a bounded representation $\pi_0 : C_0(G) \to B(H)$, it suffices to show that for all $\xi, \eta \in H$, the coefficient operator $T_{\eta, \xi}^\pi \in VN(G)$ defined in (8) is actually the convolution operator given by a measure $\mu_{\eta, \xi}^\pi \in M(G)$ with norm $\|\mu_{\eta, \xi}^\pi\|_{M(G)} \leqslant C_\pi \|\xi\|\|\eta\|$ where $C_\pi > 0$ is some constant independent of $\xi, \eta \in H$. Indeed if this is the case, we can proceed as in the proof of Theorem 8 to show that $\pi : A(G) \to B(H)$ is continuous with respect to the $\|\cdot\|_\infty$-norm on $A(G)$ (with norm bound C_π), implying that π extends continuously and uniquely to a bounded representation $\pi_0 : C_0(G) \to B(H)$. Once we have obtained π_0, we can again proceed exactly as in the proof of Theorem 8 to show that π is similar to a $*$-representation and that if π is non-degenerate, a similarity $S \in B(H)$ taking π to a $*$-representation can be chosen so that

$$\|S\|\|S^{-1}\| \leqslant \|\pi_0\|^2 \leqslant C_\pi^2.$$

We will now prove this sufficient condition with constant $C_\pi = \|\pi\|_{cb}^2$. To begin, fix $\xi, \eta \in H$. Since G is a SIN group, we can fix a neighborhood base $\mathcal{U}$ at the identity which consists of open neighborhoods $U \in \mathcal{U}$ with compact closure which are invariant under the inner automorphisms of G. For each $U \in \mathcal{U}$, let χ_U denote the characteristic function of U and define

$$\varphi_U := \frac{\chi_U * \check{\chi}_U}{\|\chi_U * \check{\chi}_U\|_2}.$$

It is easy to see that $\varphi_U \in A(G) \cap C_c(G)$ for every $U \in \mathcal{U}$. Furthermore, since each $U \in \mathcal{U}$ is inner automorphism invariant, we have

$$\varphi_U(xyx^{-1}) = \varphi_U(y) \quad (x, y \in G).$$

Since G is unimodular, this means that $\varphi_U \in A(G) \cap ZL^1(G)$ for all $U \in \mathcal{U}$.

Now for each $U \in \mathcal{U}$, let $\psi_U \geqslant 0 \in L^2(G)$ be chosen so that $\|\psi_U\|_2 = 1$ and $\|\varphi_U \psi_U\|_1 = \int_G \varphi_U(g) \psi_U(g)\, dg = 1$. Define

$$e_U = \varphi_U \psi_U \in L^1(G) \cap L^2(G),$$

and consider the net $\{e_U\}_{U \in \mathcal{U}} \subset L^1(G)$ (where $U \in \mathcal{U}$ are partially-ordered by reverse inclusion). Since $\operatorname{supp} e_U \subseteq \operatorname{supp} \varphi_U$ and $\{\operatorname{supp} \varphi_U\}_{U \in \mathcal{U}}$ forms a neighborhood base at the identity, it follows that the net $\{e_U\}_{U \in \mathcal{U}}$ is a bounded approximate identity for $L^1(G)$. Furthermore, for each $U \in \mathcal{U}$,

Lemma 18 and Remark 19 tell us that the vector $T^\pi_{\eta,\xi} e_U = T^\pi_{\eta,\xi}(\varphi_U \psi_U) \in L^2(G)$ actually lives in $L^2(G) \cap L^1(G)$ and

$$\left\| T^\pi_{\eta,\xi} e_U \right\|_{L^1(G)} \leqslant \|\pi\|^2_{cb} \|\xi\| \|\eta\| \|\varphi_U\|_2 \|\psi_U\|_2 = \|\pi\|^2_{cb} \|\xi\| \|\eta\|.$$

