

# ENTANGLEMENT AND THE TEMPERLEY-LIEB CATEGORY

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ABSTRACT. We survey some recent results from [BC18b], where a class of highly entangled subspaces of bipartite quantum systems is described, which arises from unitary fiber functors on the Temperley-Lieb category associated to the representation theory of free orthogonal quantum groups. By exploiting the rich structure of the Temperley-Lieb category and this particular fiber functor, we are able to precisely determine the largest singular values for these subspaces and obtain lower bounds for the minimum output entropy of the corresponding quantum channels. Future research directions and some open problems are also discussed.

## 1. INTRODUCTION

Entanglement is a fundamental notion in quantum mechanics that does not have an analogue in the classical world. Within the framework of quantum computation and quantum information, entanglement in bipartite or multipartite systems produces, on the one hand, many counterintuitive phenomena, while on the other hand, can be used to design new communications protocols which admit no classical analogues [Eis06, Gro96, CLSZ95, EJ96].

Throughout this paper we will focus on entanglement in bipartite quantum systems. Within the formalism of quantum mechanics, a quantum mechanical system is described by a complex Hilbert space  $H$ : the (pure) states of the system are described by unit norm vectors  $\xi \in H$ , taken up to a complex phase factor. (In this paper, all Hilbert spaces are taken to be finite-dimensional, unless otherwise specified.) Equivalently, a pure state of the system can be described by the rank one projector  $\rho = |\xi\rangle\langle\xi|$  onto the subspace  $\mathbb{C}\xi \subset H$ . The (closed) convex hull of pure states is denoted by  $\mathcal{D}(H)$ , and elements  $\rho \in \mathcal{D}(H)$  are called *mixed states*.  $\mathcal{D}(H)$  is a convex compact set (with respect to the weak  $*$ -topology induced by the trace duality  $\mathcal{B}(H) = \mathcal{S}_1(H)^*$ , where  $\mathcal{S}_1(H) = \text{span}_{\mathbb{C}}\mathcal{D}(H)$ ), and the extremal points of  $\mathcal{D}(H)$  are the rank one projectors, i.e., pure states. In the quantum context, a *bipartite system*  $AB$ , built from subsystems  $A, B$ , is modeled by the tensor product Hilbert space  $H = H_A \otimes H_B$ , where the Hilbert spaces  $H_A$  and  $H_B$  describe the states of systems  $A$  and  $B$ , respectively.

Given a bipartite system modeled by the Hilbert space tensor product  $H = H_A \otimes H_B$ , a mixed state  $\rho \in \mathcal{D}(H)$  is said to *separable* if it belongs to the convex hull of the set of product states  $\rho = \rho_A \otimes \rho_B$ , where  $\rho_A \in \mathcal{D}(H_A)$  and  $\rho_B \in \mathcal{D}(H_B)$ . A state  $\rho$  is called *entangled* if it is not separable. We shall call a Hilbert subspace  $H_0 \subset H_A \otimes H_B$  an *entangled subspace* if all of its associated pure states are entangled. In this paper, we are concerned with studying non-trivial examples of highly-entangled subspaces  $H_0 \subset H = H_A \otimes H_B$ . By highly-entangled, we shall mean that set of pure states on  $H$  associated to the subspace  $H_0$  are uniformly “far away” from the set of product states  $\rho_A \otimes \rho_B \in \mathcal{D}(H)$  with respect to some suitable measure of distance. The choice of distance here is not unique, and our choice

is based on the largest singular value of pure states - precise details will be given in the next section.

A trivial example of a highly-entangled subspace is the one-dimensional subspace  $H_0 = \mathbb{C}\xi \subset H_A \otimes H_B$  spanned by a maximally entangled (Bell) state  $\xi$ . I.e., a state  $\xi$  of the form  $\xi = d^{-1/2} \sum_{i=1}^d e_i \otimes f_i$ , where  $d = \min\{\dim H_A, \dim H_B\}$  and  $(e_i)_i \subset H_A$ ,  $(f_i)_i \subset H_B$  are orthonormal systems. Naturally, the larger the dimension of subspace  $H_0 \subseteq H$ , the less likely it will be highly entangled, as per the above notion of entanglement. In recent years it has become a very important problem in Quantum Information Theory (QIT) to do the following: *Find subspaces  $H_0$  of large relative dimension in a tensor product  $H = H_A \otimes H_B$  such that all states are highly entangled.*

One rich source of highly entangled subspaces comes from random techniques. The idea of studying random subspaces of tensor products dates back to the work of Hayden, Leung, Shor, Winter, Hastings [HLSW04, HW08, HLW06, Has09], among others, and it was explored in great detail by Aubrun, Belinschi, Collins, Fukuda, King, Nechita, Szarek, Werner [ASW11, ASY14, BCN12, FK10], and others. The general outcome of these works was the conclusion that (at least in certain asymptotic dimension regimes) highly entangled subspaces of large relative dimension are ubiquitous: with high probability, a randomly selected subspace will be highly entangled. These random constructions have had a profound impact on the field, solving several open problems, most notably the minimum output entropy additivity problem [Has09, ASW11, BCN16]. The downside to these these highly random techniques is that they provide no information on finding concrete examples that are predicted to exist by these methods. In fact, there seems to be embarrassingly few known examples of such subspaces (beyond the well-known antisymmetric subspace  $H \wedge H \subset H \otimes H$  [GHP10].) Thus, there is a need for a systematic development of non-random examples of highly entangled subspaces.

One natural place to search for examples of highly entangled subspaces is within the framework of *representation theory*. More precisely, given a (compact) group  $G$ , one can consider a pair of (irreducible) unitary representations  $H_\pi, H_\sigma$ , form their tensor product representation  $H_\pi \otimes H_\sigma$ , and then attempt to quantify the entanglement of the irreducible subrepresentations  $H_\nu \subset H_\pi \otimes H_\sigma$  that arise in the decomposition of  $H_\pi \otimes H_\sigma$  into irreducibles.

A first attempt was made in this direction by M. Al Nuwairan [AN13, AN14], by studying the entanglement of subrepresentations of tensor products of irreducible representations of the group  $SU(2)$ . Here, Al Nuwairan showed that entanglement always achieved (except when one takes the highest weight subrepresentation of a tensor product of  $SU(2)$ -irreducibles). However, as is evidenced by the results in [AN13, Section 3], a high degree of entanglement is unfortunately not achieved when working with  $SU(2)$ .

In order to use representation theory to obtain examples of entangled subspaces exhibiting a higher level of entanglement, there are two natural approaches: The first approach would be to consider more complicated examples of compact groups and their representations. The significant downside of this approach is that for most examples of groups  $G$ , one lacks the complete understanding of the representation category  $\text{Rep}(G)$  that one has for  $SU(2)$ . The second approach, which we follow in this paper, is to instead consider “ $q$ -deformations” of the representation category  $\text{Rep}(SU(2))$  arising from certain quantum group constructions.

Perhaps the most well-known examples of such deformations are the canonical realizations of (i.e., unitary fiber functors on) the Temperley-Lieb Categories  $\text{TL}(d)$  that are associated

to the Drinfeld-Jimbo-Woronowicz  $q$ -deformations of  $SU(2)$ , where  $q + q^{-1} = d \in \mathbb{C} \setminus \{0\}$  [Dri87, Jim85, Wor88, Wor87]. In this paper, we consider a very different realization of the Temperley-Lieb categories  $TL(d)$  which act on higher dimensional spaces, and come from another class of quantum groups (more closely linked with operator algebra theory and free probability theory), called *free orthogonal quantum groups*.

Given an integer  $N \geq 2$ , the free orthogonal quantum group  $O_N^+$  is the compact quantum group whose Hopf  $*$ -algebra of polynomial functions  $\mathcal{O}(O_N^+)$  is given as a certain natural non-commutative (or free) version of the algebra of polynomial functions on the classical  $N \times N$  orthogonal matrix group  $O_N$ . A remarkable observation of Banica [Ban96] showed that the representation category  $\text{Rep}(O_N^+)$  gives a faithful realization (unitary fiber functor) of the Temperley-Lieb Category  $TL(N)$  with generating object given by the  $N$ -dimensional fundamental representation space  $\mathbb{C}^N$  (in contrast to  $\mathbb{C}^2$  given by the usual  $q$ -deformation of  $SU(2)$  (or  $\mathfrak{sl}_2$ )). It is the entanglement phenomena associated to this higher dimensional fiber functor on the Temperley-Lieb category that we study here.

Our motivation to study entanglement in the context of  $\text{Rep}(O_N^+) \cong TL(N)$  comes from the pioneering work of Vergnioux [Ver07] on the seemingly unrelated *property of rapid decay* (*property RD*) for quantum groups. The property of rapid decay is a geometric-analytic property possessed by certain (quantum) groups and corresponds to the existence of polynomial bounds relating non-commutative  $L^\infty$ -norms of polynomial functions on quantum groups to their (much easier to calculate)  $L^2$ -norms. The operator algebraic notion of property RD has its origins in the groundbreaking work of Haagerup [Haa79] on approximation properties of free group  $C^*$ -algebras. Unlike in the case of ordinary groups, where property RD is connected to the combinatorial geometry of a discrete group  $G$ , in the quantum world, property RD was observed by Vergnioux to be intrinsically connected to the geometry of the relative position of a subrepresentation of a tensor product of irreducible representations of a given quantum group. More precisely, Vergnioux [Ver07, Section 4] points out that property RD for a given quantum group  $\mathbb{G}$  is related to the following geometric requirement: *Given any pair of irreducible representations  $H_A, H_B$  of  $\mathbb{G}$ , all multiplicity-free irreducible subrepresentations  $H_0 \subset H_A \otimes H_B$  must be asymptotically far from the cone of decomposable tensors in  $H_A \otimes H_B$ .*

The work [BC18b] that we survey here is largely an in-depth exploration of this passing remark of Vergnioux [Ver07], and our goal is to show how a rather modest understanding of the structure of the Temperley-Lieb category can be extremely fruitful when analyzing the entanglement problem for  $\text{Rep}(O_N^+)$ . In this context, we show that one can describe very precisely the largest singular values of states that appear in irreducible subrepresentations of tensor product representations (see Theorem 3.3). As a result we produce a new non-random class of subspaces of tensor products with the property of being highly entangled and of large relative dimension. We also deduce from our entanglement results some interesting properties for the class of quantum channels associated to these subspaces. We compute explicitly the  $\mathcal{S}^1 \rightarrow \mathcal{S}^\infty$  norms of these channels, and obtain large lower bounds on their minimum output entropies (see Section 4). We also show in Section 5 how one can use “planar algebraic” arguments to study further properties of our quantum channels, including their entanglement breaking property, and constructing positive maps on matrix algebras which are not completely positive.

It is our hope that this survey will inspire others to view quantum groups/symmetries and their associated tensor categories as a new, rich source of entangled subspaces with interesting geometric properties.

The remainder of this work is organized as follows: We recall in the first part of Section 2 some concepts related to entangled subspaces, quantum channels, and minimum output entropy of quantum channels. The second half of Section 2 introduces the free orthogonal quantum groups, describes some aspects of their representations theory, and explains the connection with the Temperley-Lieb category. Section 3 is the main section where we study the entanglement of irreducible subrepresentations of tensor products of  $O_N^+$ -representations. There we present the property RD-entanglement inequality (Proposition 3.1), establishing high entanglement for the subspaces under consideration (Theorem 3.2). We then go on to generalize this rapid decay inequality to a higher rank version (Theorem 3.3) and discuss its optimality. We apply this strengthened rapid decay property in Section 4 to study the quantum channels that are naturally associated to our entangled subspaces. Here we obtain lower bounds for the MOE's of these channels (Corollary 4.2) and discuss their sharpness. In Section 5, we use planar algebra arguments to describe the Choi maps for our quantum channels, and use these observations in subsections 5.1 and 5.2 to construct new deterministic examples of  $d$ -positive maps between matrix algebras, and study the entanglement breaking property of our channels. Finally, in Section 6 we outline some problems and future work related to our results.

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## 2. PRELIMINARIES

**2.1. Entangled subspaces of a tensor product.** Consider a pair of finite-dimensional complex Hilbert spaces  $H_A$  and  $H_B$ . We call the fundamental fact that unit vector  $\xi$  belonging to the tensor product Hilbert space  $H_A \otimes H_B$  admits a *singular value decomposition*: There are unique constants  $\lambda_1 \geq \lambda_2 \geq \dots \lambda_d \geq 0$  (with  $d = \min\{\dim H_A, \dim H_B\}$ ) and orthonormal systems  $(e_i)_{i=1}^d \subset H_A$  and  $(f_i)_{i=1}^d \subset H_B$  such that

$$\xi = \sum_{i=1}^d \sqrt{\lambda_i} e_i \otimes f_i.$$

The sequence of numbers  $(\lambda_i)_i$  is uniquely determined (as a multi-set) by  $\xi$  and these numbers are called the *singular values (or Schmidt coefficients)* of  $\xi$ . Note that  $\|\xi\|^2 = \sum_{i=1}^d \lambda_i$ .

We shall call a non-zero vector  $\xi \in H_A \otimes H_B$  *separable* if there exist vectors  $\eta \in H_A, \zeta \in H_B$  such that  $\xi = \eta \otimes \zeta$ . If  $\xi$  is not separable, it is called *entangled*. Note that a unit vector  $\xi \in H_A \otimes H_B$  is separable if and only if its corresponding sequence of Schmidt coefficients is  $(1, 0, 0, \dots, 0)$ . We shall similarly call a linear subspace  $H_0 \subseteq H_A \otimes H_B$  *separable* (resp. *entangled*) if  $H_0$  contains (resp. does not contain) separable vectors. We note that the maximally entangled unit vector in  $H_A \otimes H_B$  is the so-called *Bell vector (Bell state)*  $\xi_B$ ,

whose singular value decomposition is given by

$$\xi_B = \frac{1}{\sqrt{d}} \sum_{i=1}^d e_i \otimes f_i \quad (d \geq 2).$$

Note that the Schmidt coefficients of the Bell vector are given by  $\left(\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}, \dots, \frac{1}{\sqrt{d}}, 0, 0, \dots\right)$ . In particular, the largest Schmidt coefficient  $\lambda_1$  associated to a unit vector  $\xi \in H_A \otimes H_B$  is maximized at 1 precisely when it is separable, and it is minimized at  $d^{-1/2}$  when  $\xi = \xi_B$  is a Bell state. In this sense, the singular value decomposition is a useful tool for measuring measure how entangled a unit vector  $\xi \in H_A \otimes H_B$  is: If  $\lambda_1 \ll 1$ , then  $\xi$  is highly entangled. With this in mind, we call a linear subspace  $H_0 \subseteq H_A \otimes H_B$  *highly entangled* if the supremum of all maximal Schmidt coefficients associated to all unit vectors in  $H_0$  is bounded away from one. That is,

$$\sup_{\xi \in H_0, \|\xi\|=1} \lambda_1 \ll 1.$$

Equivalently,  $H_0 \subseteq H_A \otimes H_B$  is highly entangled if and only if

$$(1) \quad \sup_{\|\xi\|_{H_0}=\|\eta\|_{H_A}=\|\zeta\|_{H_B}=1} |\langle \xi | \eta \otimes \zeta \rangle| \ll 1.$$

**2.2. Quantum channels.** Given a finite dimensional Hilbert space  $H$ , denote by  $\mathcal{B}(H)$  the  $*$ -algebra of linear operators on  $H$ , and denote by  $\mathcal{D}(H) \subseteq \mathcal{B}(H)$  the collection of *states on  $H$* : positive semidefinite operators  $0 \leq \rho \in \mathcal{B}(H)$  satisfying  $\text{Tr}(\rho) = 1$ , where  $\text{Tr}$  denotes the canonical trace on  $\mathcal{B}(H)$ . A state  $\rho \in \mathcal{D}(H)$  is called a *pure state* if there exists a unit vector  $\xi \in H$  so that  $\rho$  is given by the rank-one projector  $\rho_\xi = |\xi\rangle\langle\xi|$ . We denote by  $\mathcal{S}_1(H)$  the Banach algebra  $\mathcal{B}(H)$ , equipped with the *trace norm*  $\|\rho\|_{\mathcal{S}_1(H)} = \text{Tr}(|\rho|)$ . At times, we will also write  $\mathcal{S}_1(H)$  for the space  $\mathcal{B}(H)$  equipped with the operator norm  $x \mapsto \|x\|_\infty = \sup_{0 \neq \xi \in H} \frac{\|x\xi\|}{\|\xi\|}$ .

Given two (finite-dimensional) Hilbert spaces  $H_A$  and  $H_B$ , a *quantum channel* is a linear, completely positive and trace-preserving map (CPTP map)  $\Phi : \mathcal{B}(H_A) \rightarrow \mathcal{B}(H_B)$  [NC00]. By definition, we have  $\Phi(\mathcal{D}(H_A)) \subseteq \mathcal{D}(H_B)$  for any quantum channel  $\Phi$ . A natural model for the construction of quantum channels comes from subspaces of Hilbert space tensor products. Given a triple of finite dimensional Hilbert spaces  $(H_A, H_B, H_C)$  and an isometric linear map  $\alpha_A^{B,C} : H_A \rightarrow H_B \otimes H_C$ , we can form a complementary pair of quantum channels

$$\begin{aligned} \Phi_A^{\bar{B},C} : \mathcal{B}(H_A) &\rightarrow \mathcal{B}(H_C); & \Phi_A^{\bar{B},C}(\rho) &= (\text{Tr}_{H_B} \otimes \iota)(\alpha_A^{B,C} \rho (\alpha_A^{B,C})^*) \\ \Phi_A^{B,\bar{C}} : \mathcal{B}(H_A) &\rightarrow \mathcal{B}(H_B); & \Phi_A^{B,\bar{C}}(\rho) &= (\iota \otimes \text{Tr}_{H_C})(\alpha_A^{B,C} \rho (\alpha_A^{B,C})^*). \end{aligned}$$

Remarkably, every quantum channel can be expressed in the above form, thanks to the well known Stinespring Dilation Theorem for completely positive maps. In other words, given any quantum channel  $\Phi : \mathcal{B}(H_A) \rightarrow \mathcal{B}(H_B)$ , the Stinespring Theorem guarantees the existence of an essentially unique *Stinespring pair*  $(H_C, \alpha_A^{B,C})$ , where  $H_C$  is an auxiliary “environment” Hilbert space and  $\alpha_A^{B,C} : H_A \rightarrow H_B \otimes H_C$  is a linear isometry, so that  $\Phi = \Phi_A^{B,\bar{C}}$  in the above notation. See [HW08], for example.