This shows that the net $\{T^\pi_{\eta,\xi} e_U\}_{U \in \mathcal{U}} \subset L^1(G)$ is uniformly bounded by $\|\pi\|^2_{cb} \|\xi\| \|\eta\|$. By passing to a subnet of $\{T^\pi_{\eta,\xi} e_U\}_{U \in \mathcal{U}}$ if necessary, we may assume that the net $\{T^\pi_{\eta,\xi} e_U\}_{U \in \mathcal{U}} \subset L^1(G)$ converges in the weak-$*$ in $M(G)$ to some measure $\mu^\pi_{\eta,\xi} \in M(G)$. Note that $\|\mu^\pi_{\eta,\xi}\|_{M(G)} \leqslant \|\pi\|^2_{cb} \|\xi\| \|\eta\|$.

It now remains to show that $T^\pi_{\eta,\xi} = \lambda(\mu^\pi_{\eta,\xi})$. To do this, note that by density of $A(G) \cap C_c(G)$ in $L^2(G)$ it suffices to show $T^\pi_{\eta,\xi} f = \lambda(\mu^\pi_{\eta,\xi}) f$ for all $f \in A(G) \cap C_c(G)$. So fix such an f, and note that since $\{e_U\}_{U \in \mathcal{U}}$ is a bounded approximate identity for $L^1(G)$, $\lim_{U \in \mathcal{U}} \|e_U * f - f\|_{A(G)} = 0$. Therefore for almost every $x \in G$ we have

$$\begin{aligned}
\left(T^\pi_{\eta,\xi} f \right)(x) &= \left(T^\pi_{\eta,\xi} \cdot f \right)(x) = \left\langle L_x \check{f}, T^\pi_{\eta,\xi} \right\rangle \\
&= \lim_{U \in \mathcal{U}} \left\langle L_x (e_U * f)^{\vee}, T^\pi_{\eta,\xi} \right\rangle = \lim_{U \in \mathcal{U}} \left[T^\pi_{\eta,\xi} \cdot (e_U * f) \right](x) \\
&= \lim_{U \in \mathcal{U}} \left[T^\pi_{\eta,\xi} (e_U * f) \right](x) \quad \left(\text{since } e_U * f \in A(G) \cap L^2(G) \right) \\
&= \lim_{U \in \mathcal{U}} \left[\left(T^\pi_{\eta,\xi} e_U \right) * f \right](x) \\
&\quad \left(\text{since } VN(G) \text{ commutes with right convolutions by } C_c(G) \right) \\
&= \lim_{U \in \mathcal{U}} \int_G \left(T^\pi_{\eta,\xi} e_U \right)(y) f\left(y^{-1} x \right) dy \\
&=: \int_G f\left(y^{-1} x \right) d\mu^\pi_{\eta,\xi}(y) \\
&= \left(\mu^\pi_{\eta,\xi} * f \right)(x) \\
&= \left(\lambda\left(\mu^\pi_{\xi,\eta} \right) f \right)(x).
\end{aligned}$$

This completes the proof. $\square$

Corollary 21. *Let G be a SIN group and let $\pi : A(G) \to B(H)$ be a completely bounded representation of $A(G)$. Then $\check{\pi}$ and π^* are also completely bounded representations.*

Proof. This is just a consequence of Theorems 20 and 8. $\square$

Corollary 22. *Let G be a SIN group and $\pi : A(G) \to B(H)$ be a completely bounded representation. Then π is non-degenerate if and only if the associated corepresentation $V_\pi \in VN(G) \overline{\otimes} B(H)$ is invertible.*

Proof. If π is non-degenerate, then Theorems 15 and 20 imply that V_π must be invertible. The converse is just Lemma 13. $\square$

In [7], it was asked whether the completely contractive representations of $A(G)$ correspond to $*$-representations, and in [25] it was asked whether or not every completely bounded representation of $A(G)$ is similar to a complete contraction. As a corollary to Theorem 20, we obtain partial answers to these questions.

Corollary 23. *Let G be a SIN group and let $\pi : A(G) \to B(H)$ be a completely bounded representation. Then π is similar to a complete contraction, and π is a $*$-representation if and only if it is a complete contraction.*

Proof. We already know that π is similar to a $*$-representation by Theorem 20, and that $*$-representations are completely contractive by the proof of Theorem 8. We therefore only need to show that $\|\pi\|_{cb} \leqslant 1$ implies that π is a $*$-representation. But in this case, Theorem 20 implies that π extends uniquely to a bounded representation $\pi_0 : C_0(G) \to B(H)$ with norm $\|\pi_0\| \leqslant \|\pi\|_{cb}^2 = 1$. Since π_0 is a contractive representation of $C_0(G)$, Remark 7 tells us that $\pi_0 = \pi_{0,e} \oplus 0_{H_e^\perp}$ relative to the decomposition $H = H_e \oplus H_e^\perp$, where H_e is the essential space of π_0 and $\pi_{0,e}$ is the essential part of π_0. Since non-degenerate contractive representations of C*-algebras are always $*$-representations, it follows that $\pi_0 = \pi_{0,e} \oplus 0_{H_e^\perp}$ is a $*$-representation. In particular, $\pi = \pi_0|_{A(G)}$ is a $*$-representation. $\quad\square$

6. Other classes of groups

In this section we examine the possibility of extending the results of Section 5 to other classes of locally compact groups. The main result of this section is that the every completely contractive representation of $A(G)$ is a $*$-representation even if we only assume that the connected component of G is a SIN group. We will consider the following terminologies.

For a locally compact group G and an open subgroup K of G, we let $G = \dot{\bigcup}_{x \in I} xK$ denote the decomposition of G to distinct left cosets of K (i.e. $xK \cap yK = \emptyset$ if $x \neq y$). For every element $u \in A(G)$ and $x \in I$, we write

$$u_x = u \chi_{xK},$$

where χ_{xK} is the characteristic function of the coset xK. Since K is open, each χ_{xK} is a norm-one idempotent in the Fourier–Stieltjes algebra $B(G)$ [10, Proposition 2.31]. Since $A(G)$ is a closed ideal in $B(G)$, $u_x \in A(G)$. We let

$$A(xK) = A(G) \chi_{xK}.$$

Since K is open, the canonical embedding of the Fourier algebra $A(K)$ into $A(G)$ (i.e. extending functions by zero outside of K) is completely isometric, allowing us to identify $A(K)$ with its image $A(eK)$ unambiguously. In what follows, we shall consider the translation operators $L_x : A(K) \to A(xK)$ defined by

$$(L_x u)(y) = u(x^{-1}y) \quad (u \in A(K)).$$

We note here that since left translation on $A(G)$ is completely isometric, $L_x : A(K) \to A(xK)$ is always a completely isometric algebra isomorphism. Finally, for any representation $\pi : A(G) \to B(H)$, we let π_x $(x \in I)$ denote the restriction of π to the ideal $A(xK)$.

Lemma 24. *Let G be a locally compact group, and let K be an open subgroup of G with the property that every completely contractive representation of $A(K)$ on a Hilbert space H is a $*$-representation. Then every completely contractive representation of $A(G)$ on H is a $*$-representation.*

Proof. Let $\pi : A(G) \to B(H)$ be a completely contractive representation. Write $G = \dot{\bigcup}_{x \in I} xK$. For every $x \in I$, the mapping $\pi_x \circ L_x : A(K) \to B(H)$ defines a completely contractive representation of $A(K)$ on H, and so, by hypothesis, $\pi_x \circ L_x$ is a $*$-representation for every $x \in I$. Now let $u \in A(G) \cap C_c(G)$ and write $u = \sum_{x \in I_0} u_x$ where $I_0 \subseteq I$ is some finite subset. Then we have

$$
\begin{aligned}
\pi(u)^* &= \pi \left(\sum_{x \in I_0} u_x \right)^* \\
&= \sum_{x \in I_0} \pi(u_x)^* \\
&= \sum_{x \in I_0} \big((\pi_x \circ L_x)(L_{x^{-1}} u_x) \big)^* \\
&= \sum_{x \in I_0} (\pi_x \circ L_x)(\overline{L_{x^{-1}} u_x}) \\
&= \sum_{x \in I_0} (\pi_x \circ L_x)(L_{x^{-1}} \bar{u}_x) \\
&= \sum_{x \in I_0} \pi(\bar{u}_x) \\
&= \pi(\bar{u}).
\end{aligned}
$$