The *minimum output entropy (MOE)* of a quantum channel  $\Phi : \mathcal{B}(H_A) \rightarrow \mathcal{B}(H_B)$  is given by

$$H_{\min}(\Phi) := \min_{\rho \in \mathcal{D}(H_A)} H(\Phi(\rho)),$$

where  $H(\cdot)$  denotes the *von Neumann entropy* of a state:  $H(\rho) = -\text{Tr}(\rho \log \rho)$ . Note that by functional calculus, we have  $H(\rho) = -\sum_i \lambda_i \log \lambda_i$ , where  $(\lambda_i)_i \subset [0, \infty)$  denotes the spectrum of  $\rho$ . In other words,  $H(\rho)$  is nothing but the Shannon entropy of the probability vector  $(\lambda_i)_i$  corresponding to the eigenvalues of  $\rho$ . Note that here we use the usual convention that  $0 \log 0 = 0$ . Since the von Neumann entropy functional  $H(\cdot)$  is well-known to be convex, it follows that the MOE  $H_{\min}(\Phi)$  is minimized on the extreme points of the compact convex set  $\mathcal{D}(H_A)$ , which corresponds to the set of all pure states on  $H$ . In particular,

$$H_{\min}(\Phi) = \min_{\xi \in H_A, \|\xi\|=1} H(\Phi(|\xi\rangle\langle\xi|)).$$

Using this fact together with the Stinespring Theorem, it follows that  $H_{\min}(\Phi)$  depends only on the relative position of the subspace  $\alpha_A^{B,C}(H_A)$  inside  $H_B \otimes H_C$  coming from the Stinespring representation  $\Phi = \Phi_A^{B,\bar{C}} = (\iota \otimes \text{Tr}_{H_C})(\alpha_A^{B,C}(\cdot)(\alpha_A^{B,C})^*)$ . Indeed, in this case, we have

$$\begin{aligned} H_{\min}(\Phi) &= \min_{\xi \in H_A, \|\xi\|=1} H(\Phi(|\xi\rangle\langle\xi|)) = \min_{\xi \in H_A, \|\xi\|=1} H((\iota \otimes \text{Tr}_C)(|\alpha_A^{B,C}(\xi)\rangle\langle\alpha_A^{B,C}(\xi)|)) \\ &= \min_{\xi \in H_A, \|\xi\|=1} -\sum_i \lambda_i \log \lambda_i, \end{aligned}$$

where  $(\lambda_i)_i$  are the Schmidt coefficients of  $\alpha_A^{B,C}(\xi) = \sum_i \sqrt{\lambda_i} e_i \otimes f_i \in H_B \otimes H_C$ . In particular,  $H_{\min}(\Phi)$  is zero if and only if  $\alpha_A^{B,C}(H_A) \subseteq H_B \otimes H_C$  is a separable subspace.

**2.3. Free orthogonal quantum groups, their representations, and the Temperley-Lieb Category.** In this section we give a very light overview of the free orthogonal quantum groups and some aspects of their finite dimensional representation theory. Much of what we state below about quantum groups and their representations can be phrased in more general terms, however this will not be needed for our purpose. The interested reader may refer to [Tim08, Wor98].

The main idea behind the concept of a free orthogonal quantum group is to formulate a *non-commutative* version of the commutative  $*$ -algebra of complex-valued polynomial functions on the real orthogonal group  $O_N$ . It turns out that if one formulates such a non-commutative  $*$ -algebra in the right way, many of the nice group-theoretic structures associated to the fact that  $O_N$  is a compact group persist (e.g., a unique ‘‘Haar measure’’, a rich finite-dimensional unitary representation theory, a Peter-Weyl theorem, and so on).

**Definition** (Free Orthogonal Quantum Groups). Let  $N \geq 2$ , let  $A$  be a unital  $*$ -algebra over  $\mathbb{C}$ , and let  $u = [u_{ij}]_{1 \leq i, j \leq N} \in M_N(A)$  be a matrix with entries in  $A$ . Write  $u^* = [u_{ji}^*] \in M_N(A)$  and  $\bar{u} = [u_{ij}^*] \in M_N(A)$ .

- (1) The matrix  $u$  is called a *quantum orthogonal matrix* if  $u$  is invertible in  $M_N(A)$ ,  $u^* = u^{-1}$ , and  $\bar{u} = u$
- (2) The *free orthogonal quantum group* (of rank  $N$ ) is given by the triple  $O_N^+ := (\mathcal{O}(O_N^+), u, \Delta)$ , where

- (a)  $\mathcal{O}(O_N^+)$  is the universal unital  $*$ -algebra (over  $\mathbb{C}$ ) generated by the coefficients  $(u_{ij})_{1 \leq i, j \leq N}$  of a quantum orthogonal matrix  $u = [u_{ij}] \in M_N(\mathcal{O}(O_N^+))$ .
- (b)  $\Delta : \mathcal{O}(O_N^+) \rightarrow \mathcal{O}(O_N^+) \otimes \mathcal{O}(O_N^+)$  is the unique unital  $*$ -algebra homomorphism, called the *co-product*, given by

$$\Delta(u_{ij}) = \sum_{k=1}^N u_{ik} \otimes u_{kj} \quad (1 \leq i, j \leq N).$$

*Remark 1.* If we quotient  $\mathcal{O}(O_N^+)$  by its commutator ideal, we obtain the abelianization of  $\mathcal{O}(O_N^+)$ , which is isomorphic to  $\mathcal{O}(O_N)$ , the  $*$ -algebra of polynomial functions on the real orthogonal group  $O_N$ . The map  $\mathcal{O}(O_N^+) \rightarrow \mathcal{O}(O_N)$  is given by  $u_{ij} \mapsto v_{ij}$ , where  $v = [v_{ij}] \in M_N(\mathcal{O}(O_N))$  forms the matrix of basic coordinate functions on  $O_N$  (a.k.a. the *fundamental representation* of  $O_N$ ). In this context, the co-product map  $\Delta$  on  $\mathcal{O}(O_N^+)$  factors through the quotient and induces a corresponding co-product map  $\Delta$  on  $\mathcal{O}(O_N)$  given by  $\Delta(f)(s, t) = f(st)$  for all  $f \in \mathcal{O}(O_N)$  and  $s, t \in O_N$ . In this sense, we are justified in calling the quantum group  $O_N^+$  a “free analogue” of the classical orthogonal group  $O_N$ .

*Remark 2.* For those readers who are familiar with the notion of a Hopf  $*$ -algebra, we note that  $\mathcal{O}(O_N^+)$  is a natural example of one. Indeed the co-inverse  $S : \mathcal{O}(O_N^+) \rightarrow \mathcal{O}(O_N^+)^{op}$  is the algebra morphism given by  $S(u_{ij}) = u_{ji}$  and the co-unit  $\epsilon : \mathcal{O}(O_N^+) \rightarrow \mathbb{C}$  is the algebra morphism given by  $\epsilon(u_{ij}) = \mathbb{C}$ . One can then readily check that the usual Hopf-algebra identities  $(\epsilon \otimes \iota)\Delta = \iota$ ,  $m(S \otimes \iota)\Delta = m(\iota \otimes S)\Delta = \epsilon(\cdot)1$  are satisfied.

We now turn to the concept of a representation of  $O_N^+$ . A (finite-dimensional unitary) *representation* of  $O_N^+$  is given by a finite dimensional Hilbert space  $H_v$  and unitary matrix  $v \in \mathcal{O}(O_N^+) \otimes \mathcal{B}(H_v)$  satisfying

$$(\Delta \otimes \iota)v = v_{13}v_{23} \in \mathcal{O}(O_N^+) \otimes \mathcal{O}(O_N^+) \otimes \mathcal{B}(H_v),$$

where above we use the standard leg numbering notation for linear maps on tensor products. If we fix an orthonormal basis  $(e_i)_{i=1}^d \subset H_v$ , then we can write  $v$  as the matrix  $[v_{ij}] \in M_d(\mathcal{O}(O_N^+))$  with respect to this basis, and the above formula translates to

$$\Delta v_{ij} = \sum_{k=1}^d v_{ik} \otimes v_{kj} \quad (1 \leq i, j \leq d).$$

Observe that the above definition corresponds precisely to our usual notion of a unitary representation if we were to assume that our Hopf  $*$ -algebra was commutative.

The first examples of representations of  $O_N^+$  that come to mind are the one-dimensional *trivial representation* (which corresponds to the unit  $1 \in \mathcal{O}(O_N^+) = M_1(\mathcal{O}(O_N^+))$ ) and the  $N$ -dimensional *fundamental representation*  $u = [u_{ij}] \in M_N(\mathcal{O}(O_N^+))$  (corresponding to the matrix of generators for  $\mathcal{O}(O_N^+)$ ). Given two representations  $v = [v_{ij}]$  and  $w = [w_{kl}]$ , we can naturally form their *direct sum*  $v \oplus w \in \mathcal{O}(O_N^+) \otimes \mathcal{B}(H_v \oplus H_w)$  and their *tensor product*  $v \otimes w = v_{12}w_{13} = [v_{ij}w_{kl}] \in \mathcal{O}(O_N^+) \otimes \mathcal{B}(H_v \otimes H_w)$  to obtain new examples of representations from known ones. From a unitary representation  $v = [v_{ij}]$ , we may also form the *contragredient representation*  $\bar{v} := [v_{ij}^*] \in \mathcal{O}(O_N^+) \otimes \mathcal{B}(\overline{H_v})$ .

In order to study the structure of various representations of  $O_N^+$ , we use the concept of intertwiner spaces. Given two representations  $u$  and  $v$  of  $O_N^+$ , define the space of *intertwiners*

between  $u$  and  $v$  as

$$\text{Hom}(u, v) = \{T \in \mathcal{B}(H_u, H_v) : (\iota \otimes T)u = v(\iota \otimes T).\}$$

Two representations  $u, v$  are called *equivalent* (written  $u \cong v$ ) if  $\text{Hom}(u, v)$  contains an invertible operator, and a representation  $u$  is called *irreducible* if  $\text{Hom}(u, u) = \mathbb{C}1$ . It is a consequence of a general fact about compact quantum groups that every unitary representation of  $O_N^+$  is equivalent to a direct sum of irreducible unitary representations [Wor87].

It is known from [Ban96] that the irreducible corepresentations of  $O_N^+$  can be labelled  $(v^k)_{k \in \mathbb{N}_0}$  (up to unitary equivalence) in such a way that  $v^0 = 1$ ,  $v^1 = u$  ( $u$  being the fundamental representation),  $v^l \cong \overline{v^l}$ , and the following fusion rules hold:

$$(2) \quad v^l \otimes v^m \cong v^{|l-m|} \oplus v^{|l-m|+2} \oplus \dots \oplus v^{l+m} = \bigoplus_{0 \leq r \leq \min\{l,m\}} v^{l+m-2r}.$$

Denote by  $H_k$  the Hilbert space associated to  $v^k$ . Then  $H_0 = \mathbb{C}$ ,  $H_1 = \mathbb{C}^N$ , and (2) shows that the dimensions  $\dim H_k$  satisfy the recursion relations  $\dim H_1 \dim H_k = \dim H_{k+1} + \dim H_{k-1}$ . Defining the quantum parameter

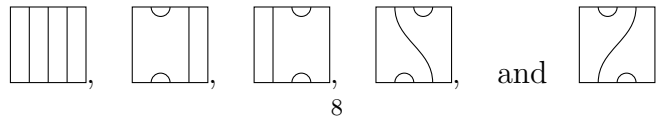
$$q = q(N) := \frac{1}{N} \left( \frac{2}{1 + \sqrt{1 - 4/N^2}} \right) \in (0, 1],$$

one can inductively show that the dimensions  $\dim H_k$  are given by the *quantum integers*

$$\dim H_k = [k + 1]_q := q^{-k} \left( \frac{1 - q^{2k+2}}{1 - q^2} \right) \quad (N \geq 3).$$

When  $N = 2$ , we have  $q = 1$ , and then  $\dim H_k = k + 1 = \lim_{q \rightarrow 1^-} [k + 1]_q$ . Note that for  $N \geq 3$ , we have the exponential growth asymptotic  $[k + 1]_q \sim N^k$  (as  $N \rightarrow \infty$ ). For our purposes, this exponential growth is crucial and therefore we generally assume  $N \geq 3$  in the sequel.

The striking resemblance of fusion rules for the irreducible representations of  $O_N^+$  to those of  $SU(2)$  is no coincidence. This turns out to be a consequence of the fact (observed by Banica) that both representation categories are described by Temperley-Lieb categories [TL71]. Let  $d \in \mathbb{C} \setminus \{0\}$ . Recall that the *Temperley-Lieb Category*  $\text{TL}(d)$  is the strict tensor category with duals whose (self-dual) irreducible objects are labelled by  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$  and whose morphism spaces  $\text{TL}_{k,l}(d) := \text{Hom}(1^{\otimes k}, 1^{\otimes l})$  are generated by the identity map  $\iota \in \text{Hom}(1, 1)$  and a unique morphism  $\cup \in \text{Hom}(0, 1 \otimes 1)$  satisfying  $\cap \circ \cup = d \in \text{Hom}(0, 0) = \mathbb{C}$ . Here  $\cap := \cup^* \in \text{Hom}(1 \otimes 1, 0)$ . The Temperley-Lieb category admits a nice diagrammatic presentation [KL94] in terms of the so-called Kauffman (or Temperley-Lieb) diagrams. Let  $k, l \in \mathbb{N}$  and  $d \in \mathbb{C} \setminus \{0\}$  be as above. If  $k + l$  is odd, we have  $\text{TL}_{k,l}(d) = 0$ . Otherwise we plot the set  $[k + l] = \{1, \dots, k + l\}$  on a rectangle clockwise with  $\{1, \dots, k\}$  on the top edge and  $\{k + l, \dots, k + 1\}$  on the bottom edge. We connect these points by a non-crossing pairing  $p \in \text{NC}_2(k + l)$ . The collection of all such Kauffman diagrams  $(D_p)_{p \in \text{NC}_2(k+l)}$  spans a basis for  $\text{TL}_{k,l}(d)$ . For example, when  $k = l = 3$  there are  $|\text{NC}_2(6)| = 5$  Kauffman diagrams spanning  $\text{TL}_{k,k}(d)$ :





In this description of the Temperley-Lieb algebra, the product  $D_p D_q$  of diagrams  $D_p$  and  $D_q$  is obtained by first stacking diagram  $D_p$  on top of  $D_q$ , connecting the bottom row of  $k$  points on  $D_p$  to the top row of  $k$  points on  $D_q$ . The result is a new planar diagram, which may have a certain number,  $c$ , of internal loops. By removing these loops, we obtain a new diagram  $D_r$  for some  $r \in NC_2(2k)$  (which is unique up to planar isotopy). The product  $D_p D_q$  is then defined to be  $d^c D_r$ . For example, we have

We leave it to the reader to verify how each of the above diagrams is obtained from sequences of the the basic operations of tensoring and composing the basic maps  $\cup, \cap$ , and  $\iota$ .

Returning to the connection with  $\text{Rep}(O_N^+)$  - observe that we can produce a natural tensor category morphism (fiber functor)  $\text{TL}(N) \rightarrow \text{Rep}(O_N^+)$  given by  $\iota \in \text{TL}_{1,1}(N) \mapsto \text{id}_{\mathbb{C}^N} \in \text{Hom}(u, u)$  and  $\cup \in \text{TL}_{0,2}(N) \mapsto \sum_{i=1}^N e_i \otimes e_i \in \text{Hom}(1, u \otimes u)$ , where  $(e_i)_{i=1}^N$  is an orthonormal basis for  $\mathbb{C}^N$ . The key point here is that the universal properties of  $O_N^+$  guarantee that this morphism is injective and surjective. More precisely, we have the following theorem of Banica.

**Theorem 2.1** (Banica [Ban96]). *The above morphism gives a faithful unitary fiber functor  $\text{TL}(N) \cong \text{Rep}(O_N^+)$ .*

With the connection between  $\text{TL}(N)$  and  $\text{Rep}(O_N^+)$ , an explicit construction of the irreducible representation spaces  $(H_k)_{k \in \mathbb{N}_0}$  of  $O_N^+$  can now proceed as follows [Ban96, VV07, BDRV06]. Denote by  $(e_i)_{i=1}^N$  a fixed orthonormal basis for  $H_1 := \mathbb{C}^N$ , and as above, put  $\cup = \sum_{i=1}^N e_i \otimes e_i \in \text{Hom}(1, u \otimes u)$ . (I.e.,  $u^{\otimes 2}(1 \otimes \cup) = (1 \otimes \cup)$ .) Next, we consider the intertwiner space  $\text{Hom}(u^{\otimes k}, u^{\otimes k}) \subseteq \mathcal{B}((\mathbb{C}^N)^{\otimes k})$ , which can be shown (using its identification with  $\text{TL}_{k,k}(N)$ ) to contain a unique non-zero self-adjoint projection  $p_k$  (*the Jones-Wenzl projection*) [Wen87] with the defining property that

$$(\iota_{H_1^{\otimes i-1}} \otimes \cup \cup^* \otimes \iota_{H_1^{\otimes k-i-1}}) p_k = 0 \quad (1 \leq i \leq k-1).$$

The projections  $p_k$  are known to satisfy the *Wenzl recursion*

$$p_1 = \iota_{H_1}, \quad p_k = \iota_{H_1} \otimes p_{k-1} - \frac{[k-1]_q}{[k]_q} (\iota_{H_1} \otimes p_{k-1}) (\cup \cup^* \otimes \iota_{H_1^{\otimes k-2}}) (\iota_{H_1} \otimes p_{k-1}) \quad (k \geq 2),$$

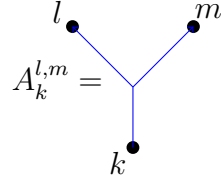
which can be used to determine  $p_k$ . In passing, we point out that the problem of obtaining explicit formulas for Jones-Wenzl projections (beyond the above recursion) has attracted a lot of attention over the years from various mathematical communities. See [BC18a, Mor15, FK97] and the references therein.