Since $A(G) \cap C_c(G)$ is dense in $A(G)$, this shows that π is a $*$-representation. $\quad\square$

Theorem 25. *Let G be a locally compact group such that G_e, the connected component of the identity, is a SIN group. Then any completely contractive representation $\pi : A(G) \to B(H)$ is a $*$-representation.*

Proof. Since any such group G has an open almost connected subgroup, by Lemma 24, we may assume that G itself is almost connected. Fix $u \in A(G)$ and $\epsilon > 0$. As it is shown in [12, Theorem 3.3], there is a compact, normal subgroup N of G and a bounded idempotent $P : A(G) \to A(G)$ such that G/N is a Lie group and

$$
\| u - Pu \|_{A(G)} \leqslant \epsilon. \tag{11}
$$

In fact, P is a projection onto $A(G : N)$, the subalgebra of $A(G)$ consisting of those functions that are constant on the left cosets of N. Since $A(G : N)^*$ can be identified as a von Neumann algebra with $VN(G/N)$, $A(G : N)$ is completely isometrically isomorphic to $A(G/N)$ [10, Proposition 3.25]. Now let $q : G \to G/N$ be the canonical quotient map. By [15, Theorems 5.18 and 7.12], the connected component $(G/N)_e$ of G/N is $q(G_e)$. Thus $(G/N)_e$ is a SIN group.

On the other hand, since G/N is a Lie group, $(G/N)_e$ is an open subgroup of G/N. Therefore, identifying $A(G : N) \cong A(G/N)$ completely isometrically, Lemma 24 implies that the completely contractive representation $\pi|_{A(G:N)} : A(G : N) \to B(H)$ is in fact a $*$-representation of $A(G : N)$. In particular $\pi(Pu)^* = \pi(\overline{Pu})$. It follows from (11) and the triangle inequality that

$$\left\| \pi(\bar{u}) - \pi(u)^* \right\| \leqslant \left\| \pi(\overline{u - Pu}) \right\| + \left\| \pi(\overline{Pu}) - \pi(u)^* \right\|$$
$$= \left\| \pi(\overline{u - Pu}) \right\| + \left\| \pi(Pu - u)^* \right\|$$
$$\leqslant \left\| \overline{u - Pu} \right\|_{A(G)} + \left\| Pu - u \right\|_{A(G)}$$
$$\leqslant 2\epsilon.$$

Since $\epsilon > 0$ and $u \in A(G)$ was arbitrary, we have the result. $\square$

The following corollary is an immediate consequence of the preceding theorem. Recall that a locally compact group G is *maximally almost periodic* if the finite-dimensional irreducible representations of G separate points in G. For connected groups, the class of maximally almost periodic groups coincides with the class of SIN groups (see [13, Theorem 2.9]). Also recall that G is said to be *totally disconnected* if $G_e = \{e\}$.

Corollary 26. *Let G be a locally compact group. Then every completely contractive representation of $A(G)$ on a Hilbert space H is a $*$-representation in either of the following cases*:

  (i) *G is a SIN group*;
 (ii) *G is maximally almost periodic*;
(iii) *G is totally disconnected*.

Acknowledgments

References

[1] S. Baaj, G. Skandalis, Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres, Ann. Sci. École Norm. Sup. (4) 26 (4) (1993) 425–488.

[2] M. Bożejko, Uniformly bounded representations of free groups, J. Reine Angew. Math. 377 (1987) 170–186.

[3] E. Christensen, On non self adjoint representations of operator algebras, Amer. J. Math. 103 (1981) 817–834.