We conclude this section with a description of the non-empty intertwiner spaces  $\text{Hom}(v^k, v^l \otimes v^m)$  that arise from the fusion rules (2). To begin, let us call a triple  $(k, l, m) \in \mathbb{N}_0^3$  *admissible* if there exists an integer  $0 \leq r \leq \min\{l, m\}$  such that  $k = l + m - 2r$ . In other words,  $(k, l, m) \in \mathbb{N}_0^3$  is admissible if and only if the tensor product representation  $v^l \otimes v^m$  contains a (multiplicity-free) subrepresentation equivalent to  $v^k$ . It is easy to see that the set of admissible triples is invariant under coordinate permutations:  $(k_1, k_2, k_3)$  is admissible **iff**  $(k_{\sigma(1)}, k_{\sigma(2)}, k_{\sigma(3)})$  is admissible for all  $\sigma \in S_3$ . Fix an admissible triple  $(k, l, m) \in \mathbb{N}_0^3$ . Then  $\text{Hom}(v^k, v^l \otimes v^m) \subseteq \mathcal{B}(H_k, H_l \otimes H_m) \subseteq \mathcal{B}(H_1^{\otimes k}, H_1^{\otimes l} \otimes H_1^{\otimes m})$  is one-dimensional and is

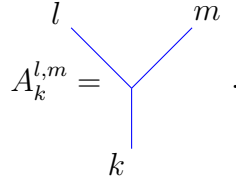
spanned by the following canonical non-zero intertwiner

$$(3) \quad A_k^{l,m} = (p_l \otimes p_m) \left( \iota_{H_{l-r}} \otimes \cup_r \otimes \iota_{m-r} \right) p_k,$$

where  $\cup_r \in \text{Hom}(1, u^{\otimes 2r})$  is defined recursively from  $\cup_1 := \cup \sum_{i=1}^N e_i \otimes e_i$  via  $\cup_r = (\iota_{H_1} \otimes \cup_1 \otimes \iota_{H_1}) \cap_{r-1}$ . In terms of the planar diagrammatics,  $\cup_r$  is simply  $r$  nested cups, viewed as an element of  $\text{TL}_{0,2r}(N)$ . The maps  $A_k^{l,m}$  are well studied in the Temperley-Lieb recoupling theory [KL94], and are known there as *three-vertices*. A three-vertex is typically diagrammatically represented as follows:



Here, the solid dots at the vertices are meant to depict the Jones-Wenzl projectors at the inputs/outputs. In the following we will simply omit these solid dots in our pictures, simply draw the three-vertex as



In order to find the unique  $O_N^+$ -equivariant *isometry*  $\alpha_k^{l,m} : H_k \rightarrow H_l \otimes H_m$  (up to multiplication by  $\mathbb{T}$ ), we simply have to renormalize  $A_k^{l,m}$ , which amounts to computing the norm of  $A_k^{l,m}$ . To do this, we define (following the terminology and diagrammatics from [KL94]) the  $\theta$ -net

$$\theta_q(k, l, m) = \text{Tr}_{H_k} ((A_k^{l,m})^* A_k^{l,m}) = l \quad \begin{array}{c} k \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ k \end{array} m.$$

Note that the trace on  $\mathcal{B}(H_k)$  corresponds to the usual Markov trace on  $\text{TL}(N)$  [KL94, Ban96].

Now, since  $A_k^{l,m}$  is a multiple of an isometry, it easily follows that  $\|A_k^{l,m}\|^2 [k+1]_q = \theta_q(k, l, m)$ .  $\theta$ -net evaluations are well known [KL94, Ver05, VV07], and are given by

$$(4) \quad \theta_q(k, l, m) := \frac{[r]_q! [l-r]_q! [m-r]_q! [k+r+1]_q!}{[l]_q! [m]_q! [k]_q!},$$

where  $k = l + m - 2r$  and  $[x]_q! = [x]_q[x-1]_q \dots [2]_q[1]_q$  denotes the quantum factorial. We thus arrive at the following formula for our isometry  $\alpha_k^{l,m}$ :

$$(5) \quad \alpha_k^{l,m} = \|A_k^{l,m}\|^{-1} A_k^{l,m} = \left( \frac{[k+1]_q}{\theta_q(k,l,m)} \right)^{1/2} A_k^{l,m}.$$

Pictorially, we have

$$\alpha_k^{l,m} = \left( \frac{[k+1]_q}{\theta_q(k,l,m)} \right)^{1/2} \begin{array}{c} l \quad m \\ \diagdown \quad / \\ \quad \quad Y \\ \diagup \quad \diagdown \\ k \end{array}$$

### 3. ENTANGLEMENT ANALYSIS

In this section we begin our study of the entanglement geometry of irreducible subrepresentations of tensor products of irreducible representations of  $O_N^+$ . The general setup we will consider is a fixed  $N \geq 3$  and an admissible triple  $(k, l, m) \in \mathbb{N}_0^3$ . This corresponds to irreducible representations  $(v^k, v^l, v^m)$  of  $O_N^+$  with corresponding representation Hilbert spaces  $(H_k, H_l, H_m)$ , and a  $O_N^+$ -equivariant isometry  $\alpha_k^{l,m} : H_k \rightarrow H_l \otimes H_m$  as constructed in the previous section. Recall that we set  $q = \frac{1}{N} \left( \frac{2}{1 + \sqrt{1 - 4/N^2}} \right) \in (0, 1)$ . Our main interest is to study the entanglement of the subspace  $\alpha_k^{l,m}(H_k) \subseteq H_l \otimes H_m$ , and the following proposition yields a measure of this.

**Proposition 3.1** ([BC18b]). *Fix  $N \geq 3$  and let  $(k, l, m) \in \mathbb{N}_0^3$  be an admissible triple. Then for any unit vectors  $\xi \in H_k, \eta \in H_l, \zeta \in H_m$ , we have*

$$|\langle \alpha_k^{l,m}(\xi) | \eta \otimes \zeta \rangle| \leq \left( \frac{[k+1]_q}{\theta_q(k,l,m)} \right)^{1/2} \leq C(q) q^{\frac{l+m-k}{4}},$$

where

$$C(q) = (1 - q^2)^{-1/2} \left( \prod_{s=1}^{\infty} \frac{1}{1 - q^{2s}} \right)^{3/2}$$

*Remark 3.* We note that the bound  $C(q) q^{\frac{l+m-k}{4}}$  appearing in Proposition 3.1 is equivalent, as  $N$  is large, to the fourth root of the relative dimension,  $\left( \frac{\dim H_k}{\dim H_l \dim H_m} \right)^{1/4}$ .

Proposition 3.1 can be interpreted as giving a general upper bound on the largest Schmidt coefficient of a unit vector belonging to the subspace  $\alpha_k^{l,m}(H_k) \subseteq H_l \otimes H_m$ . That is, if  $\xi \in H_k$  is a unit vector and  $\alpha_k^{l,m}(\xi)$  is represented by its singular value decomposition

$$\alpha_k^{l,m}(\xi) = \sum_i \sqrt{\lambda_i} e_i \otimes f_i,$$

with  $(e_i)_i \subset H_l, (f_i)_i \subset H_m$  orthonormal systems, and  $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$  satisfy  $\sum_i \lambda_i = 1$ , then

$$(6) \quad \lambda_1 \leq C(q)^2 q^{\frac{l+m-k}{2}}.$$

Since the above quantity is much smaller than 1 when  $k < l + m$ , we conclude that  $\alpha_k^{l,m}(H_k)$  is “far” from containing separable unit vectors of the form  $\eta \otimes \zeta \in H_l \otimes H_m$ . That is,  $\alpha_k^{l,m}(H_k) \subset H_l \otimes H_m$  is highly entangled. We summarize this in the following theorem.

**Theorem 3.2** ([BC18b]). *For  $k, l, m$  as above, the subspaces  $\alpha_k^{l,m}(H_k) \subseteq H_l \otimes H_m$  are (highly) entangled provided  $k < l + m$ . When  $k = l + m$ , the highest weight subspace  $\alpha_{l+m}(H_{l+m}) \subset H_l \otimes H_k$  is a separable subspace.*

*Proof.* The first statement follows from the previous proposition and the remarks that follow. The second statement follows from the observation that if one considers the elementary (separable) tensors

$$(\dots \xi \otimes \eta \otimes \xi \otimes \eta) \otimes (\xi \otimes \eta \otimes \xi \otimes \dots) \in (\mathbb{C}^N)^{\otimes l} \otimes (\mathbb{C}^N)^{\otimes m} \quad (\xi \perp \eta),$$

then they always lie in the subspace  $H_l \otimes H_m$  (thanks to the algebraic properties of the Jones-Wenzl projections!).  $\square$

In fact it turns out that one can say quite a lot more about the largest possible Schmidt coefficients for irreducible subspaces of tensor products than what is said in Proposition 3.1. The following theorem shows that the bound given above is in fact optimal in a very strong sense: For any  $d \in \mathbb{N}$ , we can find a unit vector  $\xi \in H_k$  (provided  $N$  is sufficiently large) with the property that  $\alpha_k^{l,m}(\xi)$  admits at least  $d$  Schmidt coefficients with the same magnitude as that predicted by (6).

**Theorem 3.3** ([BC18b]). *Let  $(k, l, m) \in \mathbb{N}_0^3$  be an admissible triple,  $N \geq 3$ , and  $d \leq (N - 2)(N - 1)^{\frac{l+m-k-2}{2}}$ . Then there exists a unit vector  $\xi \in H_k$  such that  $\alpha_k^{l,m}(\xi)$  has a singular value decomposition  $\alpha_k^{l,m}(\xi) = \sum_i \sqrt{\lambda_i} e_i \otimes f_i$  with  $\lambda_1 \geq \lambda_2 \geq \dots$  satisfying*

$$\lambda_1 = \lambda_2 = \dots = \lambda_d = \frac{[k+1]_q}{\theta_q(k, l, m)} \geq q^{\frac{l+m-k}{2}}.$$

*Remark 4.* For various applications of the above theorem, it is of critical importance to understand if the above result is optimal in the sense that the number  $d$  of maximal Schmidt coefficients that is obtainable is indeed given by the above bound. At this stage, we are unable to fully answer this question. However, we can show that the upper bound  $d(N) := (N - 2)(N - 1)^{\frac{m+l-k-2}{2}}$  of maximal Schmidt coefficients  $\lambda_{\max} = \frac{[k+1]_q}{\theta_q(k, l, m)}$  is *asymptotically maximal* in the sense that

$$\lim_{N \rightarrow \infty} d(N) \frac{[k+1]_q}{\theta_q(k, l, m)} = 1.$$

This shows that in the limit as  $N \rightarrow \infty$ , the vector  $\xi \in H_k$  which is asserted to exist by Theorem 3.3 becomes maximally entangled, with the bulk of its Schmidt coefficients equaling the maximal value  $\lambda_{\max}$  allowed by Proposition 3.1.

#### 4. $O_N^+$ -EQUIVARIANT QUANTUM CHANNELS AND MINIMUM OUTPUT ENTROPY ESTIMATES

In this section we consider some applications of the entanglement results of the preceding section to study the outputs of the canonical quantum channels related to our subspaces.

Following Section 2, we form, for any admissible triple  $(k, l, m) \in \mathbb{N}_0^3$ , the complementary pair of quantum channels

$$\Phi_k^{\bar{l},m} : \mathcal{B}(H_k) \rightarrow \mathcal{B}(H_m); \quad \rho \mapsto (\text{Tr} \otimes \iota)(\alpha_k^{l,m} \rho (\alpha_k^{l,m})^*),$$

$$\Phi_k^{l,\bar{m}} : \mathcal{B}(H_k) \rightarrow \mathcal{B}(H_l); \quad \rho \mapsto (\iota \otimes \text{Tr})(\alpha_k^{l,m} \rho (\alpha_k^{l,m})^*).$$

In terms of the planar diagrammatics of the Temperley-Lieb category, we have

$$\Phi_k^{l,\bar{m}}(\rho) = \frac{[k+1]_q}{\theta_q(k,l,m)} l \quad \text{and} \quad \Phi_k^{l,\bar{m}}(\rho) = \frac{[k+1]_q}{\theta_q(k,l,m)} m .$$

We then have the following proposition concerning the  $\mathcal{S}_1 \rightarrow \mathcal{S}_\infty$  behavior of these channels.

**Proposition 4.1.** *Given any admissible triple  $(k, l, m) \in \mathbb{N}_0^3$  and  $N \geq 3$ , we have*

$$\begin{aligned} \|\Phi_k^{\bar{l},m}\|_{\mathcal{S}_1(H_k) \rightarrow \mathcal{S}_\infty(H_m)} &= \|\Phi_k^{l,\bar{m}}\|_{\mathcal{S}_1(H_k) \rightarrow \mathcal{S}_\infty(H_l)} \\ &= \frac{[k+1]_q}{\theta_q(k,l,m)} \in [q^{\frac{l+m-k}{2}}, C(q)^2 q^{\frac{l+m-k}{2}}]. \end{aligned}$$

*Proof.* We shall only consider  $\Phi_k^{\bar{l},m}$  as the proof of the other case is identical. To prove the upper bound  $\|\Phi_k^{\bar{l},m}\|_{\mathcal{S}_1(H_k) \rightarrow \mathcal{S}_\infty(H_m)} \leq \frac{[k+1]_q}{\theta_q(k,l,m)}$ , note that by complete positivity, convexity and the triangle inequality, it suffices to consider a pure state  $\rho = |\xi\rangle\langle\xi| \in \mathcal{D}(H_k)$  and show that  $\|\Phi_k^{\bar{l},m}(\rho)\|_{\mathcal{S}_\infty(H_m)} \leq \frac{[k+1]_q}{\theta_q(k,l,m)}$ . But in this case, we have

$$\Phi_k^{\bar{l},m}(\rho) = (\text{Tr} \otimes \iota)(|\alpha_k^{l,m}\xi\rangle\langle\alpha_k^{l,m}\xi|) = \sum_i \lambda_i |f_i\rangle\langle f_i|,$$

where  $\alpha_k^{l,m}(\xi) = \sum_i \sqrt{\lambda_i} e_i \otimes f_i$  is the corresponding singular value decomposition. In particular,  $\|\Phi_k^{\bar{l},m}(\rho)\|_{\mathcal{S}_\infty(H_m)} = \max_i \lambda_i$ , which by Proposition 3.1 is bounded above by  $\frac{[k+1]_q}{\theta_q(k,l,m)}$ . This upper bound is obtained by taking  $\rho = |\xi\rangle\langle\xi|$ , where  $\xi$  satisfies the hypotheses of Theorem 3.3.  $\square$

The preceding norm computation for the channels  $\Phi_k^{\bar{l},m}, \Phi_k^{l,\bar{m}}$  allows for an easy estimate of a lower bound on their minimum output entropies.

**Corollary 4.2.** *Given any admissible triple  $(k, l, m) \in \mathbb{N}_0^3$  and  $N \geq 3$ , we have*

$$H_{\min}(\Phi_k^{\bar{l},m}), H_{\min}(\Phi_k^{l,\bar{m}}) \geq \log\left(\frac{\theta_q(k,l,m)}{[k+1]_q}\right) \geq -\left(\frac{l+m-k}{2}\right) \log(q) - 2 \log(C(q)).$$

*Proof.* Given a quantum channel  $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$  and  $\rho \in \mathcal{D}(H)$ , we note that  $H(\Phi(\rho)) = -\sum_i \lambda_i \log \lambda_i$ , where  $(\lambda_i)_i$  is the spectrum of  $\Phi(\rho)$ . In particular, we have the estimate

$$H(\Phi(\rho)) \geq -\log\left(\max_i \lambda_i\right) = -\log \|\Phi(\rho)\|_{\mathcal{B}(K)} \geq -\log \|\Phi\|_{\mathcal{S}_1(H) \rightarrow \mathcal{B}(K)}.$$

The first inequality in the corollary now follows immediately from Proposition 4.1. The second inequality is just a consequence of the inequality  $\frac{[k+1]_q}{\theta_q(k,l,m)} \leq C(q)^2 q^{\frac{l+m-k}{2}}$ .  $\square$

*Remark 5.* The above estimates show that for  $N$  large and  $k < l + m$  fixed, the minimum output entropy of the channels is quite large and grows logarithmically in  $N$ .

On the other hand, if we fix  $N \geq 3$  and consider, for example, the sequence of channels  $(\Phi_{k-1}^{k,\bar{1}} : \mathcal{B}(H_{k-1}) \rightarrow \mathcal{B}(H_k))_{k \in \mathbb{N}}$ , then Corollary 4.2 yields the uniform positive lower bound

$$H_{\min}(\Phi_{k-1}^{k,\bar{1}}) \geq -\log(q) - 2\log(C(q)) > 0 \quad (k \in \mathbb{N}).$$

This phenomenon stands in sharp contrast to what happens in the case of the  $SU(2)$ -equivariant quantum channels studied by Al Nuwairan in [AN13, Section 2]. Indeed, in the corresponding  $SU(2)$  setting one has  $H_{\min}(\Phi_{k-1}^{k,\bar{1}}) \approx \frac{\log(k+1)}{k+1} \rightarrow 0$  as  $k \rightarrow \infty$ .

In the case where  $k = l + m$  (the highest weight case), we note that

$$H_{\min}(\Phi_k^{\bar{l},m}) = H_{\min}(\Phi_k^{l,\bar{m}}) = 0,$$

which follows from the fact that  $\alpha_k^{l,m}(H_k) \subseteq H_l \otimes H_m$  is a separable subspace (cf. Theorem 3.2).

*Remark 6.* We expect that the lower bound for the minimum output entropies given in Corollary 4.2 to be asymptotically optimal as  $N \rightarrow \infty$ , at least in some cases (e.g.  $m$  fixed). Evidence for this is provided by Theorem 3.3 and Remark 4, which shows that  $\alpha_k^{l,m}(H_k)$  contains unit vectors which are asymptotically maximally entangled with the bulk of their Schmidt coefficients equal to  $\frac{[k+1]_q}{\theta_q(k,l,m)}$ .

## 5. THE CHOI MAP AND PLANAR ISOTOPY

In this final section we indicate how the planar structure of our representation theoretic model for highly entangled subspaces can be used to easily describe the Choi maps associated to our quantum channels. As applications of this description, we construct non-random examples of  $d$ -positive maps between matrix algebras that fail to be completely positive, and we also study the entanglement breaking property for our channels.