[4] M. Day, Means for the bounded functions and ergodicity of the bounded representations of semi-groups, Trans. Amer. Math. Soc. 69 (1950) 276–291.

[5] J. Dixmier, Les moyennes invariantes dans les semi-groupes et leurs applications, Acta Sci. Math. (Szeged) 12 (1950) 213–227.

[6] E.G. Effros, Z.-J. Ruan, Operator Spaces, Oxford Science Publications, 2000.

 [7] E.G. Effros, Z.-J. Ruan, Operator space tensor products and Hopf convolution algebras, J. Operator Theory 50 (1) (2003) 131–156.
 [8] L. Ehrenpreis, F. Mautner, Uniformly bounded representations of groups, Proc. Natl. Acad. Sci. USA 41 (1955) 231–233.
 [9] M. Enock, J.-M. Schwartz, Kac Algebras and Duality of Locally Compact Groups, Springer-Verlag, Berlin, 1992.
[10] P. Eymard, L'algèbre de Fourier d'un groupe localement compact, Bull. Soc. Math. France 92 (1964).
[11] A. Figa-Talamanca, C. Nebbia, Harmonic Analysis and Representation Theory for Groups Acting on Homogeneous Trees, London Math. Soc. Lecture Note Ser., vol. 162, Cambridge Univ. Press, Cambridge, 1990.
[12] B.E. Forrest, V. Runde, Amenability and weak amenability of the Fourier algebra, Math. Z. 250 (4) (2005) 731–744.
[13] S. Grosser, M. Moskowitz, Compactness conditions in topological groups, J. Reine Angew. Math. 246 (1971) 1–40.
[14] U. Haagerup, Solution of the similarity problem for cyclic representations of C*-algebras, Ann. of Math. 118 (1983) 215–240.
[15] E. Hewitt, K.A. Ross, Abstract Harmonic Analysis, I, Springer-Verlag, 1963.
[16] J. Kustermans, S. Vaes, Locally compact quantum groups in the von Neumann algebraic setting, Math. Scand. 92 (1) (2003) 68–92.
[17] A.M. Mantero, A. Zappa, The Poisson transform on free groups and uniformly bounded representations, J. Funct. Anal. 51 (1983) 372–399.
[18] A.M. Mantero, A. Zappa, Uniformly bounded representations and L_p convolution theorems on a free group, in: Harmonic Analysis, Proc. Cortona, 1982, in: Lecture Notes in Math., vol. 992, Springer, 1983, pp. 333–343.
[19] Y. Nakagami, M. Takesaki, Duality for Crossed Products of von Neumann Algebras, Lecture Notes in Math., vol. 731, Springer, Berlin, 1979.
[20] C.-K. Ng, Amenable representations and Reiter's property for Kac algebras, J. Funct. Anal. 187 (1) (2001) 163–182.
[21] V. Paulsen, Completely Bounded Maps and Operator Algebras, Cambridge Univ. Press, 2002.
[22] G. Pisier, The similarity degree of an operator algebra, St. Petersburg Math. J. 10 (1999) 103–146.
[23] G. Pisier, Similarity Problems and Completely Bounded Maps, second ed., Lecture Notes in Math., vol. 1618, Springer-Verlag, 2001.
[24] T. Pytlik, R. Szwarc, An analytic family of uniformly bounded representations of free groups, Acta Math. 157 (1986) 287–309.
[25] N. Spronk, On multipliers of the Fourier algebra of a locally compact group, PhD thesis, University of Waterloo, 2002.
[26] N. Spronk, Measurable Schur multipliers and completely bounded multipliers of the Fourier algebras, Proc. Lond. Math. Soc. (3) 89 (1) (2004) 161–192.
[27] J.G. Wendel, Left centralizers and isomorphisms of group algebras, Pacific J. Math. 2 (2) (1952) 251–261.
[28] J. Wysoczanski, An analytic family of uniformly bounded representations of a free product of discrete groups, Pacific J. Math. 157 (1993) 373–385.