First we recall the definition of the Choi map associated to a linear map  $\Phi : \mathcal{B}(H_A) \rightarrow \mathcal{B}(H_B)$ . Let  $(e_i)_{i \in I}, (f_i)_{i \in I}$  be two fixed orthonormal bases for  $H_A$ , and let  $(e_{ij})_{i,j \in I}, (f_{ij})_{i,j \in I}$  be the corresponding matrix units in  $\mathcal{B}(H_A)$ . Then the Choi map is the operator  $C_\Phi \in \mathcal{B}(H_A \otimes H_B)$  given by

$$(7) \quad C_\Phi = \sum_{i,j \in I} \Phi(e_{ij}) \otimes f_{ij} = (\Phi \otimes \iota)(|\psi\rangle\langle\psi|),$$

where  $\psi = \sum_{i \in I} e_i \otimes f_i \in H_A \otimes H_A$  (which is an unnormalized Bell state in  $H_A \otimes H_A$ ). Of course,  $C_\Phi$  is only defined uniquely up to the choice of matrix units  $e_{ij}$  and  $f_{ij}$ . Moreover, one could also define a “right-handed” version of  $C_\Phi$  of  $C_\Phi$  given by  $\tilde{C}_\Phi = (\iota \otimes \Phi)|\psi\rangle\langle\psi|$  (i.e., slicing on the right instead of the left). However, for our purposes, the relevant properties of  $C_\Phi$  (e.g., entanglement, positivity, etc.) do not depend the choice of matrix units or side of the tensor product on which one slices  $|\psi\rangle\langle\psi|$  by  $\Phi$ . We also note the obvious fact that the map  $\Phi \mapsto C_\Phi$  is linear in  $\Phi$ .

Turning back to our representation category  $\text{Rep}(O_N^+)$  and our quantum channels  $\Phi_k^{\bar{l},m} : \mathcal{B}(H_k) \rightarrow \mathcal{B}(H_m)$  ( $(k, l, m) \in \mathbb{N}_0^3$  admissible), we judiciously choose orthonormal bases  $(e_i)_i$  and  $(f_i)_i$  of  $H_k$  so that the unnormalized Bell vector  $\psi_k = \sum_i e_i \otimes f_i \in H_k \otimes H_k$  belongs to the one-dimensional Hom-space  $\text{Hom}(u^0, u^k \otimes u^k)$  (this is always possible, thanks to the fact that  $O_N^+$  is a compact quantum group of Kac type. See for example [Ver07]). Using our identification  $\text{Rep}(O_N^+) \cong \text{TL}(N)$ , we can depict  $\psi_k$  (in terms of planar diagrams) as a three-vertex corresponding to the admissible triple  $(0, k, k)$ , which is explicitly given by  $(p_k \otimes p_k) \circ \cup_k \in \text{TL}_{0,2k}(N)$ , where  $p_k$  is the  $k$ th Jones-Wenzl projector. Considering the projection  $|\psi_k\rangle\langle\psi_k|$ , we have

$$|\psi_k\rangle\langle\psi_k| = \begin{array}{ccc} & k & \\ & \text{---} & \\ & \text{---} & \\ & k & \end{array} \begin{array}{ccc} & k & \\ & \text{---} & \\ & \text{---} & \\ & k & \end{array}$$

Then we can compute the corresponding Choi map  $C_{\Phi_k^{\bar{l},m}} = (\Phi_k^{\bar{l},m} \otimes \iota)(|\psi_k\rangle\langle\psi_k|)$  diagrammatically by

$$\frac{\theta_q(k, l, m)}{[k+1]_q} C_{\Phi_k^{\bar{l},m}} = l \left( \begin{array}{ccc} & m & \\ & \text{---} & \\ & \text{---} & \\ & k & \\ & \text{---} & \\ & \text{---} & \\ & k & \\ & \text{---} & \\ & \text{---} & \\ & m & \end{array} \begin{array}{c} k \\ \text{---} \\ \text{---} \\ \text{---} \\ k \end{array} \right) = l \left( \begin{array}{ccc} & m & \\ & \text{---} & \\ & \text{---} & \\ & k & \\ & \text{---} & \\ & \text{---} & \\ & k & \\ & \text{---} & \\ & \text{---} & \\ & m & \end{array} \begin{array}{c} k \\ \text{---} \\ \text{---} \\ \text{---} \\ k \end{array} \right),$$

Since the linear map defined by the above planar tangle is invariant under planar isotopy (by construction it belongs to the Temperley-Lieb category!), we see that  $\frac{\theta_q(k, l, m)}{[k+1]_q} C_{\Phi_k^{\bar{l},m}}$  also corresponds to the following planar tangle:

$$\begin{array}{ccc} & m & \\ & \text{---} & \\ & \text{---} & \\ & k & \\ & \text{---} & \\ & \text{---} & \\ & l & \\ & \text{---} & \\ & \text{---} & \\ & m & \\ & \text{---} & \\ & \text{---} & \\ & k & \end{array} = \frac{\theta_q(k, l, m)}{[l+1]_q} \alpha_l^{m,k} (\alpha_l^{m,k})^*$$

Note here that  $\alpha_l^{m,k} (\alpha_l^{m,k})^*$  is simply the orthogonal equivariant projection from  $H_m \otimes H_k$  onto the unique subspace equivalent to  $H_l$ . We have therefore arrived at the following theorem.

**Theorem 5.1.** For the  $O_N^+$ -equivariant quantum channel  $\Phi_k^{\bar{l},m} : \mathcal{B}(H_k) \rightarrow \mathcal{B}(H_m)$ , we have

$$(8) \quad C_{\Phi_k^{\bar{l},m}} = \frac{[k+1]_q}{[l+1]_q} \alpha_l^{m,k} (\alpha_l^{m,k})^*.$$

A similar argument for the complementary channel  $\Phi_k^{l,\bar{m}} : \mathcal{B}(H_k) \rightarrow \mathcal{B}(H_l)$ , yields

$$(9) \quad \tilde{C}_{\Phi_k^{l,\bar{m}}} = \frac{[k+1]_q}{[m+1]_q} \alpha_m^{k,l} (\alpha_m^{k,l})^*.$$

In the following subsections, we show the utility of Theorem 5.1.

**5.1. Examples of positive but not completely positive maps.** A crucial property of the Choi map  $C_\Phi$  associated to a linear map  $\Phi : \mathcal{B}(H_A) \rightarrow \mathcal{B}(H_B)$  is that it can be used to detect positivity properties of  $\Phi$ . More precisely, we have that  $\Phi$  is completely positive if and only if  $C_\Phi$  is positive semidefinite [Cho75]. More generally,  $C_\Phi$  can be used to detect whether or not  $\Phi$  is  $d$ -positive for any  $d \in \mathbb{N}$  [HLPS12]:  $\Phi$  is  $d$ -positive if and only if

$$\langle C_\Phi x | x \rangle \geq 0$$

for all  $x \in H_A \otimes H_B$  with a *Schmidt rank* of at most  $d$ . (That is,  $x$  admits a singular value decomposition  $x = \sum_{i=1}^s \sqrt{\lambda_i} e_i \otimes f_i$  with  $\min_i \lambda_i > 0$  and  $s \leq d$ ).

Let us now return to our usual setup of an admissible triple  $(k, l, m) \in \mathbb{N}_0^3$  corresponding to a non-highest-weight inclusion  $\alpha_k^{l,m} : H_k \hookrightarrow H_l \otimes H_m$  of irreducible representations of  $O_N^+$ ,  $N \geq 3$ . For each  $t \geq 0$ , we can consider the linear map  $\Phi_t : \mathcal{B}(H_k) \rightarrow \mathcal{B}(H_m)$  given by

$$(10) \quad \Phi_t = \text{Tr}_{H_k}(\cdot) 1_{\mathcal{B}(H_l)} - t \frac{[l+1]_q}{[k+1]_q} \Phi_k^{\bar{l},m}.$$

Using Theorem 5.1 together with the simple fact that the Choi map associated to  $\mathcal{B}(H_k) \ni \rho \mapsto \text{Tr}_{H_k}(\rho) 1_{\mathcal{B}(H_l)}$  is given by  $1_{\mathcal{B}(H_m \otimes H_k)}$ , we conclude that the Choi map of  $\Phi_t$  is given by

$$(11) \quad C_{\Phi_t} = 1_{\mathcal{B}(H_m \otimes H_k)} - t \alpha_l^{m,k} (\alpha_l^{m,k})^*.$$

From this expression for  $C_{\Phi_t}$ , it is clear that  $\Phi_t$  is completely positive iff  $C_{\Phi_t} \geq 0$  iff  $t \leq 1$ . On the other hand, we can prove the following result on  $d$ -positivity of  $\Phi_t$ .

**Theorem 5.2.** Fix  $N \geq 3$  and  $(k, l, m) \in \mathbb{N}_0^3$ , and fix a natural number  $d \leq (N-2)(N-1)^{\frac{k+m-l-2}{2}}$ . Then the map  $\Phi_t : \mathcal{B}(H_k) \rightarrow \mathcal{B}(H_m)$  is  $d$ -positive (but not completely positive) if and only if

$$1 < t \leq \frac{\theta_q(k, l, m)}{d[l+1]_q} \leq C(q)^{-2} q^{-\frac{k+m-l}{2}} d^{-1}.$$

*Sketch.* We have already observed that  $\Phi_t$  is not completely positive when  $t > 1$ . Now fix  $d \in \mathbb{N}$  and  $x = \sum_{i=1}^s \sqrt{\lambda_i} e_i \otimes f_i \in H_m \otimes H_k$  with Schmidt-rank at most  $d$ . Using the inequality of Proposition 3.1, the triangle inequality, and the Cauchy-Schwarz inequality, we



have

$$\begin{aligned}
\langle C_{\Phi_t} x | x \rangle &= \|x\|^2 - t \langle \alpha_l^{m,k} (\alpha_l^{m,k})^*(x) | x \rangle \\
&\geq \|x\|^2 - t \frac{[l+1]_q}{\theta_q(k,l,m)} \left( \sum_{1 \leq i \leq s} \sqrt{\lambda_i} \|e_i\| \|f_i\| \right)^2 \\
&\geq \|x\|^2 - t \frac{[l+1]_q}{\theta_q(k,l,m)} \left( \sum_{1 \leq i \leq s} \sqrt{\lambda_i} \right)^2 \\
&\geq \|x\|^2 - t \frac{[l+1]_q}{\theta_q(k,l,m)} s \|x\|^2 \\
&\geq \|x\|^2 \left( 1 - td \frac{[l+1]_q}{\theta_q(k,l,m)} \right).
\end{aligned}$$

From this inequality, we obtain  $d$ -positivity of  $\Phi_t$  provided  $1 - td \frac{[l+1]_q}{\theta_q(k,l,m)} \geq 0$ , as claimed.

To show failure of  $d$ -positivity when  $t > \frac{\theta_q(k,l,m)}{d[l+1]_q}$ , one has to find  $x = \sum_{i=1}^d \eta_i \otimes \zeta_i \in H_l \otimes H_m$  with Schmidt rank  $d$  satisfying  $\langle C_{\Phi_t} x | x \rangle < 0$ . It turns out that such an  $x$  can be canonically constructed - see [BC18b] for details.  $\square$

*Remark 7.* The above theorem can readily be used to construct maps on matrix algebras that are  $d$  positive but not  $d+1$  positive. Indeed, one just has to choose  $t > 1$ ,  $N \geq 3$  and an admissible triple  $(k, l, m) \in \mathbb{N}_0^3$  so that

$$\frac{\theta_q(k,l,m)}{(d+1)[l+1]_q} < t \leq \frac{\theta_q(k,l,m)}{d[l+1]_q}.$$

Then the corresponding  $\Phi_t$  will do the job.

**5.2. Entanglement breaking channels.** We now turn to another application of Theorem 5.1 to the entanglement breaking property of our quantum channels  $\Phi_k^{l,m}$ .

**Definition.** A quantum channel  $\Phi : \mathcal{B}(H_A) \rightarrow \mathcal{B}(H_B)$  is called *entanglement breaking* (or *EBT*) if for any finite-dimensional auxiliary Hilbert space  $H_0$ , and any state  $\rho \in \mathcal{D}(H_0 \otimes H_A)$ , we have that  $(\iota \otimes \Phi)(\rho) \in \mathcal{D}(H_0 \otimes H_B)$  is a separable state.

The class of EBT channels are precisely those which eliminate entanglement between the input states of composite systems. These channels form an important class which are amenable to analysis. For example, it is known that for EBT channels, both the minimum output entropy and the Holevo capacity (i.e., the capacity of a quantum channel used for classical communication with product inputs) is additive [Hol01, Sho02].

In order to detect whether or not a given quantum channel is EBT, it suffices to check whether or not the corresponding Choi map is a multiple of an entangled state. The following result is well known: see for example [AN13, Proposition 3.4].

**Proposition 5.3.** *For a quantum channel  $\Phi : \mathcal{B}(H_A) \rightarrow \mathcal{B}(H_B)$ , the following conditions are equivalent.*

- (1)  $\Phi$  is EBT.
- (2) The state  $\rho := \frac{1}{\dim H_A} C_\Phi \in \mathcal{D}(H_B \otimes H_A)$  is separable.

Before coming to our main result of this section characterizing the EBT property for the channels  $\Phi_k^{\bar{l},m}$ , we first need an elementary lemma.

**Lemma 5.4.** *Let  $H_A$  and  $H_B$  be finite dimensional Hilbert spaces, let  $0 \neq p \in \mathcal{B}(H_B \otimes H_A)$  be an orthogonal projection, and let  $H_0 \subseteq H_B \otimes H_A$  denote the range of  $p$ . If  $H_0$  is an entangled subspace of  $H_B \otimes H_A$ , then the state  $\rho := \frac{1}{\dim H_0} p$  is entangled.*

*Proof.* We prove the contrapositive. If  $\rho$  is separable, then we can write

$$p = \sum_i |\xi_i\rangle\langle\xi_i| \otimes |\eta_i\rangle\langle\eta_i| \quad (0 \neq \xi_i \in H_B, 0 \neq \eta_i \in H_A).$$

For each  $i$  put  $x_i = |\xi_i\rangle\langle\xi_i| \otimes |\eta_i\rangle\langle\eta_i|$ . Then since  $x_i \leq p$  and  $p$  is a projection, it follows that  $x_i = px_i p$ , which implies that the range of  $x_i$  is contained in the range of  $p$ . In particular,  $\xi_i \otimes \eta_i \in H_0$ , so  $H_0$  is separable.  $\square$

**Theorem 5.5.** *Let  $(k, l, m) \in \mathbb{N}_0^3$  be an admissible triple. If  $k \neq l - m$ , then the quantum channel  $\Phi_k^{\bar{l},m}$  is not EBT.*

*Proof.* We have from Theorem 5.1 that  $C_{\Phi_k^{\bar{l},m}} = \frac{[k+1]_q}{[l+1]_q} \alpha_l^{m,k} (\alpha_l^{m,k})^* \in \mathcal{B}(H_m \otimes H_k)$ . Consider the orthogonal projection  $p = \alpha_l^{m,k} (\alpha_l^{m,k})^*$ . The range of  $p$  is the subrepresentation of  $H_m \otimes H_k$  equivalent to  $H_l$ , and by Theorem 3.2 this subspace is entangled iff  $l \neq k + m$ . Applying Lemma 5.4 and Proposition 5.3, we conclude that  $\Phi_k^{\bar{l},m}$  is not EBT whenever  $k \neq l - m$ .  $\square$

*Remark 8.* We note that Theorem 5.5 leaves open whether or not the channels  $\Phi_{l-m}^{\bar{l},m}$  are EBT. In this case, the corresponding Choi map is a multiple of a projection onto a separable subspace, and we do not know if this projection is a multiple of an entangled state.

## 6. FUTURE WORK AND OPEN PROBLEMS

We conclude this survey with a list of open problems and directions for future work.

- (1) A major problem in QIT is to find explicit examples of quantum channels  $\Phi, \Psi$  which are *strictly MOE-subadditive*:  $H_{\min}(\Phi \otimes \Psi) < H_{\min}(\Phi) + H_{\min}(\Psi)$ . Such channels are known to exist with high probability [Has09, ASW11, BCN16], but no explicit examples are known. It is therefore tempting to wonder whether or not the channels considered in this work might be MOE subadditive. The first step in considering this question is to have an effective means of estimating the MOE of tensor products of our channels. In this context some computations are actually possible. In particular, if one takes one of our Temperley-Lieb channels  $\Phi$ , then it is always possible to explicitly compute the von Neumann entropy  $H(\Phi \otimes \Phi^c(\rho))$  of the output of a Bell state  $\rho$ , where  $\Phi^c$  denotes the so-called *complementary channel* associated to  $\Phi$ . It turns out that this computation involves the *quantum 6j-symbols* associated to the Temperley-Lieb category. This particular calculation is the topic of work in preparation [BC17]. At the present time, it seems that in order to have any hope of witnessing strict MOE subadditivity in our channels, more tensor products beyond simply channels and their complements need to be studied, and at this time, a new idea is needed.
- (2) Another important question related to our quantum channels is the problem of computing their classical and quantum capacities. This is another completely open and important research direction.

- (3) As we have seen in this work, the Temperley-Lieb category provides a tractable concrete model for highly entangled subspaces. It is natural to wonder what other nice tensor categories or related structures give nice models of entanglement. Perhaps certain examples coming from planar algebras [Jon99] might give some interesting results?
- (4) It would be interesting to make a further study of the family of  $d$ -positive maps  $\Phi_t$  given here. The importance of such maps in QIT is for *entanglement detection* in bipartite systems: Positive maps that are not completely positive can be used to distinguish entangled states from separable ones. Of particular interest is the problem of detecting entangled states from the *positive partial transpose (PPT) states*. In this context, the relevant maps for entanglement detection are the *indecomposable maps*. I.e., positive maps  $\Phi$  which are not of the form  $\Phi = \Phi_1 + \Phi_2 \circ t$ , where  $\Phi_{1,2}$  are completely positive, and  $t$  denotes the transpose map. In this context, we ask: *Are our families of maps  $\Phi_t$  indecomposable?*

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